SLOCC CLASSIFICATION FOR NINE FAMILIES OF FOUR-QUBITS[a]

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In 2000, Dür, Vidal and Cirac indicated that there are infinitely many SLOCC classes for four qubits. Later, Verstraete, Dehaene, and Verschelde proposed nine families of states corresponding to nine different ways of entangling four qubits. And then in 2007 Lamata et al. reported that there are eight true SLOCC entanglement classes of four qubits up to permutations of the qubits. In this paper, we investigate SLOCC classification of the nine families proposed by Verstraete, Dehaene and Verschelde, and distinguish 49 true SLOCC entanglement classes from them.

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1 Introduction

Recently, many authors have exploited SLOCC (stochastic local operations and classical communication) classification. Dür et al. showed that for pure states of three qubits there are four different degenerated SLOCC entanglement classes and two different true entanglement classes [1], and for pure states of four qubits, there are infinitely many SLOCC classes. Their proof is briefly described here. A general state of n qubits depends on $2 \times 2^n - 2$ real parameters in [1]. In [1], the invertible local operator α was fixed to that $\det(\alpha) = 1$. Hence, the local operator α depends on six real parameters. Thus, the set of SLOCC classes of n qubits depends on at least $2 \times 2^n - 2 - 6n$ real parameters. Then, it was asserted [1] there exist finite SLOCC classes for three qubits while infinite SLOCC classes for $n \geq 4$ qubits. The argument is not complete due to the following two reasons. (i). By the definition, for

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any invertible operator α, $\det(\alpha) \neq 0$. The operator α with $\det(\alpha) \neq 0$ still depends on eight real parameters because we can image to remove a hyperplane $\det(\alpha) = 0$ with six real parameters from a space with eight real parameters. That is, the set of SLOCC classes of n qubits depends on at least $2 \times 2^n - 2 - 8n$ real parameters. By the argument in [1], this lower bound allows for a finite number of SLOCC classes for $n = 4$. (ii). In Sec. 7.1 of this paper, we show that two different true SLOCC entanglement classes constitute a continuous Family L_{a_4} in [2]. The state L_{a_4} with $a = 0$ and the state L_{a_4} with $a = 1$ represent these two classes. Clearly, the latter class including the state L_{a_4} with $a \neq 0$ can be labeled by a continuous parameter $a \neq 0$. It means that finite parameters may allow for finite SLOCC classes. In other words, it cannot be asserted that there exist infinite SLOCC classes for four qubits even though the set of SLOCC classes of four qubits depends on at least six real parameters.

Verstraete et al. [2] proposed that for four qubits, there exist nine families of states corresponding to nine different ways of entanglement. They gave a representative state for each family and claimed that by determinant-one SLOCC operations, a pure state of four qubits can be transformed into one of the nine families up to permutations of the qubits.

In [3][4], the authors used the partition to investigate SLOCC classification of three qubits and four qubits. The idea for the partition was originally used to analyze the separability of n qubits and multipartite pure states in [5]. In [4], the authors reported that there are 16 true SLOCC entanglement classes of four qubits, where permutation is explicitly included in the counting. Up to permutations of the qubits, there are eight true SLOCC entanglement classes of four qubits. We can show that this classification is not complete. For example, for Span $\{O_k\Psi, O_k\Psi\}$ in [4], the canonical states are

$|0000\rangle + |1100\rangle + a|0011\rangle + b|1111\rangle$ and

$|0000\rangle + |1100\rangle + a|0001\rangle + a|0010\rangle + b|1101\rangle + b|1110\rangle$,

where $a \neq b$ [4]. It was pointed out in [6] that for the former canonical state, $a = -b$ and $a \neq -b$ represent two different true SLOCC entanglement classes, while for the latter canonical state, $ab = 0$ and $ab \neq 0$ represent another two different true SLOCC entanglement classes. It says that the partition approach cannot classify the two subfamilies represented by the above canonical states under SLOCC. Using the method in this paper, we can classify each Span $\{.....\}$ in [4]. In total, the eight Spans $\{...\}$ in [4] include more than 16 true SLOCC entanglement classes.

Miyake proposed the onionlike classification of SLOCC orbits [7]. The simple criteria for the complete SLOCC classification for three qubits were given in [8]. In [9], we proposed the SLOCC invariants and semi-invariants for four qubits. Using the invariants and semi-invariants, it can be determined if two states belong to different SLOCC entanglement classes. In [6], in terms of invariants and semi-invariants we distinguished 28 distinct true entanglement classes of four qubits, where permutations of the qubits are allowed. That classification is not complete. The invariants and semi-invariants only require simple arithmetic operations. In this paper, we will investigate SLOCC classification of each family in [2] by means of invariants and semi-invariants. We want to know how many SLOCC entanglement classes there are for each family by the definition in [1]. For example, in this paper we show that Family L_{a_4} only has two SLOCC entanglement classes: L_{a_4} with $a = 0$ and L_{a_4} with $a \neq 0$, which both are true entanglement classes. The class L_{a_4} with $a \neq 0$ includes a continuous parameter of SL (determinant-one SLOCC) classes. We also show that Family $L_{a_2 0_{3\oplus 1}}$ only has two SLOCC

entanglement classes: $L_{a_2 0_{3 \oplus 1}}$ with $a = 0$ and $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$, and the latter is a true entanglement class. As well, we demonstrate that the class $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$ includes a continuous parameter of SL classes, and can be labeled by a continuous parameter $a \neq 0$.

In this paper, we distinguish at least 49 true SLOCC entanglement classes from Verstraete et al.'s nine families. For example, Family G_{abcd}, 13; Family L_{abc_2}, 19; Family $L_{a_2 b_2}$, 4; Family L_{ab_3}, 8; Family L_{a_4}, 2; Family $L_{a_2 0_{3 \oplus 1}}$, 1; Family $L_{0_{5 \oplus 3}}$, 1; Family $L_{0_{7 \oplus 1}}$, 1; Family $L_{0_{3+\bar{1}} 0_{3+\bar{1}}}$, 0. We give the complete SLOCC classifications for families L_{a_4}, $L_{a_2 0_{3 \oplus 1}}$, $L_{0_{5 \oplus 3}}$, $L_{0_{7 \oplus 1}}$, and $L_{0_{3+\bar{1}} 0_{3+\bar{1}}}$. But we cannot guarantee that the SLOCC classifications for other families are complete.

In Sec. 3 of this paper, we exploit the classification for Family G_{abcd}. In Sec. 4 of this paper, we discuss the classification for Family L_{abc_2}. In Sec. 5 and Sec. 6 of this paper, we study the classification for families $L_{a_2 b_2}$ and L_{ab_3}, respectively. For the classifications of other families, see Sec. 7.

For the readability, we list the definitions of invariants and semi-invariants in Sec. 2 of this paper.

2   SLOCC invariants and semi-invariants

The states of a four-qubit system can be generally expressed as

$$|\psi\rangle = \sum_{i=0}^{15} a_i |i\rangle. \tag{1}$$

By definition in [1], two states $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under SLOCC if and only if there exist invertible local operators α, β, γ and δ such that

$$|\psi'\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta |\psi\rangle, \tag{2}$$

where the local operators α, β, γ and δ can be expressed as 2×2 invertible matrices

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \ \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}, \delta = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}.$$

2.1   SLOCC invariant

Let $|\psi'\rangle = \sum_{i=0}^{15} a_i' |i\rangle$ in Eq. (2). If $|\psi'\rangle$ is SLOCC equivalent to $|\psi\rangle$, then the following equation was derived by induction in [9].

$$\mathcal{I}(\psi') = \mathcal{I}(\psi) \det(\alpha) \det(\beta) \det(\gamma) \det(\delta), \tag{3}$$

where

$$\mathcal{I}(\psi) = (a_0 a_{15} - a_1 a_{14}) - (a_2 a_{13} - a_3 a_{12}) - (a_4 a_{11} - a_5 a_{10}) + (a_6 a_9 - a_7 a_8) \tag{4}$$

and

$$\mathcal{I}(\psi') = (a_2' a_{13}' - a_3' a_{12}') + (a_4' a_{11}' - a_5' a_{10}') - (a_0' a_{15}' - a_1' a_{14}') - (a_6' a_9' - a_7' a_8'). \tag{5}$$

Notice that $\mathcal{I}(\psi)$ does not vary under SL-operations, i.e., determinant-one SLOCC operations, or vanish under non-determinant-one SLOCC operations. It is easy to see that if $|\psi'\rangle$ and $|\psi\rangle$ are equivalent under SLOCC, then either $\mathcal{I}(\psi') = \mathcal{I}(\psi) = 0$ or $\mathcal{I}(\psi')\mathcal{I}(\psi) \neq 0$. Otherwise, the two states belong to different SLOCC entanglement classes. Especially, if $\mathcal{I}(\psi') \neq \mathcal{I}(\psi)$, then $|\psi\rangle$ and $|\psi'\rangle$ are inequivalent under SL-operations. Eq. (3) implies that each SLOCC entanglement class has infinite SL-classes. This is also true for n qubits.

By solving matrix equation in Eq. (2), we obtain the amplitudes a_i' of $|\psi'\rangle$. By substituting a_i' into the above $\mathcal{I}(\psi')$ in Eq. (5), we obtain the values of $\mathcal{I}(\psi')$ in the Tables of this paper.

2.2   Semi-invariants D_1, D_2 and D_3

In [8][6], we defined $D_i(\psi)$ for the state $|\psi\rangle$ as follows.

$$D_1(\psi) = (a_1 a_4 - a_0 a_5)(a_{11} a_{14} - a_{10} a_{15}) - (a_3 a_6 - a_2 a_7)(a_9 a_{12} - a_8 a_{13}), \tag{6}$$

$$D_2(\psi) = (a_4 a_7 - a_5 a_6)(a_8 a_{11} - a_9 a_{10}) - (a_0 a_3 - a_1 a_2)(a_{12} a_{15} - a_{13} a_{14}), \tag{7}$$

$$D_3(\psi) = (a_3 a_5 - a_1 a_7)(a_{10} a_{12} - a_8 a_{14}) - (a_2 a_4 - a_0 a_6)(a_{11} a_{13} - a_9 a_{15}). \tag{8}$$

Let $|\psi\rangle$ be the representative states in Tables in this paper. By solving matrix equation in Eq. (2), we obtain the amplitudes a_i' of $|\psi'\rangle$. By substituting a_i' into D_1, D_2, and D_3 in Eqs. (6,7,8), we obtain the values of D_1, D_2, and D_3 in these Tables. If $D_i = 0$ for some class in these Tables, then it implies that $D_i = 0$ for for all the states of that class. If D_i is Δ for some class in these Tables, then it means that $D_i = 0$ for some states of that class while $D_i \neq 0$ for other states of that class. For example, for the class A1.2 in Table I, D_1 is 0, D_2 is Δ, and $D_3 = 0$. It says that for some state of the class A1.2 in Table I, $D_2 \neq 0$ while $D_1 = D_3 = 0$ for each state of the class A1.2.

2.3   Semi-invariants F_i

$F_1(\psi) = (a_0 a_7 - a_2 a_5 + a_1 a_6 - a_3 a_4)^2 - 4(a_2 a_4 - a_0 a_6)(a_3 a_5 - a_1 a_7),$

$F_2(\psi) = (a_8 a_{15} - a_{11} a_{12} + a_9 a_{14} - a_{10} a_{13})^2 - 4(a_{11} a_{13} - a_9 a_{15})(a_{10} a_{12} - a_8 a_{14}),$

$F_3(\psi) = (a_0 a_{11} - a_2 a_9 + a_1 a_{10} - a_3 a_8)^2 - 4(a_2 a_8 - a_0 a_{10})(a_3 a_9 - a_1 a_{11}),$

$F_4(\psi) = (a_4 a_{15} - a_6 a_{13} + a_5 a_{14} - a_7 a_{12})^2 - 4(a_6 a_{12} - a_4 a_{14})(a_7 a_{13} - a_5 a_{15}),$

$F_5(\psi) = (a_0 a_{13} - a_4 a_9 + a_1 a_{12} - a_5 a_8)^2 - 4(a_4 a_8 - a_0 a_{12})(a_5 a_9 - a_1 a_{13}),$

$F_6(\psi) = (a_2 a_{15} - a_6 a_{11} + a_3 a_{14} - a_7 a_{10})^2 - 4(a_6 a_{10} - a_2 a_{14})(a_7 a_{11} - a_3 a_{15}),$

$F_7(\psi) = (a_0 a_{14} - a_4 a_{10} + a_2 a_{12} - a_6 a_8)^2 - 4(a_4 a_8 - a_0 a_{12})(a_6 a_{10} - a_2 a_{14}),$

$F_8(\psi) = (a_1 a_{15} - a_5 a_{11} + a_3 a_{13} - a_7 a_9)^2 - 4(a_5 a_9 - a_1 a_{13})(a_7 a_{11} - a_3 a_{15}),$

$F_9(\psi) = (a_0 a_{15} - a_2 a_{13} + a_1 a_{14} - a_3 a_{12})^2 - 4(a_0 a_{14} - a_2 a_{12})(a_1 a_{15} - a_3 a_{13}),$

$F_{10}(\psi) = (a_4 a_{11} - a_7 a_8 + a_5 a_{10} - a_6 a_9)^2 - 4(a_7 a_9 - a_5 a_{11})(a_6 a_8 - a_4 a_{10})).$

Let $|\psi\rangle$ be the representative states in Tables in this paper. By solving matrix equation in Eq. (2), we obtain the amplitudes a_i' of $|\psi'\rangle$. By substituting a_i' into the above F_i, we can obtain the values of F_i. Thus, we can derive the properties of F_i for a SLOCC class.

