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CLASSICAL APPROXIMATION SCHEMES FOR THE GROUND-STATE ENERGY OF QUANTUM AND CLASSICAL ISING SPIN HAMILTONIANS ON PLANAR GRAPHS

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We describe a classical approximation algorithm for evaluating the ground state energy of the classical Ising Hamiltonian with linear terms on an arbitrary planar graph. The running time of the algorithm grows linearly with the number of spins and exponentially with $1/\epsilon$, where ϵ is the worst-case relative error. This result contrasts the well known fact that exact computation of the ground state energy for the two-dimensional Ising spin glass model is NP-hard. We also present a classical approximation algorithm for the quantum Local Hamiltonian Problem or Quantum Ising Spin Glass problem on a planar graph *with bounded degree* which is known to be a QMA-complete problem. Using a different technique we find a classical approximation algorithm for the quantum Ising spin glass problem on the simplest planar graph with unbounded degree, the star graph.

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1 Introduction

Ising spin glasses model physical spin systems with random, competing interactions due to disorder in the system [1]. In order to make meaningful predictions about such systems one can consider statistical ensembles that represent different realizations of the couplings. For a particular realization of the couplings one is generally interested in finding algorithms to determine properties such as the spectrum, partition function, the ground-state or the ground-state energy. Algorithms with a running time that is a polynomial in the problem size are called *efficient*, in contrast with inefficient procedures that take super-polynomial or exponential running times. Connections between disordered systems in statistical physics and questions of computational complexity have been extensively explored, see e.g. [2] and [3].

In this paper we find a new application of tools from approximation algorithms to spin glass problems, namely a rigorous approximation algorithm to determine the ground-state energy of a classical or quantum Ising spin glass on a planar graph. It has been shown that to determine the ground-state energy of a classical Ising spin glass on a 2D lattice with linear terms *exactly* is computationally hard, or NP-hard [4]. Terhal and Oliveira showed that determining the smallest eigenvalue of a quantum Ising spin glass of n qubits on a planar graph with $1/\mathrm{poly}(n)$ accuracy is QMA-hard [5]. Aharonov *et al.* [6] showed that even determining the smallest eigenvalue for *qudits* on a line is QMA-hard. Therefore the running time of

any algorithm for computing the ground state energy exactly must scale super-polynomially with the number of spins n (under the assumption P≠NP). Thus it is natural to look for approximation algorithms that solve the problem in polynomial time at the cost of providing a slightly non-optimal solution.

Let us formally define the classical and the quantum Ising spin glass Hamiltonians on a graph $G = (V, E)$. In the classical case we associate a classical binary variable $S_u \in \{-1, +1\}$ with every vertex $u \in V$. We shall refer to S_u as a *spin*. Let $n = |V|$ be the total number of spins. Any spin configuration $S \in \{-1, +1\}^n$ is assigned an energy

$$H(S) = \sum_{(u,v)\in E} c_{uv} S_u S_v + \sum_{u\in V} d_u S_u. \tag{1}$$

Here c_{uv} and d_u can be arbitrary real coefficients. We shall refer to $H(S)$ as the *Ising spin glass Hamiltonian*. Given a description of the graph G and a list of coefficients c_{uv} and d_u (specified as some m-bit numbers) the problem is to get an estimate of the ground-state energy $\lambda(H) = \min_S H(S)$.

In the quantum case we associate a quantum spin-1/2 (a qubit) with every vertex $u \in V$. Let $\sigma_u^x, \sigma_u^y, \sigma_u^z$ be the Pauli operators acting on a qubit u. The *quantum Ising spin glass Hamiltonian* is a hermitian operator on $(\mathbb{C}^2)^{\otimes n}$ that can be written as a linear combination of two-qubit Pauli operators associated with edges of G and arbitrary one-qubit Pauli operators:

$$H = \sum_{(u,v)\in E} Q_{u,v} + \sum_{u\in V} L_u, \quad Q_{u,v} = \sum_{\alpha,\beta\in\{x,y,z\}} c_{uv}^{\alpha\beta}\, \sigma_u^\alpha \sigma_v^\beta, \quad L_u = \sum_{\alpha\in\{x,y,z\}} d_u^\alpha\, \sigma_u^\alpha. \tag{2}$$

Here $c_{uv}^{\alpha\beta}$ and d_u^α can be arbitrary real coefficients. Given a description of the graph G and a list of coefficients $c_{uv}^{\alpha\beta}$ and d_u^α (specified as some m-bit numbers) the problem is to get an estimate of the ground-state energy $\lambda(H)$, i.e., the smallest eigenvalue of H.

Our main results are summarized by the following theorems.

Theorem 1 *There is a classical algorithm that takes as input a planar graph G with n vertices, a description of an Ising spin glass Hamiltonian H defined on G, and a real number $\epsilon > 0$. The algorithm outputs a spin configuration $S \in \{-1, +1\}^n$ such that*

$$|H(S) - \lambda(H)| \le \epsilon |\lambda(H)|, \tag{3}$$

where $\lambda(H)$ is the ground state energy of H. The algorithm has running time $O(n)2^{O(\epsilon^{-1})}$. If the graph G is a 2D square lattice, the running time is $O(n)4^{\frac{1}{\epsilon}}$.

Remark : We assume that the graph G has no parallel edges or self-loops. In this case any planar graph with n vertices has $O(n)$ edges and thus it can be specified by $O(n \log n)$ bits.

As far as the quantum problem is concerned, we can prove an analogue of Theorem 1 for planar graphs of bounded vertex degree.

Theorem 2 *There is a classical algorithm that takes as input a planar graph G with n vertices and maximum vertex degree d, a description of a quantum Ising spin glass Hamiltonian H defined on G, and a real number $\epsilon > 0$. The algorithm outputs a real number λ such that*

$$|\lambda - \lambda(H)| \le \epsilon |\lambda(H)|, \tag{4}$$

where $\lambda(H)$ is the ground state energy of H. The algorithm has running time $O(n)2^{d^{O(\epsilon^{-1})}}$.

The algorithm also outputs a classical description of a quantum state of n qubits with energy λ. In Section 3.3 we show how to improve the running time of the above algorithm to $n^{O(\epsilon^{-1})}$ for the simplest family of planar graphs with unbounded degree that we call *star graphs*. A star graph with n vertices is a tree in which the root has degree $n-1$ and all its neighbors are leaves.

Remark : For simplicity we assume that basic arithmetic operations with real numbers can be performed at a single time step. In fact our algorithm uses only the first $O(\log \epsilon^{-1})$ most significant digits in the coefficients specifying the Hamiltonian, see Corollary 2 in Section 3.1. Accordingly, taking the real cost of arithmetic operations into account would modify the running time only by a factor $poly(\log \epsilon^{-1})$.

Our approximation algorithm for the classical and quantum Ising spin glass is relevant in light of the recent research on quantum adiabatic approaches for finding the ground-state of a classical or quantum Ising spin glass and solving other NP-complete problems. The paradigm of adiabatic quantum computation (AQC) was first introduced in [8]. An adiabatic computation proceeds by slowly varying the system's Hamiltonian starting from some simple Hamiltonian H_0 at the time $t = 0$ and arriving to, for example, a classical Ising spin glass Hamiltonian H at $t = T$ (the final Hamiltonian can also capture other NP-hard problems). The adiabatic theorem, see e.g. [9], guarantees that if one starts from the ground state of H_0 and the running time T is large compared to the inverse spectral gap at all times then the final state is close the ground-state of H.

If we assume the validity of the conjecture that a quantum computer can not solve NP-hard nor QMA-hard problems (see for classical spin glasses the analysis in [10, 11]) it follows that AQC is a means to obtain an *approximation* to the ground-state and the ground-state energy. The quality of this approximation and its dependence on the hardness of the problem and the adiabatic path are at present not well understood. In order to understand the possible power of AQC it is thus of interest to consider how well an approximation to the ground-state energy can be obtained by purely classical algorithmic means.

Recently the company D-wave has claimed to have implemented the Ising spin glass Hamiltonian (with additional edges on the diagonals of the lattice) and an adiabatic evolution for 16 qubits on a 4×4 square lattice, see [12]. The hope of this endeavor is that such system outperforms classical computers in (approximately) solving optimization problems.

The area of approximation algorithms is an active area of research in computer science, see e.g. [13]. Such algorithms are often of practical importance for generic hard problems for which we are willing to trade off the relative quality of the approximation versus the running time of the algorithm. Three main types of approximation to optimization problems can be distinguished. A problem is said to have a polynomial time approximation scheme (PTAS) if given any $\epsilon > 0$, there is an algorithm A_ϵ which for any instance I produces a solution within $(1 \pm \epsilon)$ times the optimal solution. In addition A_ϵ has a running time which is a polynomial in the input size of I. Observe that the running time of a PTAS is only required to be polynomial in input size, and it can have an arbitrary dependence on ϵ (for example $n^{1/\epsilon}$ or $2^{1/\epsilon}$ poly(n) are valid running times for a PTAS). A stronger notion is that of a fully polynomial time approximation scheme (FPTAS), where the running time of the approximation scheme is required to be polynomial both in the size of the input and in $(1/\epsilon)$. For the classical Ising spin glass problem an instance is a particular graph and set of values

of the weights c_{uv} and d_u. The optimal solution is the ground-state energy $\lambda(H)$.

For the Ising spin glass problem on planar graphs, we can exclude the possibility of a FPTAS for such instances (assuming P $\neq$ NP). This is due to the fact that the problem is NP-hard even when c_{uv}, d_u are restricted to be either $-1, 0$ or $+1$ [4]. Let us assume that we have a FPTAS and set $\epsilon = \delta/\text{poly}(n)$ for some constant δ. This gives a polynomial-time algorithm to approximate $\lambda(H)$ with an error which is at most $\delta|\lambda(H)|/\text{poly}(n)$. This is sufficient accuracy to solve the NP-hard problem *exactly* since $|\lambda(H)|$ is at most $\text{poly}(n)$ and the difference between the minimum of $H(S)$ and the value right above it (i.e. the energy gap) is at least 1, that is, independent of n.

The class of classical optimization problems for which the objective function can be efficiently approximated with a relative error ϵ for some *fixed* constant ϵ is called APX. It is known that some problems in APX do not have a PTAS (under the assumption P$\neq$NP), for example, Minimum Vertex Cover and Maximum Cut problems, see [14]. In other words, for such problems no polynomial-time algorithm can make the relative error smaller than some constant threshold value ϵ_0. Problems having this property are called APX-hard. One can use the relation between the Ising spin glass problem and the Maximum Cut problem, see [15], to show that the former is APX-hard if defined on arbitrary graphs. This is the reason why the present paper focuses only on planar graphs.

Note that the existence of a PTAS for a Hamiltonian H does not imply the existence of a PTAS for the trivially related problem of finding the ground-state energy of $H + aI$, where a is a constant; this is because the PTAS produces a solution with small *relative* error. Our PTAS for the classical Ising spin glass problem can be translated to an approximation algorithm with an *absolute* error at most ϵW, where $W = \sum_{(u,v)\in E} |c_{u,v}|$, see Section 2.2. For the quantum Ising spin glass the absolute error is at most ϵW, where $W = \sum_{(u,v)\in E} \|Q_{u,v}\|$, see Section 3.1. Here $\|\cdot\|$ is the usual operator norm. We believe that an absolute error is a more adequate figure of merit to quantify the quality of approximation obtained using AQC since it is invariant under a trivial shift of energy. What approximation error can be achieved using AQC in the regime when the evolution time is $poly(n)$ while the inverse spectral gap is super-polynomial (which is likely to be the case for NP-hard problems) is a challenging open problem that goes beyond the scope of the present paper.

We will first consider the classical Ising spin glass on a graph G which is a two-dimensional lattice, and give a PTAS for this case. It has a running time $T = O(n4^{\frac{1}{\epsilon}})$, see Section 2.1. We also sketch a generalization of this algorithm to cubic lattices in $\mathbb{R}^d$ for any constant d. The intuitive idea behind these constructions is simple. Assume, for simplicity, that all couplings between spins have similar strength. Then one can subdivide a 2D lattice into blocks of size $L \times L$ by omitting the edges connecting these subblocks. The total contribution of these omitted boundary edges scales as $4L \times \frac{n}{L^2} = O(n/L)$ and hence for large, but constant, $L = 1/\epsilon$ the error that one makes by omitting these edges is bounded by at most $O(\epsilon n)$. This proves that there exists an approximation algorithm with absolute error. However one can show that the magnitude of the ground-state energy scales linearly with n (see Theorem 3) and thus the error will be proportional to the ground-state energy which is exactly what is desired for the PTAS.

In our formulation of the problem, not all edges on the 2D lattice have similar strength, hence somewhat more elaborate arguments are needed to show the existence of a PTAS.

For general planar graphs this situation is more involved. Vertices in the graph can have arbitrarily high degree and it is not clear how to divide up the graph into sub-blocks with small boundaries. In the 'classical' Theorem 3 and its quantum counterpart Theorem 4 we will show that the ground-state energy of an Ising spin glass on a planar graph is less than $-cW$ for some constant c (recall that $W = \sum_{(u,v)} ||Q_{u,v}||$ for the quantum Ising spin glass and $W = \sum_{(u,v)} |c_{uv}|$ for the classical Ising spin glass). This rigorously expresses the intuitive, physical, notion that the ground-state energy is extensive in the system size n, that is, scales linearly with n. The idea of the PTAS is then as follows. We take out a subset of edges in the planar graph for which (i) one can show that their contribution to the Hamiltonian is at most ϵW and (ii) by taking out these edges one ends up with a set of simpler disconnected graphs on which one can solve the problem efficiently (in time $O(n)$). Let $\tilde{H}$ denote the (classical or quantum) Hamiltonian for the problem where we have taken out these edges. Since we only take out edges (and no vertices), a state with minimum energy for $\tilde{H}$ is also a state for H. By Weyl's inequality $|\lambda(H) - \lambda(\tilde{H})| \leq ||H - \tilde{H}|| \leq \epsilon W$. Using Theorems 3 and 4 we can relate W back to the lowest eigenvalue of H and hence show that the PTAS outputs a (quantum) state which has energy at most $O(\epsilon|\lambda(H)|)$ higher than the true ground-state energy.

How do we take out edges from the original planar graph? In the classical case, see Section 2.3, we take out edges related to a so-called outerplanar decomposition of a graph. In this way we end up with disconnected graphs which have a constant *tree-width*. It is well known how to solve the classical spin problem on graphs with bounded tree-width using dynamic programming, see [23] and the discussion in Section 2.3.

In the quantum case we cannot choose this procedure since the quantum problem on a graph with bounded-tree width, or even on a tree, can still be hard, see [6]. This points to an interesting difference between the quantum and the classical Ising spin glass.

If, in the quantum case, we restrict ourselves to graphs with bounded-degree, we can apply a procedure that removes edges and leaves a set of disconnected graphs each of which has *constant* size (related to ϵ), see Section 3.2. Determining the ground-state energy of a Hamiltonian in constant dimension can be done classically by a brute-force method. Note that this PTAS outputs a classical description of a quantum state which has an energy $O(\epsilon|\lambda(H)|)$-close to the true ground-state energy.

Our last result finds a PTAS for the quantum problem on a star graph, see Section 3.3. The complexity of the local Hamiltonian problem on this graph is not known. For the star graph it is clear that we cannot take out edges without introducing a large error. Hence we will use a different technique which uses symmetry and the rounding of interactions. In effect, we construct a Hamiltonian $\tilde{H}$ for which $\lambda(\tilde{H})$ and its ground-state can be determined efficiently *and* $||H - \tilde{H}|| \leq \epsilon W$. The construction works as long as all terms in H have norms in a range $[a, 1]$ for a constant a. Such condition was not present in the other PTAS constructions. Extensions of this technique may be important for addressing the general quantum problem on planar graphs.

We note that our technique for the classical planar graph problem, i.e. using an outerplanar decomposition of the graph, is fairly standard for solving hard classical problems on planar graphs. In fact, many problems admit a PTAS on planar graphs even though approximating them on general graphs is known to be NP-hard, see [16, 17, 18]. While our techniques are similar to those of [17] and [18] at a conceptual level, our results do not follow directly from

their work and require some new ideas. The main difficulty is that the Hamiltonian involves both positive and negative terms which can possibly cancel out making the required precision $\epsilon|\lambda(H)|$ too small. Our proof that the ground-state energy $\lambda(H)$ is an extensive quantity is the key technical contribution of the paper which allows us to build on ideas of [17] and [18].