Denotation.   Let $P = \det^2(\beta)\det^2(\delta)\det^2(\gamma)$, $Q = \det^2(\alpha)\det^2(\gamma)\det^2(\delta)$, $R = \det^2(\alpha)\det^2(\beta)\det^2(\delta)$, $S = \det^2(\alpha)\det^2(\gamma)\det^2(\beta)$, $T = \det(\alpha)\det(\beta)\det(\gamma)\det(\delta)$.

3   Family G_{abcd}

The representative state of this family is $G_{abcd} = \frac{a+d}{2}(|0000\rangle+|1111\rangle)+\frac{a-d}{2}(|0011\rangle+|1100\rangle)+\frac{b+c}{2}(|0101\rangle+|1010\rangle)+\frac{b-c}{2}(|0110\rangle+|1001\rangle)$. G_{abcd} becomes a product state of two EPR pairs for the following cases:

$a = b = c = d$; $x = y = z = 0$ and $u \neq 0$; $x = y = z = -u$; $x = y = -z = -u$, where x, y, z, u are distinct and $x, y, z, u \in \{a, b, c, d\}$.

When either $b = c = 0$ and $a = \pm d \neq 0$ or $a = d = 0$ and $b = \pm c \neq 0$, the states obtained from G_{abcd} belong to the class $|GHZ\rangle$.

The state G_{abcd} satisfies the following equations.

$$
\begin{aligned}
\mathcal{I} &= (a^2 + b^2 + c^2 + d^2)/2, \\
D_1 &= (ac + bd)(ab + cd)/4, \\
D_2 &= (a^2 + b^2 - c^2 - d^2)(-a^2 + b^2 - c^2 + d^2)/16, \\
D_3 &= -(-ac + bd)(ab - cd)/4, \\
F_i &= (b^2 - c^2)(a^2 - d^2)/4, i = 1 \text{ to } 8, \\
F_9 &= a^2 d^2, F_{10} = b^2 c^2.
\end{aligned}
\tag{9}
$$

We consider the following four subfamilies and list the true SLOCC entanglement classes of each subfamily in Tables I (1), I (2.1), I (2.2) and I (2.3).

3.1   Subfamily G_{abcd} with $x = y = 0$ and $zu \neq 0$, where different x, y, u, $v \in \{a, b, c, d\}$

(1). G_{abcd} with $a = d = 0$ and $bc \neq 0$ is equivalent to G_{abcd} with $b = c = 0$ and $ad \neq 0$ under SLOCC $\sigma_x \otimes I \otimes I \otimes \sigma_x$, where I is the identity matrix.

(2). G_{abcd} with $c = d = 0$ and $ab \neq 0$ is equivalent to G_{abcd} with $a = b = 0$ and $cd \neq 0$ under SLOCC $I \otimes \sigma_y \otimes \sigma_y \otimes I$.

(3). G_{abcd} with $b = d = 0$ and $ac \neq 0$ is equivalent to G_{abcd} with $a = c = 0$ and $bd \neq 0$ under SLOCC $I \otimes \sigma_x \otimes I \otimes \sigma_x$.

(4). G_{abcd} with $b = d = 0$ and $ac \neq 0$ is equivalent to G_{abcd} with $c = d = 0$ and $ab \neq 0$. This is because $a(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle)$

$= \alpha \otimes \beta \otimes \gamma \otimes \delta(-a(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle + |0110\rangle + |1001\rangle)))$, where $\alpha = \gamma = diag\{i, 1\}$, $\beta = diag\{-i, 1\} = -\delta$.

Hence, we only need to consider the subfamily G_{abcd} with $b = c = 0$ and $ad \neq 0$. In this subfamily G_{abcd} with $b = c = 0$ and $ad \neq 0$, there are three true SLOCC entanglement classes, denoted as A1.1 (i.e., $|GHZ\rangle$), A1.2, and A1.3.

For the class A1.1, it includes states G_{abcd} with $b = c = 0$ and $a = \pm d \neq 0$, which are equivalent to $|GHZ\rangle$.

For the class A1.2, it includes states G_{abcd} with $b = c = 0$ and $a^2 + d^2 = 0$. A1.2 is a true SLOCC entanglement class. We can argue this as follows. It is straightforward to verify $G_{abcd}(b = c = 0, d = \pm ai) = \alpha \otimes \beta \otimes \gamma \otimes \delta \ G_{abcd}(b = c = 0, a = 1, d = \pm i)$, where $\alpha = \delta = diag(\sqrt{a}, 1)$ and $\beta = \gamma = diag(1, \sqrt{a})$. Also, $G_{abcd}(b = c = 0, a = 1, d = -i) = \alpha \otimes \beta \otimes \gamma \otimes \delta G_{abcd}(b = c = 0, a = 1, d = i)$, where $\alpha = diag(i, 1)$, $\beta = diag(1, -i)$, $\gamma = diag(-1, 1)$, and $\delta = I$.

Verstraete indicated that Family G_{abcd} includes the state $|\phi_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$. Now, we can exactly say that the state $|\phi_4\rangle$ is in the class A1.2 whose representative is the state $G_{abcd}(b = c = 0, a = 1, d = i)$. This is because the representative state is equivalent to $|\phi_4\rangle$ under SLOCC $\alpha \otimes \beta \otimes \gamma \otimes \delta$, where $\alpha = diag(i, 1)$, $\beta = I$, $\gamma = diag(1 - i, 1)$ and $\delta = diag(1, -(1 + i))$.

For the class A1.3, it includes states G_{abcd} with $b = c = 0$ and $a^2 + d^2 \neq 0$. We cannot classify A1.3 further.

The three classes A1.1, A1.2 and A1.3 are different by the values of $\mathcal{I}$, D_1, D_2, and D_3 in Table I (1).

For each state of this subfamily, the following equations hold.

$\mathcal{I} = (a^2 + d^2)/2 * T$, $D_1 = 0$, $D_2 = (a^2 - d^2)/16 * (...)$, $D_3 = 0$,

$F_1 = a^2 d^2 \alpha_1^2 \alpha_2^2 * P$, $F_2 = a^2 d^2 \alpha_3^2 \alpha_4^2 * P$, $F_3 = a^2 d^2 \beta_1^2 \beta_2^2 * Q$, $F_4 = a^2 d^2 \beta_3^2 \beta_4^2 * Q$,

$F_5 = a^2 d^2 \gamma_2^2 \gamma_1^2 * R$, $F_6 = a^2 d^2 \gamma_4^2 \gamma_3^2 * R$, $F_7 = a^2 d^2 \delta_1^2 \delta_2^2 * S$, $F_8 = a^2 d^2 \delta_3^2 \delta_4^2 * S$,

$F_9 = a^2 d^2 \left(\alpha_1 \beta_1 \alpha_4 \beta_4 - \beta_3 \alpha_3 \beta_2 \alpha_2\right)^2 \det^2(\delta) \det^2(\gamma)$,

$F_{10} = a^2 d^2 \left(-\alpha_1 \beta_3 \alpha_4 \beta_2 + \beta_1 \alpha_3 \beta_4 \alpha_2\right)^2 \det^2(\delta) \det^2(\gamma)$.

From the above equations, we have the following property.

Property 1.1.

For each state of this subfamily G_{abcd} with $b = c = 0$ and $ad \neq 0$, if $F_1 F_2 = 0 \wedge F_3 F_4 = 0$ then $F_9 = 0 \wedge F_{10} \neq 0$ or $F_9 \neq 0 \wedge F_{10} = 0$.

By computing, we obtain the Table I (1). For the first row of Table I (1), each state of the class $|GHZ\rangle$ must satisfy $\mathcal{I} \neq 0$, $D_1 = D_2 = D_3 = 0$. Δ in this paper may be zero or not.

Table I (1) The true SLOCC entanglement classes in the subfamily G_{abcd} with $b = c = 0$ and $ad \neq 0$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
A1.1 (i.e., $|GHZ\rangle$)	$a = \pm d$	$\neq 0$	0	0	0
A1.2 (i.e., $|\phi_4\rangle$)	$a \neq \pm d$, and $a^2 + d^2 = 0$	$= 0$	0	Δ	0
A1.3, *	$a \neq \pm d$, and $a^2 + d^2 \neq 0$	$\neq 0$	0	Δ	0

Note that "*" means that we cannot show that it has only one true SLOCC entanglement class.

3.2 Subfamily G_{abcd} with $x = \pm y \neq 0$ and $u = \pm v \neq 0$, where different x, y, u, $v \in \{a, b, c, d\}$

3.2.1 Subsubfamily G_{abcd} with $a = \pm d \neq 0$ and $b = \pm c \neq 0$

(1). The state G_{abcd} with $a = -d$ and $b = c$ is equivalent to G_{abcd} with $a = d$ and $b = -c$ under SLOCC $I \otimes \sigma_x \otimes I \otimes \sigma_x$.

(2). The state G_{abcd} with $a = -d$ and $b = -c$ is equivalent to G_{abcd} with $a = d$ and $b = c$ under SLOCC $I \otimes I \otimes \sigma_x \otimes \sigma_x$.

Therefore we only need to consider the subsubfamily G_{abcd} with $a = d$ and $b = \pm c$ but $a \neq \pm b$ as follows.

This subsubfamily has four true SLOCC entanglement classes denoted as A2.1, A2.2, A3.1 and A3.2.

For the class A2.1, it includes states G_{abcd} with $a = d$, $b = c$, $a \neq \pm b$, and $a^2 + b^2 = 0$. That is, A2.1 includes $a(|0000\rangle + |1111\rangle) \pm ai(|0101\rangle + |1010\rangle))$. A2.1 is a true SLOCC

entanglement class. We can argue this as follows. It is straightforward to verify $a(|0000\rangle +$
$|1111\rangle) \pm ai(|0101\rangle + |1010\rangle) = \alpha \otimes \beta \otimes \gamma \otimes \delta (|0000\rangle + |1111\rangle \pm i(|0101\rangle + |1010\rangle))$, where
$\alpha = \delta = diag(\sqrt{a}, 1)$ and $\beta = \gamma = diag(1, \sqrt{a})$. We can also show that two states $(|0000\rangle +$
$|1111\rangle \pm i(|0101\rangle + |1010\rangle))$ are equivalent under SLOCC. This is because $(|0000\rangle + |1111\rangle -$
$i(|0101\rangle + |1010\rangle)) = \alpha \otimes \beta \otimes \gamma \otimes \delta (|0000\rangle + |1111\rangle + i(|0101\rangle + |1010\rangle))$, where $\gamma = \delta = I$
and $\alpha = \beta = \sigma_z$.

For the class A2.2, it includes states G_{abcd} with $a = d$, $b = c$, $a \neq \pm b$, and $a^2 + b^2 \neq 0$.
We cannot classify A2.2 further.

For the class A3.1, it includes states G_{abcd} with $a = d$, $b = -c$, $a \neq \pm b$, and $a^2 + b^2 = 0$.
That is, A3.1 includes $a(|0000\rangle + |1111\rangle) \pm ai(|0110\rangle + |1001\rangle)$. We can show that A3.1 is a true
SLOCC entanglement class as follows. It is plain to verify that $a(|0000\rangle + |1111\rangle) \pm ai(|0110\rangle +$
$|1001\rangle) = \alpha \otimes \beta \otimes \gamma \otimes \delta (|0000\rangle + |1111\rangle \pm i(|0110\rangle + |1001\rangle))$, where $\alpha = \beta = diag(\sqrt{a}, 1)$ and
$\gamma = \delta = diag(1, \sqrt{a})$. Furthermore. we can demonstrate that $(|0000\rangle + |1111\rangle + i(|0110\rangle +$
$|1001\rangle)) = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle - i(|0110\rangle + |1001\rangle))$, where $\gamma = \delta = I$ and $\alpha = \beta = \sigma_z$.