2 Classical Hamiltonians on planar graphs

2.1 The 2D square lattice case

Consider the classical Ising spin glass Hamiltonian Eq. (1) defined on a 2D square lattice of linear size L, that is, $V = \{u = (x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq x, y \leq L - 1\}$ for open boundary conditions and $V = \mathbb{Z}_L \times \mathbb{Z}_L$ for periodic boundary conditions. Let $n = L^2$ be the total number of spins. Let $\epsilon > 0$ be a fixed small constant. Without loss of generality let us assume that $t = 1/\epsilon$ and $\epsilon\sqrt{n}$ are integers. For $i = 0, \ldots, t - 1$, let X_i denote the set of vertices $u = (x, y)$ on the horizontal lines defined by $\{y \equiv i \text{ (modulo } t)\}$. Similarly, let Y_i denote the set of vertices on the vertical lines defined by $\{ x \equiv i \text{ (modulo } t) \}$, see Figure 1. We define the Hamiltonians $H_i^x(S)$ and $H_j^y(S)$ as follows. H_i^x includes all terms $d_u S_u$, $u \in X_i$ and all terms $c_{uv} S_u S_v$ such that (u, v) is a vertical edge that has exactly one end-point in X_i. Similarly H_j^y includes all terms $d_u S_u$ for $u \in Y_j$ and and all terms $c_{uv} S_u S_v$ such that (u, v) is a horizontal edge that has exactly one end-point in Y_j. It is easy to see that $\sum_{i=0}^{t-1}(H_i^x(S) + H_i^y(S)) = 2H(S)$.[a] This implies that there exists an i and $b = x$ or y such that $H_i^b(S_{opt}) \geq H(S_{opt})/t$ or $H(S_{opt}) - H_i^b(S_{opt}) \leq (1 - \epsilon)H(S_{opt})$, where S_{opt} is a spin configuration with the minimum energy.

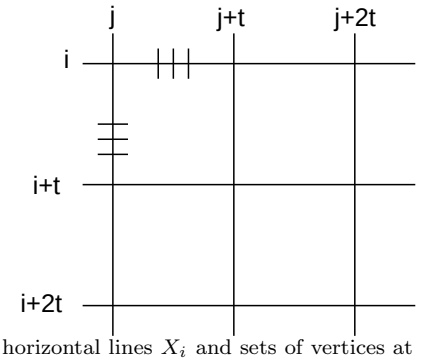

Fig. 1. Sets of vertices on horizontal lines X_i and sets of vertices at vertical lines Y_j. Drawn are also some vertical edges that are part of H_i^x and horizontal edges that are part of H_j^y.

For any i and $b = x, y$ consider a Hamiltonian $H_{sub,i}^b(S) = H(S) - H_i^b(S)$. Note that this Hamiltonian describes the Ising model defined on a set of $\epsilon\sqrt{n}$ disconnected strips of size $\frac{1}{\epsilon} \times \sqrt{n}$ and $\epsilon\sqrt{n}$ disconnected lines of size $\sqrt{n}$. The latter corresponds to sets of edges that have both endpoints in X_i or Y_i and thus contains no linear terms. It means that $H_{sub,i}^b(S)$ is invariant under flipping all the spins in any connected component of X_i (if $b = x$) or Y_i (if $b = y$). Besides, as shown in the previous paragraph, there exists a choice of i and b such that $H_{sub,i}^b(S_{opt}) \leq (1 - \epsilon)H(S_{opt})$.

[a] We note that a similar decomposition could be obtained for qubits on the square lattice with additional diagonal interactions. In such case we would define four Hamiltonians corresponding to lines of vertical, horizontal, diagonal-\, and diagonal-/ vertices with the edges incident on these vertices.

Let S'_{opt} be a spin configuration that achieves the minimum of $H^b_{sub,i}(S)$ for some fixed i and b. Note that S'_{opt} assigns values to all the vertices of the lattice. Using the symmetry of $H^b_{sub,i}(S)$ mentioned above, one can choose S'_{opt} such that $H^b_i(S'_{opt}) \leq 0$. Indeed, if $b = x$ then $H^x_i(S'_{opt})$ changes a sign under flipping all the spins in X_i while $H^x_{sub,i}(S)$ is invariant under this flip (the same argument applies to $b = y$). Therefore, for any i and b one has $H(S_{opt}) \leq H(S'_{opt}) \leq H^b_{sub,i}(S'_{opt})$ and for some i and b one has $H^b_{sub,i}(S'_{opt}) \leq H^b_{sub,i}(S_{opt}) \leq (1-\epsilon)H(S_{opt})$. Thus the minimum energy of $H^b_{sub,i}(S)$ over all i, b, and S approximates $H(S_{opt})$ within a factor $1-\epsilon$ (one can easily show that the minimum energy $H(S_{opt})$ is always non-positive, see Claim 1 in Section 2.2).

It follows that we can get a PTAS by finding the minimum of $H^b_{sub,i}(S)$ over all choices of $i = 0, \ldots, t-1$, $b = x, y$, and all spin configurations S. The running time of this PTAS is $T = \frac{2}{\epsilon} \cdot \left[T_{strip}(\sqrt{n}, \frac{1}{\epsilon}) + T_{strip}(\sqrt{n}, 1)\right] \cdot O(\epsilon\sqrt{n})$, where $T_{strip}(r, s)$ is the running time needed to find the optimal solution for the Ising spin glass Hamiltonian on a $r \times s$ strip. One can easily show that

Lemma 1 *There is a dynamic programming algorithm that computes the optimum solution for a $r \times s$ lattice-blocks using space $O(2^s)$ and time $O(r4^s)$.*

Proof: Let B_i denote an $r \times s$ lattice B restricted to rows $1, \ldots, i$ ($B_r = B$). Let S be $\{-1, +1\}$ vector of size s and for $i \in \{1, \ldots, r\}$, let $V(i, S)$ be the optimum value of the Hamiltonian restricted to B_i such that the variables on the i-th row have assignment S. The dynamic program sequentially computes $V(i, S)$ for all i and S starting from $i = 1$. This suffices as the optimum solution for B is exactly $\min_S V(r, S)$. At each step i one has to store values of $V(i, S)$ and $V(i+1, S)$ only.

For $i = 1$, the quantity $V(1, S)$ can be easily computed for each S. Suppose $V(i, S)$ has been computed and stored for all S. For an assignment S of row $i+1$ and an assignment S' of row i, let $Z(i+1, S, S')$ denote the contribution of all terms to the Hamiltonian corresponding to vertices on row $i+1$, and edges in B_{i+1} with at least one end point in row $i+1$. Since the assignment S on row $i+1$ can only affect Z, it follows that

$$V(i+1, S) = \min_{S'}(V(i, S') + Z(i+1, S, S')). \tag{5}$$

Since $Z(i+1, S, S')$ can be computed in time $O(s)$ and $V(i, S')$ are already stored, computing $V(i+1, S)$ for all S takes time $O(s2^s \cdot 2^s)$. We can speed up this procedure somewhat to $O(4^s)$ by considering the assignments S' in say the Gray code order (where successive assignments differ in exactly 1 variable), and hence only $O(1)$ work needs to be done per assignment S'. □

Thus using the dynamic program leads to a PTAS with an overall running time $T = O(n4^{\frac{1}{\epsilon}})$.

We can consider how a quantum computer could improve these running times. A possible application of quantum searching is inside the dynamic programming. Since many classical approximation algorithms rely on dynamic programming, this could be an important area of applications. For each row of s spins the dynamic program performs 2^s minimizations and the minimum is over a function which takes 2^s values. There exists a quantum algorithm for finding the minimum of a function [19] with square-root speed-up over a brute-force classical minimization. This algorithm uses the Grover search algorithm as a subroutine. However in the dynamic program, part of the input to this function is *stored* which implies that this

is a problem of searching in a real database. In the database setting one has to consider the additional time/hardware overhead in accessing the spatially extended database as well as the overhead associated with implementation of the oracle in the Grover search. Optical or classical wave implementations of this type of searching have been considered, see [20]. One can use the Grover search subroutine for solving the lattice strips and one obtains a running time of $T_q = O(n2^{3/2\epsilon})$ not taking into account the additional overheads. Whether this application of Grover's algorithm in dynamic programming is of genuine interest will depend how its physical implementation competes in practice with the capabilities of classical computers.

Let us briefly sketch how similar ideas provide a PTAS for the Ising spin glass problem defined on a d-dimensional cubic lattice. For simplicity we consider the case $d_u = 0$, that is $H(S)$ contains only quadratic terms (the analysis extends directly to the case when $d_u \neq 0$). For every integer vector $\mathbf{i} = (i_1, \ldots, i_d)$ with components defined modulo $t \equiv 1/\epsilon$ consider a subset of vertices $M_\mathbf{i}$ such that $v = (v_1, \ldots, v_d) \in M_\mathbf{i}$ iff $v_p = i_p \pmod t$ for at least one coordinate $1 \leq p \leq d$. There are t^d such subsets. Note that for every edge (u, v) the number of subsets $M_\mathbf{i}$ such that $u \in M_\mathbf{i}$ or $v \in M_\mathbf{i}$ is $N_{t,d} = t(t^{d-1} - (t-1)^{d-1}) + 2(t-1)^{d-1} = (d+1)t^{d-1} + O(t^{d-2})$. Now define $H_{sub,\mathbf{i}}(S)$ as a sum of all terms $c_{uv}S_uS_v$ such that $u \notin M_\mathbf{i}$ and $v \notin M_\mathbf{i}$. Since $H_{sub,\mathbf{i}}$ describes $O(n\epsilon^d)$ independent blocks of spins of linear size t, its ground state energy can be computed exactly in time $O(n\epsilon^d 2^{t^d})$. On the other hand, $\sum_\mathbf{i}(H(S_{opt}) - H_{sub,\mathbf{i}}(S_{opt})) = N_{t,d}H(S_{opt}) \approx (d+1)t^{d-1}H(S_{opt})$ and thus there exists an $\mathbf{i}$ such that $H_{sub,\mathbf{i}}(S_{opt}) \leq (1-(d+1)\epsilon)H(S_{opt})$. Here for simplicity we retain only the terms linear in ϵ. Furthermore, for any fixed $\mathbf{i}$ let S'_{opt} be the minimum energy configuration of $H_{sub,\mathbf{i}}$. Note that S'_{opt} is not defined on spins from $M_\mathbf{i}$. Let us show that S'_{opt} can be extended to a spin configuration S''_{opt} on the whole lattice such that $H(S''_{opt}) \leq H_{sub,\mathbf{i}}(S'_{opt})$. Indeed, let $M^0_\mathbf{i}$ and $M^1_\mathbf{i}$ include all vertices in $M_\mathbf{i}$ which have even and odd sum of coordinates respectively. For any $\alpha, \beta \in \{+1, -1\}$ extend S'_{opt} by setting value α to every spin in $M^0_\mathbf{i}$ and value β to every spin in $M^1_\mathbf{i}$. Let $S''_{opt}(\alpha, \beta)$ be the resulting spin configuration. Since every edge has at most one endpoint in $M^0_\mathbf{i}$ and $M^1_\mathbf{i}$ we have $\sum_{\alpha,\beta=\pm1} H(S''_{opt}(\alpha,\beta)) = 4H_{sub,\mathbf{i}}(S'_{opt})$. Thus one can choose some α, β such that $S''_{opt} = S''_{opt}(\alpha,\beta)$ satisfies $H(S''_{opt}) \leq H_{sub,\mathbf{i}}(S'_{opt})$. We conclude that there exists an $\mathbf{i}$ such that $H(S''_{opt}) \leq H_{sub,\mathbf{i}}(S'_{opt}) \leq H_{sub,\mathbf{i}}(S_{opt}) \leq (1-(d+1)\epsilon)H(S_{opt})$.

Therefore the minimum of $H(S''_{opt})$ over all $\mathbf{i}$ provides an approximation to $H(S_{opt})$ with a relative error at most $(d+1)\epsilon$. Trying all possible t^d sublattices we get a PTAS with a running time $T = O(n2^{\epsilon^{-d}})$ and a relative error $(d+1)\epsilon$.

2.2 General planar graphs

Let us now consider the general case of planar graphs. In the lattice case we used the symmetry of the lattice to argue that there exists a small subset of edges such that they have a small contribution to the Hamiltonian and removing them decomposes the lattice into small disjoint blocks. For general planar graphs we cannot use this argument due to the lack of symmetry and hence we argue indirectly. We show that the magnitude of the optimal solution is at least a constant fraction of the sum of the absolute values of quadratic terms corresponding to the edges. This allows us to find a subset of edges with relatively small weight such that removing them decomposes the graph into simpler disjoint graphs for which the problem can be solved directly. Since the removed edges have small weight adding them back in does not

increase the value of the Hamiltonian by too much.

For a planar graph $G = (V, E)$ let $W = \sum_{(u,v)\in E} |c_{uv}|$. Let $S_{opt} \in \{-1, +1\}^n$ be a ground-state spin configuration such that $\lambda(H) = H(S_{opt})$. The key to our PTAS is the following result that shows that the ground-state energy scales linearly with W:

Theorem 3 *If G is planar, then $H(S_{opt}) \leq -W/3$.*

We begin with a simple property of the optimal solution (the one that minimizes $H(S)$) that holds for an arbitrary graph. Let us write the Hamiltonian as $H(S) = Q(S) + L(S)$, where $Q(S)$ is the overall contribution of the quadratic terms $c_{u,v}S_uS_v$ and $L(S)$ is the overall contribution of the linear terms d_uS_u.

Claim 1 *There exists an optimal solution S_{opt} such that $H(S_{opt}) \leq 0$ and $L(S_{opt}) \leq 0$.*

Proof: It is clear that $\sum_S H(S) = 0$. This implies that there must exist a spin configuration with non-positive energy and thus $H(S_{opt})$ is non-positive. The second part can be argued by assuming the contrary ($L(S_{opt}) > 0$) and then noting that the solution with opposite signs $-S_{opt}$ is better than S_{opt} itself. □

Thus we note that it suffices to show that $\min_S Q(S) \leq -W/3$ since Claim 1 shows that $\min_S H(S) \leq \min_S Q(S)$. Hence we consider planar graphs with only quadratic terms in $H(S)$, i.e. we assume that $d_u = 0$ for all u. Recall that Bieche et al. [7] have shown that this problem can be solved exactly in polynomial time. Our proof of Theorem 3 builds on the ideas of Bieche et al. and so we first describe these ideas. We begin with some notation.

A graph is planar if it can be drawn in the plane such that no edges cross. This drawing defines disjoint regions in the plane that are called faces. A cycle in a graph is a subset of edges $C \subseteq E$ such that every vertex has even number of incident edges from C. Given two cycles C_1 and C_2, their sum $C_1 \oplus C_2$ is defined as the symmetric difference of C_1 and C_2. The faces of planar graph form a cycle basis, that is, every cycle can be expressed as a sum of faces. Given an assignment S, an edge (u, v) is called unsatisfied if u and v are not assigned according to the sign of c_{uv} (i.e. if $c_{uv} \geq 0$ but $S_uS_v = 1$ or if $c_{uv} < 0$ but $S_uS_v = -1$). A face F is called frustrated if its boundary contains an odd number of edges with positive weight. A key observation is that for any assignment, a frustrated face must always contain an odd number (hence at least one) of unsatisfied edges. Conversely, if J is a subset of edges such that each frustrated (resp. non-frustrated) face contains exactly an odd (resp. even) number of edges in J, then there is an assignment S such that the unsatisfied edges are exactly those in J. Thus, $Q(S) = -W + 2\sum_{(u,v)\in J} |c_{uv}|$. We shall see below that finding such a set J with minimum weight is equivalent to finding the minimum weight T-join in the dual graph G^* (these terms are defined below). Let us remark that the total number of frustrated faces is even because every edge (u, v) with $c_{u,v} > 0$ either has exactly two adjacent faces or it does not belong to a boundary of any face (edges of the latter type may correspond to "dangling" paths or trees).

For a planar graph G, its dual graph G^* is defined as follows: G^* has a vertex for each face in G. Vertices u and v in G^* are connected by an edge if and only if the faces corresponding to u and v in G share a common edge. Note that for a given graph G, the dual G^* is not necessarily unique (it depends on the drawing of G). A subset of edges E' is called a cut-set if removing them disconnects the graph into two or more components. For planar graphs, every cut-set in the dual graph G^* corresponds to a cycle in G.

Let $G = (V, E)$ be a graph with edge weights, and let T be a subset of vertices $T \subseteq V$ such

that $|T|$ is even. A *T-join* is a collection of edges J such that each vertex in T is adjacent to an odd number of edges in J and each vertex in $V \setminus T$ is adjacent to an even number of edges in J. It follows from these definitions that finding the optimal assignment is equivalent to finding the minimum weight T-join in G^* where T is taken to be the set of vertices corresponding to the frustrated faces in G, and where an edge e corresponding to (u, v) in G has weight $w(e) = |c_{uv}|$.