For the class A3.2, it includes states G_{abcd} with $a = d$, $b = -c$, $a \neq \pm b$, and $a^2 + b^2 \neq 0$.
We cannot classify A3.2 further.

The four classes are different by the values of $\mathcal{I}$, D_1, D_2, and D_3 in (2.1).

Table I (2.1). The true SLOCC entanglement classes in the subsubfamily G_{abcd} with $a = d$ and
$b = \pm c$ but $a \neq \pm b$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
A2.1	$b = c$, $a^2 + b^2 = 0$	$= 0$	Δ	0	0
A2.2, *	$b = c$, $a^2 + b^2 \neq 0$	$\neq 0$	Δ	0	0
A3.1	$b = -c$, $a^2 + b^2 = 0$	$= 0$	0	0	Δ
A3.2, *	$b = -c$, $a^2 + b^2 \neq 0$	$\neq 0$	0	0	Δ

Let us show that this subsubfamily is different from the subfamily G_{abcd} with $x = y = 0$
and $zu \neq 0$ as follows.

The representative state of this subsubfamily is G_{abcd} with $a = d$ and $b = \pm c$. From
Eq. (9), this representative state satisfies $F_i = 0$, $i = 1$ to 8, $F_9 = a^4$, and $F_{10} = b^4$. This
violates property 1.1. Therefore this subsubfamily is different from the subfamily G_{abcd} with
$x = y = 0$ and $zu \neq 0$.

By calculating, each state of classes A2.1 and A2.2 satisfies the following equations.
$\mathcal{I} = (a^2 + b^2)$, $D_1 = -ab * (...)$, $D_2 = D_3 = 0$,
$F_1 = (a^2 - b^2)\alpha_1^2\alpha_2^2 * P$, $F_2 = (a^2 - b^2)\alpha_3^2\alpha_4^2 * P$,
$F_3 = (a^2 - b^2)\beta_1^2\beta_2^2 * Q$, $F_4 = (a^2 - b^2)\beta_3^2\beta_4^2 * Q$,
$F_5 = (a^2 - b^2)\gamma_2^2\gamma_1^2 * R$, $F_6 = (a^2 - b^2)\gamma_4^2\gamma_3^2 * R$,
$F_7 = (a^2 - b^2)\delta_1^2\delta_2^2 * S$, $F_8 = (a^2 - b^2)\delta_3^2\delta_4^2 * S$,

We omit F_9 and F_{10} because they are too complicated. From the above F_i, we can derive
the following property 1.2.

Property 1.2.

(1). For each state of the subsubfamily G_{abcd} with $a = d$ and $b = c$ but $a \neq \pm b$,

   (i). if $\mathcal{I} = 0$, then $F_9 = F_{10}$,

   (ii). if $F_1F_2 = 0 \wedge F_3F_4 = 0$, then $F_9 \neq 0 \wedge F_{10} \neq 0$,

   (iii). if $\mathcal{I} \neq 0$ and $F_1F_2 = 0 \wedge F_3F_4 = 0$, then $F_9 \neq F_{10}$.

(2). For each state of the subsubfamily G_{abcd} with $a = d$ and $b = c$ but $a \neq \pm b$, if $F_1 F_2 = 0 \wedge F_3 = F_4 = 0$ or $F_3 F_4 = 0 \wedge F_1 = F_2 = 0$, then $D_1 \neq 0$.

3.2.2 Subsubfamily G_{abcd} with $a = \pm b \neq 0$, and $c = \pm d \neq 0$

(1). When $a = b \wedge c = d$ or $a = -b \wedge c = -d$, and $a^2 + c^2 = 0$, the states are in A2.1.

(2). When $a = b \wedge c = d$ or $a = -b \wedge c = -d$, and $a^2 + c^2 \neq 0$, the states are in A2.2.

(3). When $a = b \wedge c = -d$ or $a = -b \wedge c = d$, and $a^2 + c^2 = 0$, the states are in A3.1.

(4). When $a = b \wedge c = -d$ or $a = -b \wedge c = d$, and $a^2 + c^2 \neq 0$, the states are in A3.2.

3.2.3 Subsubfamily G_{abcd} with $a = \pm c \neq 0$ and $b = \pm d \neq 0$

The states G_{abcd} with $a = \pm c$ and $b = \pm d$ are equivalent to the states G_{abcd} with $a = \pm b$, and $c = \pm d$ in the above subsection 3.2.2 under SLOCC. For example, the state G_{abcd} with $a = c$ and $b = d$, denoted as $G_{abcd}(a = c$ and $b = d)$, is equivalent to the state G_{abcd} with $a = b$, and $c = d$, denoted as $G_{abcd}(a = b$ and $c = d)$. This fact can be verified as follows. Let $\alpha = \delta = diag\{i, 1\}$, $\beta = \gamma = diag\{-i, 1\}$. Then, it is easy to see that $G_{abcd}(a = b$ and $c = d) = \alpha \otimes \beta \otimes \gamma \otimes \delta G_{abcd}(a = c$ and $b = d)$.

3.3 Subfamily G_{abcd} with either $a = \pm d \neq 0$ and $b \neq \pm c$, or $b = \pm c \neq 0$ and $a \neq \pm d$

Note that

(1). The state G_{abcd} with $b = c$ and $a \neq \pm d$ is equivalent to state G_{abcd} with $a = d$ and $b \neq \pm c$ under $\sigma_x \otimes I \otimes \sigma_x \otimes I$.

(2). The state G_{abcd} with $b = -c$ and $a \neq \pm d$ is equivalent to the state G_{abcd} with $a = d$ and $b \neq \pm c$ under $\sigma_x \otimes I \otimes I \otimes \sigma_x$.

(3). Furthermore, the state G_{abcd} with $a = -d$ and $b \neq \pm c$ is equivalent to the state G_{abcd} with $a = d$ and $b \neq \pm c$ under $I \otimes I \otimes \sigma_x \otimes \sigma_x$.

Therefore, we only need to consider the subfamily G_{abcd} with $a = d$ and $b \neq \pm c$ as follows.

This subfamily has four true SLOCC entanglement classes denoted as A4.1, A4.2, A4.3 and A4.4.

For the class A4.1, it includes states G_{abcd} with $a = d$, either $a = \pm b$ or $\pm c$, and $2a^2 + b^2 + c^2 = 0$.

For the class A4.2, it includes states G_{abcd} with $a = d$, either $a = \pm b$ or $\pm c$, and $2a^2 + b^2 + c^2 \neq 0$.

For the class A4.3, it includes states G_{abcd} with $a = d$, $a \neq \pm b$, $a \neq \pm c$, and $2a^2 + b^2 + c^2 = 0$.

For the class A4.4, it includes states G_{abcd} with $a = d$, $a \neq \pm b$, $a \neq \pm c$, and $2a^2 + b^2 + c^2 \neq 0$.

We cannot classify A4.1, A4.2, A4.3 or A4.4 further.

Table I (2.2). The true SLOCC entanglement classes in the subfamily G_{abcd} with $a = d$ and $b \neq \pm c$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
A4.1, *	either $a = \pm b$ or $\pm c$, and $2a^2 + b^2 + c^2 = 0$	$= 0$	Δ	Δ	Δ
A4.2, *	either $a = \pm b$ or $\pm c$, and $2a^2 + b^2 + c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
A4.3, *	$a \neq \pm b$, $a \neq \pm c$, and $2a^2 + b^2 + c^2 = 0$	$= 0$	Δ	Δ	Δ
A4.4, *	$a \neq \pm b$, $a \neq \pm c$, and $2a^2 + b^2 + c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ

We demonstrate the following properties of this subfamily.

When $a = d$, G_{abcd} becomes $G_{abcd}(a = d) = a(|0000\rangle + |1111\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle)$. Any state connected with $G_{abcd}(a = d)$ by SLOCC satisfies the following: $\mathcal{I} = 1/2(2a^2 + b^2 + c^2)T$,

$$
\begin{aligned}
F_1 &= (a^2 - b^2)(a^2 - c^2)^2 \alpha_1^2 \alpha_2^2 P, F_2 = (a^2 - b^2)(a^2 - c^2)^2 \alpha_3^2 \alpha_4^2 P, \\
F_3 &= (a^2 - b^2)(a^2 - c^2)^2 \beta_1^2 \beta_2^2 Q, F_4 = (a^2 - b^2)(a^2 - c^2)^2 \beta_3^2 \beta_4^2 Q, \\
F_5 &= (a^2 - b^2)(a^2 - c^2)^2 \gamma_2^2 \gamma_1^2 R, F_6 = (a^2 - b^2)(a^2 - c^2)^2 \gamma_4^2 \gamma_3^2 R, \\
F_7 &= (a^2 - b^2)(a^2 - c^2)^2 \delta_1^2 \delta_2^2 S, F_8 = (a^2 - b^2)(a^2 - c^2)^2 \delta_3^2 \delta_4^2 S.
\end{aligned}
\tag{10}
$$

We omit the complicated expressions of D_1, D_2, D_3, F_9 and F_{10}.

From Eq. (10), we have the following property 1.3.

Property 1.3.

(1). From Eq. (10), each state of the subfamily G_{abcd} with $a = d$, $b \neq \pm c$, and either $a = \pm b$ or $\pm c$ satisfies $F_i = 0$, $i = 1$ to 8, and when $b^2 + c^2 \neq 0$, $|F_9| + |F_{10}| \neq 0$.

(2). From Eq. (10), for each state of the subfamily G_{abcd} with $a = d$, $b \neq \pm c$, $a \neq \pm b$ and $a \neq \pm c$, if $F_1 = F_2 = F_3 = F_4 = 0$, then one can obtain (i) $F_9 \neq 0$ and $F_{10} \neq 0$ whenever $abcd \neq 0$, and (ii) $F_9 \neq F_{10}$ whenever $a^4 \neq b^2 c^2$.

From Eq. (10), $F_i \neq 0$, $i = 1, ..., 8$ for some states of the subfamily G_{abcd} with $a = d$, $b \neq \pm c$. Therefore, classes A4.1, A4.2, A4.3 and A4.4 are different from each other. Also see Table I (2.2).

Let us argue that this subfamily is different from the subfamily G_{abcd} with $x = y = 0$ and $zu \neq 0$ and the subfamily G_{abcd} with $a = \pm d$ and $b = \pm c$ as follows.

The representative state of this subfamily is G_{abcd} with $a = d$ and $b \neq \pm c$. From Eq. (9), this representative state satisfies $F_i = 0$, $i = 1$ to 8, $F_9 = a^4$, $F_{10} = b^2 c^2$, $D_1 = a^2(b+c)^2/4 \neq 0$, $D_2 = (b^2 - c^2)^2/16 \neq 0$, $D_3 = -a^2(b-c)^2/4 \neq 0$. These F_i violate property 1.1. Hence, this subfamily is different from the subfamily G_{abcd} with $x = y = 0$ and $zu \neq 0$. These D_i also violate the criteria for D_i in Table I (2.1), therefore this subfamily is different from the subfamily G_{abcd} with $a = \pm d$ and $b = \pm c$.

3.4   Subfamily G_{abcd} with $x \neq \pm y$, or $x \neq \pm y$ but only one $r = s$, where x, $y \in \{a, b, c, d\}$, $r \in \{\pm a, \pm d\}$, and $s \in \{\pm b, \pm c\}$

Class A4.5 includes states G_{abcd} with $a^2 + b^2 + c^2 + d^2 = 0$. The class A4.6 includes states G_{abcd} with $a^2 + b^2 + c^2 + d^2 \neq 0$. We cannot classify A4.5 or A4.6 further.

Table I (2.3). The true SLOCC entanglement classes in the subfamily G_{abcd} with $x \neq \pm y$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
A4.5, *	$a^2 + b^2 + c^2 + d^2 = 0$	$= 0$	Δ	Δ	Δ
A4.6, *	$a^2 + b^2 + c^2 + d^2 \neq 0$	$\neq 0$	Δ	Δ	Δ

For this subfamily, $\mathcal{I} = \frac{1}{2}(a^2 + b^2 + c^2 + d^2)T$. The two classes are different because of their different values of $\mathcal{I}$. See Table I (2.3).

4   Family L_{abc_2}

The representative state of this family is $L_{abc_2} = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle$. When $a = b = c = 0$, this becomes a full separable state. We

divide Family L_{abc_2} into three subfamilies. They are the subfamily L_{abc_2} with $c = 0$, the subfamily L_{abc_2} with $abc \neq 0$, and the subfamily L_{abc_2} with $c \neq 0$, $ab = 0$. We list the true SLOCC entanglement classes of each subfamily in Tables II (1), II (2), II (3), and II (4), and demonstrate that these classes of each subfamily are distinct true SLOCC entanglement classes in Appendix A.