We will use a polyhedral description of T-joins. For a subset of edges J, let $\vec{J}$ denote the corresponding $|E|$-dimensional incidence vector (with 1 in the i-th coordinate if edge i lies in J and 0 otherwise). For a subset of vertices X, let $\delta(X)$ denote the set of edges with one end point in X and other in $V \setminus X$. Given a graph G and the set T, we say that a subset of edges J is an upper T-join if some subset J' of J is a T-join for G. Let P be the convex hull of all vectors $\vec{J}$ corresponding to the incidence vector of upper T-joins. P is called the up-polyhedra of T-joins. Edmonds and Johnson [21] gave the following exact description of P (see the book by Schrijver [22], Chapter 29, par. 1-6 for further details).

$$\sum_{e \in \delta(X)} x(e) \geq 1, \quad \text{for all sets } X \text{ s.t. } |X \cap T| \text{ is odd}, \tag{6}$$

$$0 \leq x(e) \leq 1 \qquad \text{for all edges } e. \tag{7}$$

This implies that any feasible solution x to the system of inequalities above can be written as a convex combination of upper T-joins, i.e. $x = \sum \alpha_i \vec{J}_i$ where $0 \leq \alpha_i \leq 1$ and $\sum_i \alpha_i = 1$. In particular this implies that

Corollary 1 *If all the edge weights $w(e)$ are non-negative, then given any feasible assignment $x(e)$ satisfying the inequalities above, there exists a T-join with cost at most $\sum_e w(e)x(e)$.*

We are now ready to prove Theorem 3. Recall that a graph is called *simple* iff it has no self-loops and no multiple edges between the same pair of vertices.

Lemma 2 *Let G be a simple planar graph with weights $|c_{uv}|$ on the edges, and let G^* be its dual graph. For any subset of vertices T of G^* such that $|T|$ is even, the minimum weighted T-join has weight at most $(\sum_{(u,v)} |c_{uv}|)/3$.*

Proof: Each cut-set J of G^* corresponds to a cycle in G. Since G is simple, each cycle has length at least 3, and hence each cut-set J of G^* contains at least 3 edges. Consider the assignment $x(e) = 1/3$ for all $e \in E$. It clearly satisfies Eq. (7). Moreover it also satisfies Eq. (6) as $\delta(X) \geq 3$ for all $X \subset V$, $X \neq V$. The result then follows from Corollary 1.

□

Lemma 2 implies Theorem 3 immediately, since the value of the optimal assignment is $-W$ plus twice the weight of the optimal T-join which is at most $-W + 2W/3 = -W/3$. Observe that Theorem 3 is tight, as seen from the example where G is a triangle with edge weights $+1, +1$ and -1. Here $W = 3$, but the optimal spin assignment has value -1. The condition that G is simple is necessary. Otherwise, consider the graph on two vertices with two edges, one with weight -1 and other with weight $+1$. Here $W = 2$, but the optimal assignment has value 0.

2.3 The approximation algorithm

Given Theorem 3, the PTAS follows using some ideas in [18]. We begin by describing the notions of p-outerplanar graphs and tree-widths. An outerface of a planar graph drawn in

the plane is the set of edges that constitute the boundary of the drawing. For a tree, the outerface is the set of all the edges. One defines a p-outerplanar graph recursively as follows.

Definition 1 (p-outerplanar graphs) *An outerplanar or 1-outerplanar graph is a planar graph that has an embedding in the plane with all vertices appearing on the outerface. A p-outerplanar graph is a planar graph that has an embedding in the plane such that removing all the vertices on the outerface gives a $(p-1)$-outerplanar graph.*

Examples of 1-outerplanar graphs include any tree or a graph consisting of two cycles that share a common vertex. An example of a 4-outerplanar graph is shown on Fig. 2.

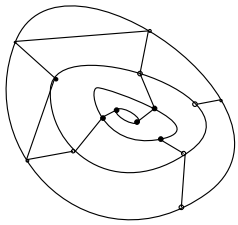

Fig. 2. A planar graph which is drawn as a 4-outerplanar graph.

The notion of tree decompositions (TD) was introduced by Robertson and Seymour [23], see also [18, 24]. Roughly speaking, a TD allows one to map the Ising spin Hamiltonian $H(S)$ on a graph $G=(V,E)$ to a new spin Hamiltonian $H_{tree}(\Theta)$ that depends on spins $\Theta_t \in \{0,1\}^b$ living at vertices t of some tree T. The spin Θ_t represents the "opinion" that a vertex t has about value of spins S_u in some subset of vertices $B_t \subseteq V$ called a *bag*. Accordingly, Θ_t may take 2^b values, where b is the number of spins in B_t. Suppose we can form m bags $B_1,\ldots,B_m \subseteq V$ associated with the m vertices of T such that (i) every vertex $u \in V$ is contained in some bag B_t; (ii) for every edge $(u,v)\in E$, some bag B_t contains both u and v; (iii) a set of bags containing any given vertex of G forms a subtree of T. These data specify a TD of G with a size m and a width $b-1$. The *tree-width* of a graph G is the minimum integer w such that G has a TD of width w. The rules (i),(ii) guarantee that one can distribute the terms $c_{uv}S_uS_v$ and d_uS_u of the Hamiltonian $H(S)$ over the bags $B_1,\ldots,B_m$ such that every term appears in exactly one bag. This distribution defines Hamiltonians $H_{tree,t}(\Theta_t)=\sum_{(u,v)\in B_t} c_{uv}S_uS_v+\sum_{u\in B_t} d_uS_u$, where the spins S_u are determined by Θ_t. Define $H_{tree}(\Theta)=\sum_{t=1}^m H_{tree,t}(\Theta_t)$. For every edge $(s,t)\in E(T)$ let us say that Θ_s and Θ_t are consistent on (s,t) iff their opinions about any spin $u\in B_s\cap B_t$ agree. The rule (iii) guarantees that Θ is consistent with some spin configuration S iff (Θ_s,Θ_t) are consistent on every edge of T. Accordingly, $\min_S H(S)=\min'_\Theta H_{tree}(\Theta)$, where $\min'$ means that the consistency condition on every edge of T is imposed. The optimal solution Θ can be found very efficiently using the standard dynamic programming approach since the problem is defined on a tree. For the sake of completeness we sketch the main steps of this approach below.

Lemma 3 *One can compute* $\min'_\Theta H_{tree}(\Theta)$ *in time* $O(m4^b)$.

Proof: Let r be the root of T. For any vertex s let T_s be the subtree of T rooted at s. Consider a quantity

$$V(s,\Theta_s)=H_{tree,s}(\Theta_s)+\min'_\Theta \sum_{t\in T_s\setminus s} H_{tree,t}(\Theta_t)$$

which is the minimum energy of the Hamiltonian H_{tree} restricted to the subtree T_s where the minimization is subject to the consistency conditions on edges of T_s and the spin Θ_s at the root of T_s is fixed. One can easily check that $V(s,\Theta_s)$ obeys the following recursive equation:

$$V(s,\Theta_s) = H_{tree,s}(\Theta_s) + \sum_{t=\mathrm{child}(s)} \min'_{\Theta_t} V(t,\Theta_t),$$

where the minimization is over all Θ_t consistent with Θ_s. Thus given a table of values of $V(t,\Theta_t)$ for every child of s, one can compute a table of values of $V(s,\Theta_s)$ in time $O(c(s)\cdot 2^b\cdot 2^b)$, where $c(s)$ is the number of children of s. Thus one can compute $\min'_\Theta H_{tree}(\Theta) = \min_{\Theta_r} V(r,\Theta_r)$ in time $O(m4^b)$.

□

It is known that a p-outerplanar graph has a TD with a size $m = 2n-1$ and a tree-width at most $3p-1$. Such a TD can be computed in time $O(pn)$, see [25]. Summarizing, the minimum energy of the Ising spin glass Hamiltonian on a p-outerplanar graph with n vertices can be found in time $T = O(n2^{6p})$.

Our algorithm works as follows. Given the planar graph G, one first constructs a drawing of G in the plane. This can be done in linear time, using for example the algorithm of Hopcroft and Tarjan [26]. This gives an outerplanar decomposition of G, see e.g. Fig 2. Say it is h-outerplanar (h could be as large as $O(n)$). Partition the vertices into levels $V_1,\ldots,V_h$ where V_1 is the outer face and V_i is the outer face obtained by removing $V_1,\ldots,V_{i-1}$. Let E_i be a set of edges that go from V_i to V_{i+1}. For $j=0,\ldots,t-1$, let G_j be the union of sets E_i for all $i = j\,(\mathrm{modulo}\,t)$. (Recall that $t\equiv 1/\epsilon$.) As each edge lies in at most one set G_j, there exists some index j such that the sum of $|c_{uv}|$ over all edges in G_j is at most ϵW. Remove all the edges in G_j from the graph G. This decomposes G into a disjoint collection of t-outerplanar graphs $F_1,F_2,\ldots,F_{\epsilon h}$. We find the minimum energy separately on each of these subgraphs.

Now consider the quality of the solution obtained for the decomposed problem. Let $H_{sub,j}(S) = H(S) - \sum_{(u,v)\in G_j} c_{uv}S_uS_v$ and let the optimal solution for $H_{sub,j}$ be S'_{opt}. By the reasoning above there exists j such that

$$H_{sub,j}(S'_{opt}) \le H_{sub,j}(S_{opt}) \le H(S_{opt}) + \epsilon W. \tag{8}$$

Furthermore, $H(S'_{opt}) \le H_{sub,j}(S'_{opt}) + \epsilon W$. Thus $H(S'_{opt}) \le H(S_{opt}) + 2\epsilon W \le (1-6\epsilon)H(S_{opt})$ by Theorem 3. It follows that we can get a PTAS with a relative error 6ϵ by trying all possible $j=0,\ldots,t-1$ and choosing the optimal solution S'_{opt} that yields the smallest value of $H(S'_{opt})$.

For a fixed j finding S'_{opt} requires time $T_j = \sum_{a=1}^{\epsilon h} O(|F_a|2^{6t}) = O(n2^{6t})$. Thus the overall running time of the PTAS is $T = \sum_{j=0}^{t-1} T_j = O(n\epsilon^{-1}2^{6/\epsilon})$. Choosing $\delta = 6\epsilon$ this implies that the algorithm obtains an assignment with the energy at most $(1-\delta)H(S_{opt})$ in time $O(n\delta^{-1}2^{36/\delta})$.

3 Quantum Hamiltonians on planar graphs

3.1 Upper bound on the ground-state energy

Let us start from proving the quantum equivalent of Theorem 3 which will be instrumental in proving our results. It asserts that the ground-state energy of a quantum Ising spin glass Hamiltonian on a planar graph is an extensive quantity, that is, its magnitude is at least a constant fraction of the sum of the absolute values of all quadratic and linear terms.

Theorem 4 *The minimum eigenvalue $\lambda(H)$ of H for a planar graph $\lambda(H) \leq -\sum_u ||L_u||/5 - W/(5 \cdot 3^5)$, where $W = \sum_{(u,v)\in E} \|Q_{u,v}\|$.*

Proof: The strategy will be to upper bound $\lambda(H)$ by $\lambda_{sep}(H)$, where λ_{sep} is the minimal energy achieved on tensor products of states $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$, where

$$|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}, \quad |\pm i\rangle = (|0\rangle \pm i\,|1\rangle)/\sqrt{2}.$$

It is enough to prove that λ_{sep} is an extensive quantity and this can be achieved using Theorem 3 — the result for the classical case. Let us first prove the theorem for the special case when $L = 0$, that is when the Hamiltonian involves only interactions quadratic in Pauli operators. For every edge $(u, v) \in E$ the interaction $Q_{u,v}$ generally involves all 9 combinations of Pauli operators. We will choose one of them that has the coefficient with the largest absolute value (ties are broken arbitrarily). The corresponding term in the Hamiltonian proportional to a tensor product of two Pauli operators will be called a *dominating coupling*. For example, if $Q_{u,v} = 3\sigma_u^x \otimes \sigma_v^y - 4\sigma_u^z \otimes \sigma_v^x$, then the edge (u, v) has dominating coupling $-4\sigma_u^z \otimes \sigma_v^x$. We have

Lemma 4 *Suppose $Q_{u,v}$ has a dominating coupling $c_{uv}\,\sigma_u^\alpha \otimes \sigma_v^\beta$, where $\alpha, \beta \in \{X, Y, Z\}$. Then*

$$|c_{uv}| \geq \frac{1}{9}\|Q_{u,v}\|. \tag{9}$$

Proof: Indeed, otherwise the triangle inequality would imply $\|Q_{u,v}\| \leq 9|c_{uv}| < \|Q_{u,v}\|$. □

We shall now partition edges E into several subsets $E = \cup_j E_j$, such that the dominating couplings in each subset E_j commute with each other, that is, the sum of dominating couplings over E_j is equivalent to a classical Ising Hamiltonian up to a local change of basis. First of all, since G is a planar graph we can color its vertices by $\{1, 2, 3, 4\}$ such that adjacent vertices have different colors. A map $f : V \to \{X, Y, Z\}$ that assigns a Pauli operator to every vertex of G will be called a *Pauli frame* if $f(u)$ depends only on the color of u. Consider the following Pauli frames.

1	2	3	4
X	X	X	X
X	Y	Z	Y
X	Z	Y	Z
Y	X	Z	Z
Y	Y	Y	X
Y	Z	X	Y
Z	X	Y	Y
Z	Y	X	Z
Z	Z	Z	X

This table forms an orthogonal array of strength two with alphabet $\{X, Y, Z\}$, that is every pair of columns contains every possible combination of two Pauli operators exactly one time (for a general theory of orthogonal arrays see [27]). Let $f_1, \ldots, f_9$ be the corresponding Pauli frames. Denote by E_j the subset of edges $(u, v) \in E$ such that (u, v) has a dominating coupling

$$c_{uv}\,\sigma_u^{f_j(u)} \otimes \sigma_v^{f_j(v)}.$$

Then we conclude that

$$E_j \cap E_k = \emptyset \quad \text{if} \quad j \neq k, \quad \text{and} \quad \cup_{j=1}^{9} E_j = E. \tag{10}$$

For every Pauli frame f_j we can introduce a classical Ising Hamiltonian Q_j obtained from Q by restricting the whole Hilbert space to classical states in the Pauli frame f_j (that is, if $f_j(u) = X$ for some qubit u, we allow this qubit to be in either of states $|+\rangle$ and $|-\rangle$; if $f_j(u) = Z$, we allow u to be in either of states $|0\rangle$, $|1\rangle$, etc.). By definition,

$$\lambda(Q) \leq \lambda(Q_j). \tag{11}$$

The Hamiltonian Q_j has a single interaction $c_{uv}^{\alpha\beta}\sigma_u^\alpha\sigma_v^\beta$ associated with every edge $(u,v) \in E$, where $\alpha = f_j(u)$ and $\beta = f_j(v)$. Note that for every edge $(u,v) \in E_j$ the dominating coupling in $Q_{u,v}$ is diagonal in the Pauli frame f_j and thus it is included to Q_j (for edges that are not in E_j the dominating coupling is not included to Q_j but some other coupling may be included). Thus, applying Theorem 3 to Q_j we obtain

$$\lambda(Q_j) \leq -\frac{1}{3}\sum_{(u,v)\in E_j} |c_{uv}| \leq -\frac{1}{3^3}\sum_{(u,v)\in E_j} \|Q_{u,v}\|, \tag{12}$$

where the second inequality follows from Lemma 4. Combining Eqs. (10,11,12) we arrive at

$$\lambda(Q) \leq \frac{1}{9}\sum_{j=1}^{9}\lambda(Q_j) \leq -\frac{1}{3^5}\sum_{(u,v)\in E} \|Q_{u,v}\|. \tag{13}$$

It remains to generalize this bound to the case $L \neq 0$. We can show that as in the classical case (see Claim 1), the following holds:

Lemma 5 *One can choose a ground-state $|\psi_0\rangle$ of Q such that $\langle\psi_0|L|\psi_0\rangle \leq 0$.*

Proof: Consider the anti-unitary[b] operator W (known as the time reversal operator) defined by

$$W|\phi\rangle = \bigotimes_{u\in V} Y_u\,|\phi^*\rangle$$

for all $|\phi\rangle$, where $|\phi^*\rangle$ is the complex conjugate of $|\phi\rangle$ such that the complex conjugation is performed with respect to the $|0\rangle, |1\rangle$ basis. One can check that $P_u^\alpha W = -WP_u^\alpha$ for any single-qubit Pauli operator P_u^α. It follows that

$$QW = WQ, \quad LW = -WL.$$

Thus, if one defines a state $|\phi_0\rangle = W|\psi_0\rangle$, one gets

$$\langle\phi_0|L|\phi_0\rangle = -\langle\psi_0|L|\psi_0\rangle, \quad \langle\phi_0|Q|\phi_0\rangle = \langle\psi_0|Q|\psi_0\rangle.$$

It follows that $|\phi_0\rangle$ is also a ground state of Q and one of the inner products $\langle\phi_0|L|\phi_0\rangle$ or $\langle\psi_0|L|\psi_0\rangle$ is non-positive.