4.1 The classification for the subfamily L_{abc_2} with $c = 0$

Here the state L_{abc_2} with $c = 0$ represents this subfamily. Let $L_{abc_2}(c = 0) = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + |0110\rangle$. The subfamily $L_{abc_2}(c = 0)$ includes the following five distinct true SLOCC entanglement classes. The five classes are denoted as B1.1, B1.2, B1.3, B1.4, and B1.5. See Table II (1).

For the class B1.1, it includes states L_{abc_2} with $c = 0$ and $a = b \neq 0$. That is, B1.1 includes $a(|0000\rangle + |1111\rangle) + |0110\rangle$. B1.1 is a true SLOCC entanglement class because $a(|0000\rangle + |1111\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle + |0110\rangle)$, where $\gamma = \delta = I$, $\alpha = diag(1, a)$, and $\beta = diag(a, 1)$.

For the class B1.2, it includes states L_{abc_2} with $c = 0$ and $a = -b \neq 0$. That is, B1.2 includes $a(|0011\rangle + |1100\rangle) + |0110\rangle$. B1.2 is a true SLOCC entanglement class because $a(|0011\rangle + |1100\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0011\rangle + |1100\rangle + |0110\rangle)$, where $\gamma = \delta = I$, $\alpha = diag(1, a)$, and $\beta = diag(a, 1)$.

For the class B1.3, it includes states L_{abc_2} with $c = 0$ and $a \neq \pm b$ and $a^2 + b^2 = 0$. That is, B1.3 includes $\frac{a(1\pm i)}{2}(|0000\rangle + |1111\rangle) + \frac{a(1\mp i)}{2}(|0011\rangle + |1100\rangle) + |0110\rangle$. We can prove that B1.3 is a true SLOCC entanglement class as follows. It is easy to verify that $\frac{a(1\pm i)}{2}(|0000\rangle + |1111\rangle) + \frac{a(1\mp i)}{2}(|0011\rangle + |1100\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(\frac{(1\pm i)}{2}(|0000\rangle + |1111\rangle) + \frac{(1\mp i)}{2}(|0011\rangle + |1100\rangle) + |0110\rangle)$, where $\alpha = diag(1, a)$, $\beta = diag(a, 1)$, $\gamma = \delta = I$. Also, we can show that $\frac{(1+i)}{2}(|0000\rangle + |1111\rangle) + \frac{(1-i)}{2}(|0011\rangle + |1100\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(\frac{(1-i)}{2}(|0000\rangle + |1111\rangle) + \frac{(1+i)}{2}(|0011\rangle + |1100\rangle) + |0110\rangle)$, where $\alpha = diag(1, i)$, $\beta = diag(-i, 1)$, $\gamma = diag(-1, 1)$, and $\delta = I$.

For the class B1.4, it includes states L_{abc_2} with $c = 0$, $a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 \neq 0$. We cannot classify B1.4 further.

For the class B1.5, it includes states L_{abc_2} with $c = 0$, $a \neq \pm b$, and $ab = 0$. B1.5 is a true SLOCC entanglement class. The following is our argument. $\frac{a}{2}(|0000\rangle + |1111\rangle) \pm \frac{a}{2}(|0011\rangle + |1100\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(\frac{1}{2}(|0000\rangle + |1111\rangle) \pm \frac{1}{2}(|0011\rangle + |1100\rangle) + |0110\rangle)$, where $\alpha = diag(1, a)$, $\beta = diag(a, 1)$, $\gamma = \delta = I$. We can also show $\frac{1}{2}(|0000\rangle + |1111\rangle) + \frac{1}{2}(|0011\rangle + |1100\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(\frac{1}{2}(|0000\rangle + |1111\rangle) - \frac{1}{2}(|0011\rangle + |1100\rangle) + |0110\rangle)$, where $\alpha = \delta = \sigma_z$, $\beta = \gamma = I$.

Classes B1.1-B1.5 are distinct because of their different values of $\mathcal{I}$, D_1, D_2, and D_3 in Table II (1).

Table II (1). The true SLOCC entanglement classes in the subfamily L_{abc_2} with $c = 0$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
B1.1	$a = b \neq 0$	$\neq 0$	0	0	Δ
B1.2	$a = -b \neq 0$	$\neq 0$	Δ	0	0
B1.3	$a \neq \pm b$, and $a^2 + b^2 = 0$	$= 0$	Δ	Δ	Δ
B1.4, *	$a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
B1.5	$a \neq \pm b$, and $ab = 0$	$\neq 0$	0	Δ	0

4.2 The classification for the subfamily L_{abc_2} with $abc \neq 0$

This subfamily has three inequivalent subsubfamilies under SLOCC. They are the subsubfamily L_{abc_2} with $abc \neq 0$ and $a = b$, the subsubfamily L_{abc_2} with $abc \neq 0$ and $a = -b$, and the subsubfamily L_{abc_2} with $abc \neq 0$, $a \neq \pm b$.

Table II (2). The true SLOCC entanglement classes in the subfamily L_{abc_2} with $abc \neq 0$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
B2.1	$a = b$, $a = \pm c$	$\neq 0$	Δ	0	0
B2.2	$a = b$, $a \neq \pm c$, and $a^2 + c^2 = 0$	$= 0$	Δ	Δ	Δ
B2.3, *	$a = b$, $a \neq \pm c$, and $a^2 + c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
B3.1	$a = -b$, and $a = \pm c$	$\neq 0$	0	0	Δ
B3.2	$a = -b$, $a \neq \pm c$, and $a^2 + c^2 = 0$	$= 0$	Δ	Δ	Δ
B3.3, *	$a = -b$, $a \neq \pm c$, and $a^2 + c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
B4.1, *	$a \neq \pm b$, $a = c$, and $(3a^2 + b^2) = 0$	$= 0$	Δ	Δ	Δ
B4.2, *	$a \neq \pm b$, $a = c$, and $(3a^2 + b^2) \neq 0$	$\neq 0$	Δ	Δ	Δ

4.2.1 Subsubfamily L_{abc_2} with $abc \neq 0$ and $a = b$

This subsubfamily has the following three distinct true SLOCC entanglement classes. They are named as B2.1, B2.2, and B2.3. See Table II (2).

For the class B2.1, it includes states L_{abc_2} with $abc \neq 0$, $a = b$, and $a = \pm c$. That is, B2.1 includes $a\,(|0000\rangle + |1111\rangle) \pm a\,(\,|0101\rangle + |1010\rangle) + |0110\rangle$. We can show that B2.1 is a true entanglement class as follows. $a\,(|0000\rangle + |1111\rangle) \pm a\,(\,|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle \pm (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = diag(1, a)$, $\gamma = diag(a, 1)$, $\beta = \delta = I$. Also, $|0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle - (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \delta = \sigma_z$, and $\beta = \gamma = I$.

For the class B2.2, it includes states L_{abc_2} with $abc \neq 0$, $a = b$, $a \neq \pm c$, and $a^2 + c^2 = 0$. That is, B2.2 includes $a\,(|0000\rangle + |1111\rangle) \pm ai\,(\,|0101\rangle + |1010\rangle) + |0110\rangle$. We can also show that B2.2 is a true entanglement class as follows. $a(|0000\rangle + |1111\rangle) \pm ai\,(\,|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle \pm i\,(|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = diag(1, a)$, $\gamma = diag(a, 1)$, $\beta = \delta = I$. Also, $|0000\rangle + |1111\rangle + i\,(\,|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0000\rangle + |1111\rangle - i\,(|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \delta = \sigma_z$, and $\beta = \gamma = I$.

For the class B2.3, it includes states L_{abc_2} with $abc \neq 0$, $a = b$, $a \neq \pm c$, and $a^2 + c^2 \neq 0$. We cannot classify B2.3 further.

Classes B2.1, B2.2, and B2.3 are different from each other by the values of $\mathcal{I}$, D_1, D_2, and D_3 in Table II (2).

4.2.2 The classification for the subsubfamily L_{abc_2} with $abc \neq 0$ and $a = -b$

This subsubfamily has the following three distinct true SLOCC entanglement classes. They are denoted as B3.1, B3.2, and B3.3. See Table II (2).

For the class B3.1, it includes states L_{abc_2} with $abc \neq 0$ and $a = -b$ and $a = \pm c$. That is, B3.1 includes $a\,(|0011\rangle + |1100\rangle) \pm a\,(\,|0101\rangle + |1010\rangle) + |0110\rangle$. We demonstrate that B3.1 is a true entanglement class below.

$a\,(|0011\rangle + |1100\rangle) \pm a\,(\,|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0011\rangle + |1100\rangle \pm (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \delta = I$, $\beta = \gamma = diag(a, 1)$.

Also, $|0011\rangle + |1100\rangle + |0101\rangle + |1010\rangle + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0011\rangle + |1100\rangle - (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \gamma = I$, $\beta = \sigma_z$, and $\delta = -\sigma_z$.

For the class B3.2, it includes states L_{abc_2} with $abc \neq 0$ and $a = -b$ and $a \neq \pm c$, $a^2 + c^2 = 0$. That is, B3.2 includes $a (|0011\rangle + |1100\rangle) \pm ai (|0101\rangle + |1010\rangle) + |0110\rangle$. We can argue that B3.2 is a true entanglement class as follows.

$a (|0011\rangle + |1100\rangle) \pm ai (|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0011\rangle + |1100\rangle \pm i (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \delta = I$, $\beta = \gamma = diag(a, 1)$.

Also, $|0011\rangle + |1100\rangle + i (|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0011\rangle + |1100\rangle - i (|0101\rangle + |1010\rangle) + |0110\rangle)$, where $\alpha = \gamma = I$, $\beta = \sigma_z$, and $\delta = -\sigma_z$.

For the class B3.3, it includes states L_{abc_2} with $abc \neq 0$ and $a = -b$ and $a \neq \pm c$, $a^2 + c^2 \neq 0$. We cannot classify B3.3 further.

Classes B3.1, B3.2 and B3.3 are different from each other by the values of $\mathcal{I}$, D_1, D_2, and D_3 in Table II (2).

4.2.3 Subsubfamily L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, but $c = \pm a$ or $c = \pm b$

Note that the states L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, and $c = \pm b$ can be obtained by SLOCC $I \otimes \sigma_z \otimes \sigma_z \otimes I$ from the states L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, and $c = \mp a$, respectively.

Furthermore, the state L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, and $c = -a$ is equivalent to the L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, and $c = a$ under SLOCC. The following is our argument. Let $p = -a$ and $q = -b$. Then, from the state $\frac{p+q}{2}(|0000\rangle + |1111\rangle) + \frac{p-q}{2}(|0011\rangle + |1100\rangle) + p(|0101\rangle + |1010\rangle) + |0110\rangle$, we can obtain $\frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) - a(|0101\rangle + |1010\rangle) + |0110\rangle$ under SLOCC $(-\sigma_z) \otimes \sigma_z \otimes I \otimes I$.

Therefore, we only need to consider states L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, $a = c$. This subsubfamily has the following two different true SLOCC entanglement classes: B4.1 and B4.2. See Table II (2).

For the class B4.1, it includes states L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, $c = a$ and $(3a^2 + b^2) = 0$.

For the class B4.2, it includes states L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, $a = c$, $(3a^2 + b^2) \neq 0$.

We cannot classify B4.1 or B4.2 further.

Classes B4.1 and B4.2 are different because they have different values of $\mathcal{I}$ in Table II (2).

4.2.4 Subsubfamily L_{abc_2} with $abc \neq 0$, and $x \neq \pm y$, where x, $y \in \{a, b, c\}$

We can distinguish two classes B4.3 and B4.4 from this subsubfamily.

For the class B4.3, it includes states L_{abc_2} with $abc \neq 0$ and $x \neq \pm y$ and $a^2 + b^2 + 2c^2 = 0$.

For the class B4.4, it includes states L_{abc_2} with $abc \neq 0$ and $x \neq \pm y$ and $a^2 + b^2 + 2c^2 \neq 0$.

We cannot classify B4.3 or B4.4 further.

Table II (3). The true SLOCC entanglement classes in the subfamily L_{abc_2} with $abc \neq 0$ and $x \neq \pm y$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
B4.3, *	$x \neq \pm y$, $a^2 + b^2 + 2c^2 = 0$	$= 0$	Δ	Δ	Δ
B4.4, *	$x \neq \pm y$, $a^2 + b^2 + 2c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ

These two classes are different because of their different values of $\mathcal{I}$.

4.3   The classification for the subfamily L_{abc_2} with $c \neq 0$, $ab = 0$

First we show that the state L_{abc_2} with $c \neq 0$, $b = 0$ is equivalent to the state L_{abc_2} with $c \neq 0$, $a = 0$ under SLOCC as follows. From the state $\frac{a}{2}(|0000\rangle + |1111\rangle) - \frac{a}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle$, we can obtain $\frac{a}{2}(|0000\rangle + |1111\rangle) + \frac{a}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle$ under SLOCC $\alpha \otimes \beta \otimes \gamma \otimes \delta$, where $\alpha = \beta = diag\{i, 1\}$, $\gamma = \delta = diag\{-i, 1\}$. Therefore, we only consider L_{abc_2} with $c \neq 0$, $a = 0$ below.