[b] Recall that an anti-unitary operator on a Hilbert space $\mathcal{H}$ is a bijective map $W : \mathcal{H} \to \mathcal{H}$ such that $\langle W\phi|W\psi\rangle = \langle\psi|\phi\rangle$ for all vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.

□

Remark: As far as the analysis of the approximation algorithm is concerned (see the next section), one can use a weaker form of Theorem 4, namely, $\lambda(H) \leq -3^{-5} \sum_{(u,v)\in E} \|Q_{u,v}\|$ which follows from Eq. (13) and Lemma 5.

Now define 5 states $\rho_1, \ldots, \rho_5$ such that
(i) For $j = 1, \ldots, 4$ a state ρ_j sets qubits $u \in V$ of color j to the ground state of L_u; all other qubits are set to the maximally mixed state.
(ii) ρ_5 is a ground state of Q such that $\mathrm{Tr}(\rho_5 L) \leq 0$.
Then one has the following inequalities

$$\mathrm{Tr}(Q\,\rho_j) = 0 \quad \text{for} \quad j = 1, \ldots, 4, \quad \text{and} \quad \mathrm{Tr}(Q\,\rho_5) = \lambda(Q). \tag{14}$$

$$\mathrm{Tr}(L\,\rho_j) = -\sum_{u\,:\,\mathrm{color}(u)=j} \|L_u\| \quad \text{for} \quad j = 1, \ldots, 4, \quad \text{and} \quad \mathrm{Tr}(L\,\rho_5) \leq 0. \tag{15}$$

Let ρ be the uniform probabilistic mixture of $\rho_1, \ldots, \rho_5$. Then Eqs. (13,14,15) imply that

$$\mathrm{Tr}(\rho\, H) \leq -\frac{1}{5} \sum_{u\in V} \|L_u\| - \frac{1}{5 \cdot 3^5} \sum_{(u,v)\in E} \|Q_{u,v}\|.$$

□

Corollary 2 *One can estimate the ground-state energy $\lambda(H)$ with a relative error ϵ by specifying the coefficients in H only up to the first $m = O(\log \epsilon^{-1})$ most significant digits.*

Indeed, Theorem 4 implies that it suffices to estimate $\lambda(H)$ with an absolute error $\delta = O(\epsilon)(\sum_u \|L_u\| + \sum_{(u,v)} \|Q_{u,v}\|)$. Setting the i-th digit in coefficients specifying L_u and $Q_{u,v}$ to zero for all $i > m$ will change the Hamiltonian at most by $O(\delta)$ in operator norm and thus it will change $\lambda(H)$ at most by $O(\delta)$. Note that this observation applies also to the classical Ising spin glass problem.

3.2 The approximation algorithm for planar graphs with bounded degree

Let us start from defining a weak diameter of a subgraph.

Definition 2 (Weak Diameter): *Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be a subgraph of G. We say that G' has weak diameter d with respect to G, if for any two vertices $v, w \in V'$, their distance in G is at most d.*

We shall use the following result of Klein, Plotkin and Rao [28] on decomposing planar graphs.

Theorem 5 *Let $G = (V, E)$ be an undirected planar graph with non-negative edge weights, and let W denote the total edge weight. Then, given any $\epsilon > 0$, there is a subset of edges E' with total weight at most ϵW such the removing these edges decomposes the graph G into components each of which has weak diameter at most c/ϵ with respect to G. Here c is a fixed constant independent of ϵ.*

If G is a planar graph with maximum degree d, Theorem 5 implies that each component can have at most $d^{O(1/\epsilon)}$ vertices and hence the minimum eigenvalue problem for a Hamiltononian H restricted to every component can be solved in time $2^{d^{O(1/\epsilon)}}$ by a direct calculation of the minimum eigenvalue, for example, using the Lanczos algorithm [29].

There is a linear time algorithm to determine the set of edges E'. The algorithm works as follows. Let $\delta = \epsilon/3$. Root the graph G at an arbitrary vertex and consider the breadth first tree. A vertex is said to be at level i, if it is at distance i from the root. For $j = 0, \ldots, 1/\delta - 1$, let E_j denote the set of edges that connect two vertices at level i and $i+1$ where $i \equiv j$ (modulo $1/\delta$). Choose the set E_j with least weight and remove these edges from G. Let G_1 denote the obtained graph. Now consider each of the components of G_1 and apply this procedure again to obtain the graph G_2. Finally, apply the same procedure to G_2 to obtain G_3 (one applies the procedure three times because planar graphs are $K_{3,3}$ minor free). The result of Klein, Plotkin and Rao [28] shows that G_3 has weak diameter at most $O(1/\delta) = O(1/\epsilon)$. Moreover the weight of edges removed is at most $3\delta = \epsilon$ fraction of the total weight.

Let $H = \sum_{(u,v)\in E} Q_{u,v} + \sum_{u\in V} L_u$ be a quantum Ising spin glass Hamiltonian defined on a graph $G = (V, E)$. Define a weight associated with an edge (u, v) as $\|Q_{u,v}\|$. Let H_{sub} be a Hamiltonian obtained from H by keeping all the linear terms L_u and the quadratic terms $Q_{u,v}$ associated with edges of a subgraph G_3 defined above. By definition of G_3 one has $\|H - H_{sub}\| \leq \epsilon W$ and thus $|\lambda(H_{sub}) - \lambda(H)| \leq \epsilon W$. Theorem 4 implies that $|\lambda(H_{sub}) - \lambda(H)| \leq c\epsilon|\lambda(H)|$ for some numeric constant c. Thus one can approximate $\lambda(H)$ with any fixed relative error ϵ in time $n2^{d^{O(1/\epsilon)}}$.

3.3 Quantum Ising spin glass problem on a star graph

The construction of a PTAS for classical Hamiltonians on planar graphs presented in Section 2.3 relies on the fact that the classical problem on a tree (or any graph of constant treewidth) can be solved efficiently using the dynamic programming. Unfortunately, this method does not work for quantum Hamiltonians. In this section we develop a new technique that allows one to obtain a PTAS for the quantum Ising spin glass Hamiltonian on a *star graph* — a tree that consists of $n + 1$ vertices with one vertex having degree n and n vertices having degree 1. The corresponding graph is $G = (V, E)$, where $V = \{0, 1, \ldots, n\}$ and $E = \{(0, 1), (0, 2), \ldots, (0, n)\}$. We shall refer to spins sitting at vertices $1, 2, \ldots, n$ as *bath spins* and the spin sitting at the vertex 0 as *central spin* (by analogy with the central spin problem studied in the condensed matter physics [32]). Let the Hamiltonian be

$$H = H_0 + \sum_{j=1}^{n} H_{0,j} \tag{16}$$

where H_0 is a linear term acting on the central spin and $H_{0,j}$ is the interaction between the central spin and j-th bath spin (which can include both quadratic and linear terms).

Theorem 6 *Suppose there exist constants $0 < a \leq b$ such that $a \leq \|H_{0,j}\| \leq b$ for all j. Then one can approximate the smallest eigenvalue $\lambda(H)$ with a relative error ϵ in time $n^{\epsilon^{-O(1)}}$.*

In the rest of the section we prove Theorem 6. We start from proving that $\lambda(H)$ can be computed *exactly* in time $\mathrm{poly}(n)$ as long as the number of distinct interactions $H_{0,j}$ is bounded by a constant. We shall use

Lemma 6 *Suppose the interactions $H_{0,j}$ are the same for some subset of bath spins $S \subseteq \{1, \ldots, n\}$. Then one can choose a pure ground state of H symmetric under permutations of spins in S.*

Proof: Suppose one chooses a ground state $|\Psi\rangle$ and symmetrizes it over all permutations of spins in S. Since H commutes with all such permutations, the symmetrized state $|\Psi'\rangle$ is a pure ground state of H. The main difficulty in following this approach is that $|\Psi'\rangle$ may be a zero vector, so we need somewhat more elaborate arguments.