Furthermore, note that the state L_{abc_2} with $c \neq 0$, $a = 0$, and $b = -c$ is equivalent to the state L_{abc_2} with $c \neq 0$, $a = 0$, and $b = c$ under SLOCC. We can argue this as follows. From the state $\frac{-b}{2}(|0000\rangle + |1111\rangle) - \frac{-b}{2}(|0011\rangle + |1100\rangle) - b(|0101\rangle + |1010\rangle) + |0110\rangle$, we can obtain $\frac{b}{2}(|0000\rangle + |1111\rangle) - \frac{b}{2}(|0011\rangle + |1100\rangle) - b(|0101\rangle + |1010\rangle) + |0110\rangle$ under SLOCC $I \otimes I \otimes \sigma_z \otimes (-\sigma_z)$.

This subfamily has the following four true SLOCC entanglement classes: B5.1, B5.2, B5.3 and B5.4.

Class B5.1, it includes states L_{abc_2} with $c \neq 0$ and $a = b = 0$. That is, B5.1 includes $c(|0101\rangle + |1010\rangle) + |0110\rangle$. B5.1 is a true SLOCC entanglement class because $c(|0101\rangle + |1010\rangle) + |0110\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta(|0101\rangle + |1010\rangle + |0110\rangle)$, where $\alpha = \delta = I$, $\beta = \gamma = diag(c, 1)$.

Class B5.2, it includes states L_{abc_2} with $c \neq 0$, $a = 0$, and $b = c$. That is, B5.2 includes $\frac{b}{2}(|0000\rangle + |1111\rangle) - \frac{b}{2}(|0011\rangle + |1100\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle$. B5.2 is a true SLOCC entanglement class because $L_{abc_2}(a = 0, b = c) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{abc_2}(a = 0, b = c = 1)$, where $\alpha = \delta = diag(1, \sqrt{b})$, and $\beta = \gamma = diag(\sqrt{b}, 1)$.

Class B5.3, it includes states L_{abc_2} with $c \neq 0$, $a = 0$, $b \neq \pm c$, and $b^2 + 2c^2 = 0$. That is, B5.3 includes $\frac{b}{2}(|0000\rangle + |1111\rangle) - \frac{b}{2}(|0011\rangle + |1100\rangle) \pm \frac{bi}{\sqrt{2}}(|0101\rangle + |1010\rangle) + |0110\rangle$. We can show that B5.3 is a true SLOCC entanglement class as follows. $L_{abc_2}(a = 0, c = \pm bi/\sqrt{2}) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{abc_2}(a = 0, b = 1, c = \pm i/\sqrt{2})$, where $\alpha = \delta = diag(1, \sqrt{b})$, and $\beta = \gamma = diag(\sqrt{b}, 1)$. Also, $L_{abc_2}(a = 0, b = 1, c = i/\sqrt{2}) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{abc_2}(a = 0, b = 1, c = -i/\sqrt{2})$, where $\alpha = \sigma_z$, $\beta = diag(i, 1)$, $\gamma = diag(-i, 1)$, and $\delta = diag(1, -i)$.

Class B5.4, it includes states L_{abc_2} with $c \neq 0$, $a = 0$, $b \neq \pm c$, and $b^2 + 2c^2 \neq 0$. We cannot classify B5.4 further.

Table II (4). The true SLOCC entanglement classes in the subfamily L_{abc_2} with $c \neq 0$, $ab = 0$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
B5.1	$a = b = 0$	$\neq 0$	0	Δ	0
B5.2	$a = 0$, and $b = c$	$\neq 0$	Δ	Δ	Δ
B5.3	$a = 0$, $b \neq \pm c$, and $b^2 + 2c^2 = 0$	$= 0$	Δ	Δ	Δ
B5.4, *	$a = 0$, $b \neq \pm c$, and $b^2 + 2c^2 \neq 0$	$\neq 0$	Δ	Δ	Δ

5   Family $L_{a_2 b_2}$

The representative state of this family is $L_{a_2 b_2} = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle$. When $a = b = 0$, this becomes a product state: $|01\rangle_{13} \otimes (|01\rangle + |10\rangle)_{24}$. We can distinguish four true SLOCC entanglement classes in this family, which are denoted as V1, V2, V3, and V4.

For the class V1, it includes states $L_{a_2 b_2}$ with $a = \pm b \neq 0$. V1 is a true SLOCC entanglement class. The representative state is denoted as $L_{a_2 b_2}(a = b = 1) = |0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle + |0110\rangle + |0011\rangle$. The following is our argument. Let $\alpha = diag\{1, a\}$,

$\gamma = \{a, 1\}$, and $\beta = \delta = I$. Then $L_{a_2b_2}(a = b \neq 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2b_2}(a = b = 1)$. Let $\alpha = diag\{-i, a\}$, $\gamma = diag\{-ai, 1\}$, and $\beta = \delta = diag\{i, 1\}$. Then $L_{a_2b_2}(a = -b \neq 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2b_2}(a = b = 1)$.

For the class V2, it includes states $L_{a_2b_2}$ with $a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 = 0$. We can argue that V2 is a true SLOCC entanglement class as follows. Let $\alpha = diag\{1, a\}$, $\beta = I$, $\gamma = diag\{a, 1\}$, and $\delta = I$. Then, it is easy to verify that $L_{a_2b_2}$(with $b = \pm ai$) $= \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2b_2}(a = 1, b = \pm i)$. Furthermore, we can demonstrate that $L_{a_2b_2}(a = 1, b = -i) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2b_2}(a = 1, b = i)$, where $\alpha = diag(1, i)$, $\beta = diag(i, 1)$, $\gamma = diag(-i, 1)$, and $\delta = diag(1, -i)$.

For the class V3, it includes states $L_{a_2b_2}$ with $a \neq \pm b, ab \neq 0$, and $a^2 + b^2 \neq 0$. Each state of V3 is a true entangled state. However we cannot classify V3 further.

For the class V4, it includes states $L_{a_2b_2}$ with $a \neq \pm b$ and $ab = 0$. V4 is a true SLOCC entanglement class. The representative state is denoted as $L_{a_2b_2}(a = 1, b = 0) = |0000\rangle + |1111\rangle + |0110\rangle + |0011\rangle$. We can argue this as follows. We can show that for any a and b, $L_{a_2b_2}$ with $a \neq \pm b$ and $ab = 0$ is equivalent to the representative state $L_{a_2b_2}(a = 1, b = 0)$ under SLOCC as follows. Let $\alpha = diag\{1, a\}$, $\gamma = diag\{a, 1\}$, $\beta = \delta = I$. Then $L_{a_2b_2}(a \neq 0, b = 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta\ L_{a_2b_2}(a = 1, b = 0)$. Let $\alpha = diag\{1, b\}$, $\gamma = diag\{b, 1\}$, and $\beta = \delta = \sigma_x$. Then, $L_{a_2b_2}(a = 0, b \neq 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2b_2}(a = 1, b = 0)$.

Note that the state $L_{a_2b_2}$ with $a = 1$ and $b = 0$, and the state $L_{a_2b_2}$ with $a = 0$ and $b = 1$ are different representative states of the class V4 because $|0101\rangle + |1010\rangle + |0110\rangle + |0011\rangle = I \otimes \sigma_x \otimes I \otimes \sigma_x(\ |0000\rangle + |1111\rangle + |0110\rangle + |0011\rangle)$.

Table III. Four true SLOCC entanglement classes in Family $L_{a_2b_2}$

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
V1	$L_{a_2b_2}$ with $a = \pm b \neq 0$	$\neq 0$	Δ	0	0
V2	$L_{a_2b_2}$ with $a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 = 0$	$= 0$	Δ	Δ	Δ
V3, *	$L_{a_2b_2}$ with $a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
V4	$L_{a_2b_2}$ with $a \neq \pm b$ and $ab = 0$	$\neq 0$	Δ	Δ	Δ

To demonstrate that the four classes in Table III are distinct true SLOCC entanglement classes, we only need to show that the class V4 is different from the class V3 due to the properties of $\mathcal{I}$, D_1, D_2, and D_3. For the class V4, each state of the class V4 has the following F_i:

$F_1 = a^4\alpha_1^2\alpha_2^2 P$, $F_2 = a^4\alpha_3^2\alpha_4^2 P$, $F_3 = a^4\beta_1^2\beta_2^2 Q$, $F_4 = a^4\beta_3^2\beta_4^2 Q$, $F_5 = a^4\gamma_2^2\gamma_1^2 R$, $F_6 = a^4\gamma_4^2\gamma_3^2 R$, $F_7 = a^4\delta_1^2\delta_2^2 S$, $F_8 = a^4\delta_3^2\delta_4^2 S$,

$F_9 = a^4(\alpha_1\alpha_4\beta_1\beta_4 - \alpha_2\alpha_3\beta_2\beta_3)^2 \det^2(\delta)\det^2(\gamma)$,

$F_{10} = a^4(-\alpha_1\alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_1\beta_4)^2 \det^2(\delta)\det^2(\gamma)$.

Clearly, for each state of the class V4, the above F_i satisfy Properties 2.1 and 2.2 in Appendix A. However, in the class V3 the state $L_{a_2b_2}$ with $a \neq \pm b$, $ab \neq 0$, and $a^2 + b^2 \neq 0$ satisfies the following

$$F_i = 0, i = 2, 3, 4, 5, 7, 8; \tag{11}$$

$$F_1 = F_6 = 4ab, F_9 = a^4, F_{10} = b^4. \tag{12}$$

Clearly, these F_i in Eqs. (11) and Eq. (12) do not satisfy property 2.2 in appendix A.

Therefore the class V4 is different from the class V3.

6 Family L_{ab_3}

The representative state of this family is $L_{ab_3} = a\,(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(\,|0101\rangle + |1010\rangle) + \frac{a-b}{2}(\,|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle).$

Let us consider three subfamilies, which are the subfamily L_{ab_3} with $a = \pm b$, the subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$, and the subfamily L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$. We demonstrate that these three subfamilies are inequivalent under SLOCC and there are at least eight true SLOCC entanglement classes in this family below.

The state L_{ab_3} satisfies
$\mathcal{I} = -(b^2 + 3a^2),\ D_1 = a^2(a+b)^2/4,\ D_2 = (a^2 - b^2)^2/16,\ D_3 = -a^2(a-b)^2/4,$

$$
\begin{aligned}
F_1 &= F_3 = F_6 = F_8 = (a^2 - b^2)/2, \\
F_2 &= F_4 = F_5 = F_7 = 0, \\
F_9 &= a^4, F_{10} = a^2 b^2.
\end{aligned}
\tag{13}
$$

Let $|\psi\rangle$ in Eq. (2) be the state L_{ab_3}. Let $|\psi'\rangle$ be any state which is equivalent to $|\psi\rangle$ under SLOCC. By solving matrix equation in Eq. (2), we obtain the amplitudes a_i of the state $|\psi'\rangle$. By substituting a_i into F_i, we obtain the following F_i. That is, each state in Family L_{ab_3} satisfies

$$
\begin{aligned}
F_1 &= (a^2 - b^2)\alpha_1^4 P/2,\ F_2 = (a^2 - b^2)\alpha_3^4 P/2, \\
F_3 &= (a^2 - b^2)\beta_1^4 Q/2,\ F_4 = (a^2 - b^2)\beta_3^4 Q/2, \\
F_5 &= (a^2 - b^2)\gamma_2^4 R/2,\ F_6 = (a^2 - b^2)\gamma_4^4 R/2, \\
F_7 &= (a^2 - b^2)\delta_2^4 S/2,\ F_8 = (a^2 - b^2)\delta_4^4 S/2.
\end{aligned}
\tag{14}
$$

6.1 Subfamily L_{ab_3} with $a = \pm b$

This subfamily has the following three classes: R1.1, R1.2 and R1.3.

For the class R1.1, it includes the state L_{ab_3} with $a = b = 0$. Let $L_{ab_3}(a = b = 0) = |0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle$. Clearly, $L_{ab_3}(a = b = 0)$ is equivalent to the state $|W\rangle$ because $L_{ab_3}(a = b = 0) = I \otimes I \otimes \sigma_x \otimes \sigma_x |W\rangle$.

For the class R1.2, it includes states L_{ab_3} with $a = b \neq 0$. That is, R1.2 includes $L_{ab_3}(a = b) = a(|0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. A representative state is $L_{ab_3}(a = b = 1) = |0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. R1.2 is a true SLOCC entanglement class because $L_{ab_3}(a = b) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{ab_3}(a = b = 1)$, where $\alpha = \beta = diag\{1, a\}$, $\gamma = diag\{a, 1\}$, and $\delta = diag\{1, 1/a\}$.

For the class R1.3, it includes states L_{ab_3} with $a = -b \neq 0$. That is, R1.3 includes $L_{ab_3}(a = -b) = a(|0000\rangle + |1111\rangle + |0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. A representative state is $L_{ab_3}(a = -b = 1) = |0000\rangle + |1111\rangle + |0110\rangle + |1001\rangle + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. R1.3 is a true SLOCC entanglement class because $L_{ab_3}(a = -b) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{ab_3}(a = -b = 1)$, where $\alpha = \beta = diag\{1, a\}$, $\gamma = diag\{a, 1\}$, and $\delta = diag\{1, 1/a\}$.

Classes R1.1, R1.2, and R1.3 are different under SLOCC by the values of $\mathcal{I}$, D_1, D_2, and D_3 in Table IV.

6.2 Subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$

Subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$ has two classes denoted as R2.1 and R2.2.

For the class R2.1, it includes states L_{ab_3} with $a = 0, b \neq 0$. That is, R2.1 includes $L_{ab_3}(a = 0, b \neq 0) = \frac{b}{2}(|0101\rangle + |1010\rangle) - \frac{b}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. R2.1 is a true entanglement class because $L_{ab_3}(a = 0, b \neq 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{ab_3}(a = 0, b = 1)$, where $\alpha = \beta = diag\{1, b\}$, $\gamma = diag\{b, 1\}$, and $\delta = diag\{1, 1/b\}$.

For the class R2.2, it includes states L_{ab_3} with $a \neq 0, b = 0$. That is, R2.2 includes $L_{ab_3}(b = 0) = a (|0000\rangle + |1111\rangle) + \frac{a}{2}(|0101\rangle + |1010\rangle) + \frac{a}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. R2.2 is a true entanglement class because $L_{ab_3}(b = 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{ab_3}(a = 1, b = 0)$, where $\alpha = \beta = diag\{1, a\}$, $\gamma = diag\{a, 1\}$, and $\delta = diag\{1, 1/a\}$.

We argue that classes R2.1 and R2.2 are different under SLOCC as follows.

For each state of the class R2.1, F_i $(i = 1, ..., 8)$ can be obtained from Eq. (14) by letting $a = 0$,

$F_9 = -\frac{1}{2}b^2(\alpha_2\alpha_3\beta_1\beta_3 - \alpha_1\alpha_4\beta_1\beta_3 - \alpha_1\alpha_3\beta_2\beta_3 + \alpha_1\alpha_3\beta_1\beta_4)^2 \det^2(\gamma)\det^2(\delta)$,

$F_{10} = -\frac{1}{2}b^2(\alpha_2\alpha_3\beta_1\beta_3 - \alpha_1\alpha_4\beta_1\beta_3 + \alpha_1\alpha_3\beta_2\beta_3 - \alpha_1\alpha_3\beta_1\beta_4)^2 \det^2(\gamma)\det^2(\delta)$.

Then, we derive that for each state in the class R2.1, F_i satisfy the following:

Property 7.1. If $F_i = 0$ then $F_9 = F_{10}$, $i = 1, 2, 3$, or 4.

Property 7.2. If $F_1 F_2 = 0 \wedge F_3 F_4 = 0$ then $F_9 = F_{10} = 0$.

For each state of the class R2.2, F_i $(i = 1, ..., 8)$ can be obtained from Eq. (14) by letting $b = 0$. F_9 and F_{10} are omitted because they are too complicated. We can derive that for each state of the class R2.2, F_i satisfy the following properties:

Property 8.1. If $F_1 = F_3 = 0$ or $F_2 = F_4 = 0$ then $F_9 \neq 0 \wedge F_{10} = 0$.

Property 8.2. If $F_1 = F_4 = 0$ or $F_2 = F_3 = 0$ then $F_9 = 0 \wedge F_{10} \neq 0$.

Classes R2.1 and R2.2 are different under SLOCC by the following argument.

From Eq. (13), the state $L_{ab_3}(b = 0)$ in the class R2.2 satisfies $F_1 = F_3 = F_6 = F_8 = a^2/2$, $F_2 = F_4 = F_5 = F_7 = 0$, $F_9 = a^4$, $F_{10} = 0$. It is obvious that the state $L_{ab_3}(b = 0)$ in the class R2.2 does not satisfy property 7.2. Therefore, classes R2.1 and R2.2 are different under SLOCC.

6.3 Subfamily L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$

This subfamily has two subsubfamilies denoted as R3.1 and R3.2.

Subsubfamily R3.1 includes states L_{ab_3} with $a \neq \pm b, ab \neq 0, 3a^2 + b^2 \neq 0$. Each one of this subsubfamily is a true entanglement state. However, we cannot classify R3.1 further.

Subsubfamily R3.2 includes states L_{ab_3} with $a \neq \pm b, ab \neq 0, 3a^2 + b^2 = 0$. This subsubfamily consists of two inequivalent true SLOCC entanglement classes. Their representative states are denoted as $L_{ab_3}(a = 1, b = \pm\sqrt{3}i) = |0000\rangle + |1111\rangle + \frac{(1\pm\sqrt{3}i)}{2}(|0101\rangle + |1010\rangle) + \frac{(1\mp\sqrt{3}i)}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$. We can argue this as follows. Let $\alpha = \beta = diag\{1, a\}$, $\gamma = diag\{a, 1\}$, and $\delta = diag\{1, 1/a\}$. Then, it is easy to verify that the state L_{ab_3} (with $a \neq \pm b, ab \neq 0$ and $b = \pm\sqrt{3}ai) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{ab_3}(a = 1, b = \pm\sqrt{3}i)$. Furthermore, we can demonstrate that these two representative states are inequivalent under SLOCC.

Subfamilies R3.1 and R3.2 are different under SLOCC because they have different values of $\mathcal{I}$ in Table IV.

6.4   Inequivalent subfamilies

Lemma 6.1. The subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$, and the subfamily L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$ are inequivalent to the subfamily L_{ab_3} with $a = \pm b$ under SLOCC.

Proof. From Eq. (14), it is easy to see that $F_i = 0$, $i = 1, ..., 8$, for any state in the subfamily L_{ab_3} with $a = \pm b$. While $F_i \neq 0$, $i = 1, ..., 8$, for the state L_{ab_3} with $a \neq \pm b$ and $ab = 0$ and the state L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$. Hence, the subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$ and the subfamily L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$ are inequivalent to the subfamily L_{ab_3} with $a = \pm b$ under SLOCC, respectively.

Lemma 6.2. The subfamily L_{ab_3} with $a \neq \pm b$ and $ab = 0$, and the subfamily L_{ab_3} with $a \neq \pm b$ and $ab \neq 0$ are inequivalent under SLOCC.

Proof.

When $a \neq \pm b$ and $ab \neq 0$, F_i in Eq. (13) do not satisfy properties 7.2 or 8.1. Hence, lemma 6.2 holds.

Conclusively, Family L_{ab_3} has at least eight distinct true entanglement classes. See Table IV. It means that Family L_{ab_3} does not include any product state.

Table IV. Eight True SLOCC entanglement classes in Family L_{ab_3}

classes	criteria for classification	$\mathcal{I}$	D_1	D_2	D_3
R1.1 (i.e., $|W\rangle$)	L_{ab_3} with $a = b = 0$	$= 0$	0	0	0
R1.2	L_{ab_3} with $a = b \neq 0$	$\neq 0$	Δ	0	0
R1.3	L_{ab_3} with $a = -b \neq 0$	$\neq 0$	0	0	Δ
R2.1	L_{ab_3} with $a = 0, b \neq 0$	$\neq 0$	Δ	Δ	Δ
R2.2	L_{ab_3} with $a \neq 0, b = 0$	$\neq 0$	Δ	Δ	Δ
R3.1, *	L_{ab_3} with $a \neq \pm b, ab \neq 0, 3a^2 + b^2 \neq 0$	$\neq 0$	Δ	Δ	Δ
subsubfamily R3.2 (two classes)	L_{ab_3} with $a \neq \pm b, ab \neq 0, 3a^2 + b^2 = 0$	$= 0$	Δ	Δ	Δ

7   Other five families

7.1   Family L_{a_4}

The representative state of this family is $L_{a_4} = a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + (i|0001\rangle + |0110\rangle - i|1011\rangle)$. There are two true SLOCC entanglement classes in this family.

(1). The class L_{a_4} with $a = 0$. In this case, L_{a_4} reduces to $i|0001\rangle + |0110\rangle - i|1011\rangle$. Clearly, this is a true entanglement class.

(2). The class L_{a_4} with $a \neq 0$. We show that this is a true SLOCC entanglement class as follows. Let $L_{a_4}(a = 1) = |0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle + (i|0001\rangle + |0110\rangle - i|1011\rangle)$. Let $\alpha = diag\{1, a^2\}$, $\beta = diag\{1, a\}$; $\gamma = diag\{1, 1/a^2\}$, and $\delta = diag\{a, 1\}$. Then, it is easy to verify that for any $a \neq 0$, $L_{a_4} = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_4}(a = 1)$.

Therefore, there are two true SLOCC entanglement classes in Family L_{a_4} and these two classes are different because they have different values of $\mathcal{I}$. See Table V.

For the class L_{a_4} with $a \neq 0$, $\mathcal{I} = 2a^2 T$. For SL-operations, $\mathcal{I} = 2a^2$. It means that the SLOCC entanglement class L_{a_4} with $a \neq 0$ includes a continuous parameter of SL classes. In other words, the class L_{a_4} with $a \neq 0$ can be described by a continuous parameter.

Table V. Two true SLOCC entanglement classes in Family L_{a_4}

	$\mathcal{I}$	D_1	D_2	D_3
L_{a_4} with $a = 0$	0	Δ	0	0
L_{a_4} with $a \neq 0$	$\neq 0$	Δ	Δ	Δ

7.2   Family $L_{a_2 0_{3 \oplus 1}}$

The representative state of this family is $L_{a_2 0_{3 \oplus 1}} = a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle$. We argue there are only two SLOCC entanglement classes in this family as follows.

(1). The class $L_{a_2 0_{3 \oplus 1}}$ with $a = 0$. In this case, $L_{a_2 0_{3 \oplus 1}}$ becomes $|0\rangle \otimes (|011\rangle + |101\rangle + |110\rangle)$, which is a product state of the one-qubit state $|0\rangle$ and the 3-qubit $|W\rangle$.

(2). The class $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$. We show that this is a true SLOCC entanglement class as follows.

The state $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$ is a true entanglement state. Let $L_{a_2 0_{3 \oplus 1}}(a = 1) = |0000\rangle + |1111\rangle + |0011\rangle + |0101\rangle + |0110\rangle$. Let $\alpha = diag\{\sqrt{a}, a^2\}$, $\beta = diag\{1/\sqrt{a}, 1/a\}$, $\gamma = diag\{\sqrt{a}, 1\}$, and $\delta = diag\{\sqrt{a}, 1\}$. Then it is easy to verify that $L_{a_2 0_{3 \oplus 1}}(a \neq 0) = \alpha \otimes \beta \otimes \gamma \otimes \delta L_{a_2 0_{3 \oplus 1}}(a = 1)$.

For the class $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$, $\mathcal{I} = a^2 T$. For SL-operations, $\mathcal{I} = a^2$. It implies that the SLOCC entanglement class $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$ includes a continuous parameter of SL classes. It says that the class $L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$ can be characterized by a continuous parameter.

Table VI. One true SLOCC entanglement class in Family $L_{a_2 0_{3 \oplus 1}}$

criterion for classification	$\mathcal{I}$	D_1	D_2	D_3
$L_{a_2 0_{3 \oplus 1}}$ with $a \neq 0$	$\neq 0$	Δ	Δ	Δ

7.3   Family $L_{0_{5 \oplus 3}}$

The representative state of this family is $L_{0_{5 \oplus 3}} = |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle$. This family is a true SLOCC entanglement class. Each state of this family satisfies the following:
$\mathcal{I} = 0$, $D_1 = D_2 = D_3 = 0$, $F_1 = F_2 = 0$, $|F_3| + |F_4| \neq 0$, $|F_5| + |F_6| \neq 0$, $|F_7| + |F_8| \neq 0$, $F_9 = F_{10}$, $F_3 F_4 = (F_9)^2$.

7.4   Family $L_{0_{7 \oplus 1}}$

The representative state of this family is $L_{0_{7 \oplus 1}} = |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle$. This family is a true SLOCC entanglement class. Each state of this family satisfies the following:
$\mathcal{I} = 0$, D_1 is Δ, D_2 is Δ, D_3 is Δ, $|F_3| + |F_4| \neq 0$, $|F_5| + |F_6| \neq 0$, $|F_7| + |F_8| \neq 0$.