Without loss of generality $S = \{1, 2, \ldots, k\}$. Let $|\Psi\rangle$ be a ground state of H. Denote $W_{j,k}$ the swap of qubits j and k. Assume that $|\Psi\rangle$ is not symmetric under permutations of spins in S. Without loss of generality, $W_{1,2}|\Psi\rangle \neq |\Psi\rangle$. There are two cases: (i) $W_{1,2}|\Psi\rangle$ is proportional to $|\Psi\rangle$. Then $W_{1,2}|\Psi\rangle = -|\Psi\rangle$, since $W_{1,2}$ has eigenvalues ± 1. Therefore $|\Psi\rangle = |\Psi^-\rangle_{1,2} \otimes |\Psi\rangle_{else}$, where $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle$ is the singlet state and $|\Psi\rangle_{else}$ is some state of spins $\{0, 3, 4, \ldots, n\}$. The second case is (ii) $W_{1,2}|\Psi\rangle$ and $|\Psi\rangle$ are linearly independent. Then the anti-symmetrized state $|\Psi'\rangle = |\Psi\rangle - W_{1,2}|\Psi\rangle$ is non-zero. On the other hand, $|\Psi'\rangle$ is a ground state of H since $W_{1,2}$ commutes with H. We conclude that $|\Psi'\rangle = |\Psi^-\rangle_{1,2} \otimes |\Psi\rangle_{else}$. In both case we conclude that H has a ground state $|\Psi\rangle = |\Psi^-\rangle_{1,2} \otimes |\Psi\rangle_{else}$.

Since the energy of a state depends only upon the reduced density matrices $\rho_{0,j}$, we can replace the antisymmetric singlet $|\Psi^-\rangle$ by the symmetric EPR state $|\Psi^+\rangle$ without changing the energy. On the other hand, any state with energy equal to the ground-state energy must be a ground state. We conclude that H has a ground-state $|\Psi\rangle = |\Psi^+\rangle_{1,2} \otimes |\Psi\rangle_{else}$.

By iterating the arguments above one concludes that H has a ground state

$$|\Psi\rangle = |\Psi^+\rangle_{1,2} \otimes \cdots \otimes |\Psi^+\rangle_{k-1,k} \otimes |\Psi\rangle_{else} \equiv |\Psi\rangle_S \otimes |\Psi\rangle_{else}. \tag{17}$$

where $|\Psi\rangle_{else}$ is some state of all spins $j \notin S$ (if k is odd then there will be one unpaired spin in S; this will not change the arguments below very much). Now we can symmetrize $|\Psi\rangle$ by a brute force method. Let

$$\Pi = \frac{1}{k!} \sum_{\tau \in S_k} W(\tau)$$

be the projector onto the symmetric subspace, where $W(\tau)$ is the unitary operator implementing a permutation τ of k spins in S. Note that the state $|\Psi\rangle_S$ in Eq. (17) has non-negative amplitudes in the standard basis. Therefore $W(\tau)|\Psi\rangle_S$ also has non-negative amplitudes. Therefore $\Pi|\Psi\rangle_S \neq 0$, and, accordingly, $|\Psi'\rangle = (\Pi_S \otimes I_{else})|\Psi\rangle$ is a non-zero state symmetric under permutations of spins in S. On the other hand, since $W(\tau)$ commutes with H, $|\Psi'\rangle$ is a ground state of H.

□

This result implies that we can look for a ground state that "occupies" only a $|S| + 1$ dimensional subspace of the $2^{|S|}$ dimensional Hilbert space describing spins in S. If we have a constant number M of distinct interactions, the dimension of the space in which the optimization takes place is $\Pi_{i=1}^{M}(|S_i| + 1) \leq n^M$ which is polynomial in n. Thus the optimization problem for constant M can be solved efficiently in n.

In order to map the general problem onto one in which we have constant number of distinct interaction, we apply a coarse-graining procedure to the general Hamiltonian Eq. (16). One can show

Lemma 7 *For any $0 < a < 1$ define a set M_a of 2-qubit Hamiltonians H satisfying $a \leq \|H\| \leq 1$. For any $\epsilon > 0$ there exist 2-qubit Hamiltonians $G_1, \ldots, G_M$, $M = O((a\epsilon)^{-32})$ such that $\min_\alpha \|G_\alpha - H\| \leq \epsilon \|H\|$ for any $H \in M_a$.*

Proof: It is enough to satisfy $\min_\alpha \|G_\alpha - H\| \leq \epsilon a$. A 2-qubit Hamiltonian satisfying $\|H\| \leq 1$ lives in a $2 \times 2 \times \cdots \times 2$ cube in $\mathbb{R}^{32}$ (where 32 is the dimension over $\mathbb{R}$ of the vector space of 4×4 complex matrices). Construct an ϵa-mesh, count the number of points.

□

Now we are ready to prove Theorem 6. Without loss of generality, $b = 1$ (otherwise multiply H by b^{-1}). Applying Lemma 7 to every interaction $H_{0,j}$ one can partition the n bath spins into $M = O((a\epsilon)^{-32})$ subsets $S_1, \ldots, S_M$ such that $\|H_{0,j} - G_\alpha\| \leq \epsilon \|H_{0,j}\|$ for all $j \in S_\alpha$. We define a coarse-grained Hamiltonian $\tilde{H} = H_0 + \sum_{\alpha=1}^{M} \sum_{j \in S_\alpha} G_\alpha[j]$, where the notation $G_\alpha[j]$ means that G_α acts on the spins 0 and j. We have $\|H - \tilde{H}\| \leq \epsilon \sum_{j=1}^{n} \|H_{0,j}\| \leq \epsilon 5 \cdot 3^5 |\lambda(H)|$ where the second inequality follows from Theorem 4. Therefore, $|\lambda(H) - \lambda(\tilde{H})| \leq c\epsilon|\lambda(H)|$ for some numeric constant c. The classical PTAS will find the ground-state (that is, a $poly(n)$-sized classical description of this state) and the ground-state energy $\lambda(\tilde{H})$ of the coarse-grained Hamiltonian. Lemma 6 implies that it requires time $n^{\epsilon^{-O(1)}}$.

4 Discussion

An important open question is whether there exists a classical or quantum PTAS for the quantum Ising spin glass problem on general planar graphs (with arbitrary vertex degree). It is clear that some new techniques will be needed to settle this problem. A simpler problem in this realm would the quantum Ising spin glass problem on a tree with unbounded degree. Note that even in the simplest case of star graphs the existence of PTAS for the quantum problem is not proved (Theorem 6 assumes the constant lower and upper bounds on the norm of interactions $H_{0,j}$).

One interesting approach to address these quantum problems may be to consider quantum or classical algorithms that output a thermal state $Z^{-1} e^{-H/T}$ at temperature T. One can show that such thermal state is in fact providing a PTAS. One proves this by showing that the average energy $\langle H \rangle_T = Z^{-1} \operatorname{Tr}(He^{-H/T})$ is bounded as $|\langle H \rangle_T - \lambda(H)| \leq 2nT$. This bound follows from the fact that for the free-energy $F(T) = -T \log Z = \langle H \rangle_T - TS(T)$ we have $|F(T) - \langle H \rangle_T| \leq nT$ and $|F(T) - \lambda(H)| = |F(T) - F(0)| \leq nT$. When the ground-state energy $\lambda(H)$ scales with n, see e.g. Theorems 3 and 4, the error in the approximation can be made $\epsilon\lambda(H)$ for $T = O(\epsilon)$. This also shows that finite but small temperature implementation of adiabatic quantum computation will generally provide a PTAS-approximation to the ground-state energy problem (assuming that, say, for bounded-degree graphs beyond the planar ones, the ground-state energy will be extensive, scales with n).

The difference between classical and quantum behavior on tree graphs is also witnessed by the fact that the algorithm of classical belief propagation (for zero temperature this essentially corresponds to dynamic programming) converges efficiently on trees, whereas quantum belief propagation will only work when additional conditions are fulfilled [30, 31]. It is expected that for bounded-degree trees the quantum belief propagation algorithm of [31] at finite temperature T will give rise to a PTAS.

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