7.5   Family $L_{0_{3 + \bar{1}} 0_{3 + \bar{1}}}$

The representative state of this family is $L_{0_{3 + \bar{1}} 0_{3 + \bar{1}}} = |0\rangle(|000\rangle + |111\rangle)$, which is a product state of the one-qubit state $|0\rangle$ and the three-qubit $|GHZ\rangle$ state.

8   Summary

Verstraete, Dehaene, and Verschelde proposed nine families of states corresponding to nine different ways of entangling four qubits [2]. In this paper, we investigate SLOCC classification of each of the nine families, and distinguish 49 true SLOCC entanglement classes from them. We give complete SLOCC classifications for Families L_{a_4}, $L_{a_2 0_{3 \oplus 1}}$, $L_{0_{5 \oplus 3}}$, $L_{0_{7 \oplus 1}}$ and $L_{0_{3 + \bar{1}} 0_{3 + \bar{1}}}$. But we cannot guarantee that SLOCC classifications for Families G_{abcd}, L_{abc_2}, $L_{a_2 b_2}$ and L_{ab_3} are complete.

References

1. W. Dür, G.Vidal and J.I. Cirac (2000), Three qubits can be entangled in two inequivalent ways, Phys. Rev. A. 62, 062314.
2. F. Verstraete, J.Dehaene, B. De Moor and H. Verschelde (2002), Four qubits can be entangled in nine different ways, Phys. Rev. A. 65, 052112.
3. L. Lamata, J. León, D. Salgado and E. Solano (2006), Inductive entanglement classification of four qubits under SLOCC, Phys. Rev. A. 74 , 052336 .
4. L. Lamata, J. León, D. Salgado and E. Solano (2007), Inductive classification of multipartite entanglement under SLOCC, Phys. Rev. A. 75, 022318. Also, see quant-ph/0610233.
5. D. Li, X. Li, H. Huang and X. Li (2008), Necessary and sufficient conditions of separability for multipartite pure states, Commun. Theor. Phys., Vol. 49, 1211-1216. Submitted to Phys. Rev. Lett. (Sep. 2004), the paper No. LV9637. Late, quant-ph/0604147.
6. D. Li, X. Li, H. Huang and X. Li (2007), Classification of four-qubit states by means of a stochastic operation and the classical communication invariant and semi-invariants, Phys. Rev. A 76, 052311. Also see, quant-ph/ 0701032.
7. A. Miyake (2003), Classification of multipartite entangled states by multidimensional determinants, Phys. Rev. A. 67, 012108 .
8. D. Li, X. Li, H. Huang and X. Li (2006), Simple criteria for the SLOCC classification, Phys. Lett. A 359, 428.
9. D. Li, X. Li, H. Huang and X. Li (2007), Stochastic local operations and classical communication invariant and the residual entanglement for n qubits, Phys. Rev. A 76, 032304. Also see, quant-ph/0704.2087.

Appendix A Classification for Family L_{abc_2}

The representative state of Family L_{abc_2} satisfies the following:
$\mathcal{I} = -(a^2 + b^2 + 2c^2)/2,$

$$
\begin{aligned}
D_1 &= (a+b)^2 c^2/4, D_2 = -(a^2 - b^2)^2/16, D_3 = (a-b)^2 c/4, \\
F_1 &= F_4 = F_6 = F_7 = (a^2 - b^2)c, \\
F_2 &= F_3 = F_5 = F_8 = 0, F_9 = a^2 b^2, F_{10} = c^4.
\end{aligned} \tag{A.1}
$$

8.1   Subfamily L_{abc_2} with $c = 0$.

L_{abc_2} becomes $L_{abc_2}(c = 0) = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + |0110\rangle$. Each state of this subfamily satisfies the following:

$\mathcal{I} = \frac{1}{2}(a^2 + b^2)T,$
$D_1 = \frac{1}{2}ab(a - b)\alpha_1\alpha_3\gamma_2\gamma_4 \det(\alpha) \det^2(\beta) \det(\gamma) \det^2(\delta),$
$D_2 = \frac{1}{16}(a^2 - b^2) \det(\alpha) \det(\beta) \det^2(\gamma) \det^2(\delta)$
$(2a^2\alpha_1\alpha_3\beta_1\beta_3 + 2b^2\alpha_1\alpha_3\beta_1\beta_3 + a^2\alpha_2\alpha_3\beta_2\beta_3 - b^2\alpha_2\alpha_3\beta_2\beta_3 + a^2\alpha_1\alpha_4\beta_2\beta_3 - b^2\alpha_1\alpha_4\beta_2\beta_3$
$+a^2\alpha_2\alpha_3\beta_1\beta_4 - b^2\alpha_2\alpha_3\beta_1\beta_4 + a^2\alpha_1\alpha_4\beta_1\beta_4 - b^2\alpha_1\alpha_4\beta_1\beta_4 + 2a^2\alpha_2\alpha_4\beta_2\beta_4 + 2b^2\alpha_2\alpha_4\beta_2\beta_4)$
$D_3 = -\frac{1}{2}ab(a + b)\alpha_1\alpha_3\delta_1\delta_3 \det(\alpha) \det^2(\beta) \det^2(\gamma) \det(\delta),$

$$
\begin{aligned}
F_1 &= a^2 b^2 \alpha_1^2 \alpha_2^2 P, F_2 = a^2 b^2 \alpha_3^2 \alpha_4^2 P, F_3 = a^2 b^2 \beta_1^2 \beta_2^2 Q, F_4 = a^2 b^2 \beta_3^2 \beta_4^2 Q, \\
F_5 &= a^2 b^2 \gamma_2^2 \gamma_1^2 R, F_6 = a^2 b^2 \gamma_4^2 \gamma_3^2 R, F_7 = a^2 b^2 \delta_1^2 \delta_2^2 S, F_8 = a^2 b^2 \delta_3^2 \delta_4^2 S, \\
F_9 &= a^2 b^2 (\alpha_2\alpha_3\beta_2\beta_3 - \alpha_1\alpha_4\beta_1\beta_4)^2 \overset{2}{\det}(\gamma) \overset{2}{\det}(\delta), \\
F_{10} &= a^2 b^2 (\alpha_1\alpha_4\beta_2\beta_3 - \alpha_2\alpha_3\beta_1\beta_4)^2 \overset{2}{\det}(\gamma) \overset{2}{\det}(\delta).
\end{aligned} \tag{A.2}
$$

From Eq. (A.2), for each state of this subfamily L_{abc_2} with $c = 0$, F_i have the following properties.

Property 2.1. If $ab \neq 0$ and $F_i = 0$ then $|F_9| + |F_{10}| \neq 0$, $i = 1, 2, 3,$ or 4.

Property 2.2. If $ab \neq 0$ and $F_1 F_2 = F_3 F_4 = 0$ then $F_9 = 0 \wedge F_{10} \neq 0$ or $F_9 \neq 0 \wedge F_{10} = 0$.

Property 2.3. If $ab \neq 0$, $a \neq \pm b$, and $F_1 = F_2 = F_3 = F_4 = 0$, then $D_2 \neq 0$.

Property 2.4. If $F_1 = F_2 = 0$ or $F_5 = F_6 = 0$ then $D_1 = 0$.

Property 2.5. If $F_1 = F_2 = 0$ or $F_7 = F_8 = 0$ then $D_3 = 0$.

From the Table II (1), it is not hard to see that the classes B1.1, B1.2, B1.3, B1.4, and B1.5 are different from each other by the values of $\mathcal{I}$, D_1, D_2, D_3. Note that the state L_{abc_2} with $c = a = 0 \wedge b \neq 0$ and the state L_{abc_2} with $c = b = 0 \wedge a \neq 0$ are different representatives of the class B1.5 because these two states are obtained from each other by SLOCC $I \otimes \sigma_z \otimes \sigma_z \otimes I$.

8.2   Subfamily L_{abc_2} with $abc \neq 0$

This subfamily is different from the subfamily L_{abc_2} with $c = 0$ under SLOCC. This is because from Eq. (A.1), it is straightforward that for the representative state L_{abc_2} with $abc \neq 0$, F_i do not satisfy property 2.2. It implies that classes B2.1-B2.3, B3.1-B3.3, B4.1 and B4.2 in Table II (2) are different from classes B1.1-B1.5 in Table II (1).

8.2.1   Subsubfamily L_{abc_2} with $abc \neq 0$ and $a = b$

L_{abc_2} becomes $L_{abc_2}(a = b) = a\,(|0000\rangle + |1111\rangle) + c\,(\,|0101\rangle + |1010\rangle) + |0110\rangle$. For any state which is connected with $L_{abc_2}(a = b)$ by SLOCC, we obtain the following F_i and D_i.

$$
\begin{aligned}
F_1 &= (a^2 - c^2)^2 \alpha_1^2 \alpha_2^2 P, \; F_2 = (a^2 - c^2)^2 \alpha_3^2 \alpha_4^2 P, \\
F_3 &= (a^2 - c^2)^2 \beta_1^2 \beta_2^2 Q, \; F_4 = (a^2 - c^2)^2 \beta_3^2 \beta_4^2 Q, \\
F_5 &= (a^2 - c^2)^2 \gamma_2^2 \gamma_1^2 R, \; F_6 = (a^2 - c^2)^2 \gamma_4^2 \gamma_3^2 R, \\
F_7 &= (a^2 - c^2)^2 \delta_1^2 \delta_2^2 S, \; F_8 = (a^2 - c^2)^2 \delta_3^2 \delta_4^2 S.
\end{aligned}
\tag{A.3}
$$

$D_1 = ac\,\det(\alpha)\,\det^2(\beta)\,\det(\gamma)\,\det^2(\delta)(a^2 \alpha_1 \alpha_3 \gamma_1 \gamma_3 + c^2 \alpha_1 \alpha_3 \gamma_1 \gamma_3 + ac\alpha_2 \alpha_3 \gamma_2 \gamma_3 + ac\alpha_1 \alpha_4 \gamma_2 \gamma_3 + ac\alpha_2 \alpha_3 \gamma_1 \gamma_4 + ac\alpha_1 \alpha_4 \gamma_1 \gamma_4 + a^2 \alpha_2 \alpha_4 \gamma_2 \gamma_4 + c^2 \alpha_2 \alpha_4 \gamma_2 \gamma_4)$,

$D_2 = -c(a^2 - c^2)\alpha_1 \alpha_3 \beta_2 \beta_4 \det(\alpha)\,\det(\beta)\,\det^2(\gamma)\,\det^2(\delta)$,

$D_3 = -a(a^2 - c^2)\alpha_1 \alpha_3 \delta_1 \delta_3 \det(\alpha)\,\det^2(\beta)\,\det^2(\gamma)\,\det(\delta)$,

When $a \neq \pm c$, from Eq. (A.3) and the above D_1, D_2 and D_3, it can be verified that the following properties 3.1-3.5 hold for each state which is connected with L_{abc_2} with $abc \neq 0$, $a = b$ and $a \neq \pm c$ by SLOCC.

Property 3.1. If $F_1 = F_2 = 0$, then $D_2 = D_3 = 0$.

Property 3.2. If $F_3 = F_4 = 0$, then $D_2 = 0$.

Property 3.3. If $F_7 = F_8 = 0$, then $D_3 = 0$.

Property 3.4. If $F_1 = F_2 = F_5 = F_6 = 0$, then $D_1 \neq 0$.

Property 3.5. If $F_1 = F_2 = F_3 = F_4 = 0$, then $F_9 \neq 0$ and $F_{10} \neq 0$.

We argue that property 3.4 holds as follows.

When $F_1 = F_2 = 0$ and $F_5 = F_6 = 0$, there are following four cases: case 1. $\alpha_1 = \alpha_4 = 0$ and $\gamma_1 = \gamma_4 = 0$, case 2. $\alpha_1 = \alpha_4 = 0$ and $\gamma_2 = \gamma_3 = 0$, case 3. $\alpha_2 = \alpha_3 = 0$ and $\gamma_1 = \gamma_4 = 0$, case 4. $\alpha_2 = \alpha_3 = 0$ and $\gamma_1 = \gamma_4 = 0$. From the above four cases, any state, which is

connected with L_{abc_2} with $abc \neq 0$, $a = b$ and $a \neq \pm c$ by SLOCC, satisfies the above property 3.4:

8.2.2   Subsubfamily L_{abc_2} with $abc \neq 0$ and $a = -b$

When $a \neq \pm c$, L_{abc_2} becomes $L_{abc_2}(a = -b) = a\,(|0011\rangle + |1100\rangle) + c\,(\,|0101\rangle + |1010\rangle) + |0110\rangle$. For any state which is connected with L_{abc_2} with $abc \neq 0$, $a = -b$ and $a \neq \pm c$ by SLOCC, F_i ($i = 1, ..., 8$) satisfy Eq. (A.3) and the following D_1, D_2, and D_3 hold.

$D_1 = -a(a^2 - c^2)\alpha_1\alpha_3\gamma_2\gamma_4 \det(\alpha)\det^2(\beta)\det(\gamma)\det^2(\delta)$,

$D_2 = -c(a^2 - c^2)\alpha_1\alpha_3\beta_2\beta_4 \det(\alpha)\det(\beta)\det^2(\gamma)\det^2(\delta)$,

$D_3 = ac\det(\alpha)\det^2(\beta)\det^2(\gamma)\det(\delta)*$

$(a^2\alpha_2\alpha_4\delta_1\delta_3 + c^2\alpha_2\alpha_4\delta_1\delta_3 + ac\alpha_2\alpha_3\delta_2\delta_3 + ac\alpha_1\alpha_4\delta_2\delta_3$

$+ac\alpha_2\alpha_3\delta_1\delta_4 + ac\alpha_1\alpha_4\delta_1\delta_4 + a^2\alpha_1\alpha_3\delta_2\delta_4 + c^2\alpha_1\alpha_3\delta_2\delta_4)$.

Property 3.5 also holds. We can derive the following properties 4.1-4.4 for each state which is connected with L_{abc_2} with $abc \neq 0$, $a = -b$ and $a \neq \pm c$ by SLOCC.

Property 4.1. If $F_3 = F_4 = 0$, then $D_2 = 0$.

Property 4.2. If $F_1 = F_2 = 0$, then $D_1 = D_2 = 0$.

Property 4.3. If $F_5 = F_6 = 0$, then $D_1 = 0$.

Property 4.4. If $F_1 = F_2 = F_7 = F_8 = 0$, then $D_3 \neq 0$.

From Eq. (A.1), for the state $L_{abc_2}(a = -b \neq 0)$, $F_i = 0$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, and $D_1 = D_2 = 0$. Therefore, the state $L_{abc_2}(a = -b \neq 0)$ does not satisfy Property 3.4. It means that classes B3.2 and B3.3 are different from classes B2.2 and B2.3, respectively.

8.2.3   Subsubfamily L_{abc_2} with $abc \neq 0$ and $a \neq \pm b$, but $c = \pm a$ or $c = \pm b$

By discussion in Sec. 4.2.3, we only need to consider $c = a$ here.

When $c = a$, L_{abc_2} becomes $\frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + a(\,|0101\rangle + |1010\rangle) + |0110\rangle$. Thus, each state in this subsubfamily satisfies the following equations.

$F_1 = a(a^2 - b^2)\alpha_1^4 P$, $F_2 = a(a^2 - b^2)\alpha_3^4 P$, $F_3 = a(a^2 - b^2)\beta_2^4 Q$, $F_4 = a(a^2 - b^2)\beta_4^4 Q$, $F_5 = a(a^2 - b^2)\gamma_2^4 R$, $F_6 = a(a^2 - b^2)\gamma_4^4 R$, $F_7 = a(a^2 - b^2)\delta_1^4 S$, $F_8 = a(a^2 - b^2)\delta_3^4 S$.

We omit the complicated expressions of F_9, F_{10}, D_1, D_2, and D_3.

It is plain to see that F_i satisfy the following inequality.

$$|F_i| + |F_{i+1}| \neq 0, i = 1, 3, 5, 7. \tag{A.4}$$

We also have the following properties for each state which is connected with L_{abc_2} with $abc \neq 0$, $a \neq \pm b$, and $c = a$ by SLOCC.

Property 5.1. If $F_1 F_2 = 0$ and $F_5 F_6 = 0$, then $D_1 \neq 0$.

Property 5.2. If $F_1 F_2 = 0$ and $F_3 F_4 = 0$, then $D_2 \neq 0$.

Property 5.3. If $F_1 F_2 = 0$ and $F_7 F_8 = 0$, then $D_3 \neq 0$.

Property 5.4. If $F_2 = F_3 = 0$, then $F_9 \neq 0$ and $F_{10} \neq 0$.

In Eq. (A.1), F_i and D_2 obtained by letting $a = \pm b$ do not satisfy Properties 5.2. It implies that classes B2.2, B2.3, B3.2 and B3.3 are different from classes B4.1 and B4.2.

8.2.4   Subsubfamily L_{abc_2} with $abc \neq 0$, and $x \neq \pm y$, where $x, y \in \{a, b, c\}$

Each state in this subsubfamily satisfies the following.

$F_1 = \alpha_1^2[c(a^2 - b^2)\alpha_1^2 + (a^2 - c^2)(b^2 - c^2)\alpha_2^2]P$,

$$F_2 = \alpha_3^2[c(a^2 - b^2)\alpha_3^2 + (a^2 - c^2)(b^2 - c^2)\alpha_4^2]P,$$
$$F_3 = \beta_2^2[c(a^2 - b^2)\beta_2^2 + (a^2 - c^2)(b^2 - c^2)\beta_1^2]P,$$
$$F_4 = \beta_4^2[c(a^2 - b^2)\beta_4^2 + (a^2 - c^2)(b^2 - c^2)\beta_3^2]P,$$
$$F_5 = \gamma_2^2[c(a^2 - b^2)\gamma_2^2 + (a^2 - c^2)(b^2 - c^2)\gamma_1^2]P,$$
$$F_6 = \gamma_4^2[c(a^2 - b^2)\gamma_4^2 + (a^2 - c^2)(b^2 - c^2)\gamma_3^2]P,$$
$$F_7 = \delta_1^2[c(a^2 - b^2)\delta_1^2 + (a^2 - c^2)(b^2 - c^2)\delta_2^2]P,$$
$$F_8 = \delta_3^2[c(a^2 - b^2)\delta_3^2 + (a^2 - c^2)(b^2 - c^2)\delta_4^2]P,$$
$$\mathcal{I} = \tfrac{1}{2}(a^2 + b^2 + 2c^2)T.$$

We want to show that some states in this subsubfamily violates Eq. (A.4). So, this subsubfamily is inequivalent to the subsubfamily L_{abc_2} with $abc \neq 0$ and $a \neq \pm b$, but $c = \pm a$ or $c = \pm b$ under SLOCC. The following is our argument.

For the operator α, let $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_3^2 = \frac{(a^2 - c^2)(c^2 - b^2)}{c(a^2 - b^2)}\alpha_4^2$, and $\alpha_4 \neq 0$. Clearly, $\det(\alpha) \neq 0$, but $F_1 = F_2 = 0$. So, it violates Eq. (A.4).

8.3   Subfamily L_{abc_2} with $c \neq 0$ and $ab = 0$

There are three subsubfamilies. They are the subsubfamily L_{abc_2} with $c \neq 0$ and $a = b = 0$, the subsubfamily L_{abc_2} with $c \neq 0$ and $a = 0$ and $b = c$, and the subsubfamily L_{abc_2} with $bc \neq 0$ and $a = 0$ and $b \neq \pm c$.

8.3.1   Subsubfamily L_{abc_2} with $c \neq 0$ and $a = b = 0$

L_{abc_2} becomes $L_{abc_2}(a = b = 0) = c (|0101\rangle + |1010\rangle) + |0110\rangle$. Each state, which is connected $L_{abc_2}(a = b = 0)$ by SLOCC, satisfies the following D_i and F_i.

$D_1 = D_3 = 0$, $D_2 = c^3\alpha_1\alpha_3\beta_2\beta_4 \det(\alpha) \det(\beta) \det^2(\gamma) \det^2(\delta)$,

F_i $(i = 1, ..., 8)$ can be obtained from Eq. (A.3) by letting $a = b = 0$.

$F_9 = c^4(\alpha_1\alpha_4\beta_2\beta_3 - \alpha_2\alpha_3\beta_1\beta_4)^2 \det^2(\gamma) \det^2(\delta)$,

$F_{10} = c^4(\alpha_2\alpha_3\beta_2\beta_3 - \alpha_1\alpha_4\beta_1\beta_4)^2 \det^2(\gamma) \det^2(\delta)$.

It is easy to know that the above F_i satisfy Properties 2.1 and 2.2.

Each state, which is connected with L_{abc_2} with $c \neq 0$ and $a = b = 0$ under SLOCC, satisfies the following Properties 6.1 and 6.2.

Property 6.1. If $F_1 = F_2 = 0$ or $F_3 = F_4 = 0$, then $D_2 = 0$.

Property 6.2. If $F_1F_2 = 0$ and $F_3F_4 = 0$, then $F_9 = 0 \wedge F_{10} \neq 0$ or $F_9 \neq 0 \wedge F_{10} = 0$.

Remark 1. Classes B1.5 and B5.1 are different though they have the same information for $\mathcal{I}$, D_1, D_2, and D_3.

We argue remark 1 as follows. From Eq. (A.2), when $ab = 0$, any state in class B1.5 satisfies $F_i = 0$, $i = 1, ..., 10$. But a representative of class B5.1 is $c(|0101\rangle + |1010\rangle) + |0110\rangle$, for which $F_{10} = c^4 \neq 0$ from Eq. (A.1). Therefore classes B1.5 and B5.1 are different.

8.3.2   Subsubfamily L_{abc_2} with $c \neq 0$, $a = 0$, and $b = c$

This subsubfamily is a true SLOCC entanglement class.

For each state of this class, we have the following F_i.

$F_1 = -b^3\alpha_1^4 P$, $F_2 = -b^3\alpha_3^4 P$, $F_3 = -b^3\beta_2^4 Q$, $F_4 = -b^3\beta_4^4 Q$,

$F_5 = -b^3\gamma_2^4 R$, $F_6 = -b^3\gamma_4^4 R$, $F_7 = -b^3\delta_1^4 S$, $F_8 = -b^3\delta_3^4 S$.

F_i satisfy Eq. (A.4). Therefore, this class is different from the classes obtained from $c = 0$ and the classes obtained from $c \neq 0 \wedge a = \pm b$ because the F_i $(i = 1, ..., 8)$ of the latter representative states vanish. See Eq. (A.1). We also have the following properties.

Properties 5.1, 5.2, and 5.3 hold for this case.

Each state, which is connected with L_{abc_2} with $c \neq 0$ and $a = 0$ and $b = c$ under SLOCC, satisfies the following property 6.3

Property 6.3. If $F_2 = F_3 = 0$, then $F_9 = 0$ and $F_{10} \neq 0$.

From Eq. (A.1), it is obvious that the representative state with $a = 0$ and $b = c \neq 0$ does not satisfy property 5.4. Hence, the class B5.2 is different from classes B4.1 and B4.2, respectively.

For the state L_{abc_2} with $c = 0$ and $ab \neq 0$, we obtain $F_9 \neq 0$ and $F_{10} = 0$ from Eq. (A.1). Thus, it violates property 6.3. It means that classes B1.3 and B1.4 are different from the class B5.2.

From Eq. (A.1), the representative state L_{abc_2} with $abc \neq 0$ does not satisfy property 6.2 or 6.3. Hence, the classes in the subfamily L_{abc_2} with $abc \neq 0$ are different from the classes B5.1 and B5.2, respectively.

8.3.3 Subsubfamily L_{abc_2} with $bc \neq 0$ and $a = 0$ and $b \neq \pm c$

Each state of this subsubfamily satisfies the following equations.

$\mathcal{I} = (b^2 + 2c^2)/2 * T$,

$F_1 = -c\alpha_1^2(b^2\alpha_1^2 + c(b^2 - c^2)\alpha_2^2) * P$, $F_2 = -c\alpha_3^2(b^2\alpha_3^2 + c(b^2 - c^2)\alpha_4^2) * P$,

$F_3 = -c\beta_2^2(b^2\beta_2^2 + c(b^2 - c^2)\beta_1^2) * Q$, $F_4 = -c\beta_4^2(b^2\beta_4^2 + c(b^2 - c^2)\beta_3^2) * Q$,

$F_5 = -c\gamma_2^2(b^2\gamma_2^2 + c(b^2 - c^2)\gamma_1^2) * R$, $F_6 = -c\gamma_4^2(b^2\gamma_4^2 + c(b^2 - c^2)\gamma_3^2) * R$,

$F_7 = -c\delta_1^2(b^2\delta_1^2 + c(b^2 - c^2)\delta_2^2) * S$, $F_8 = -c\delta_3^2(b^2\delta_3^2 + c(b^2 - c^2)\delta_4^2) * S$.

We can choose $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_4 \neq 0$, and $\alpha_3^2 = \frac{c(c^2 - b^2)}{b^2}\alpha_4^2$ such that $\det(\alpha) \neq 0$ but $F_1 = F_2 = 0$. This violates Eq. (A.4). Therefore this subsubfamily is different from the subsubfamily L_{abc_2} with $c \neq 0$ and $a = 0$ and $b = c$.