

QUANTUM MARGULIS EXPANDERS

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We present a simple way to quantize the well-known Margulis expander map. The result is a quantum expander which acts on discrete Wigner functions in the same way the classical Margulis expander acts on probability distributions. The quantum version shares all essential properties of the classical counterpart, e.g., it has the same degree and spectrum. Unlike previous constructions of quantum expanders, our method does not rely on non-Abelian harmonic analysis. Analogues for continuous variable systems are mentioned. Indeed, the construction seems one of the few instances where applications based on discrete and continuous phase space methods can be developed in complete analogy.

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1 Introduction

Motivated by the prominent role expander graphs play in theoretical computer science [1], quantum expanders have recently received a great deal of attention [2, 3, 4, 5, 6, 7]. In this short work, we report an observation which allows for the simple explicit construction of such quantum expanders. The method relies heavily on quantum phase space techniques: Once familiar with this techniques, the result is an almost trivial corollary of the analogous classical statement. We further discuss continuous analogues of quantum expanders, where again, phase space methods render this an obvious generalization. Hence, the present work can equally be regarded as the presentation of a simple quantum expander, as as a short exposition of the strengths of the phase space formalism as such.

2 Preliminaries

2.1 Expanders

Expander graphs turn up in various areas of combinatorics and computer science (for all claims made in this section, the reader is referred to the excellent survey article Ref. [1]).

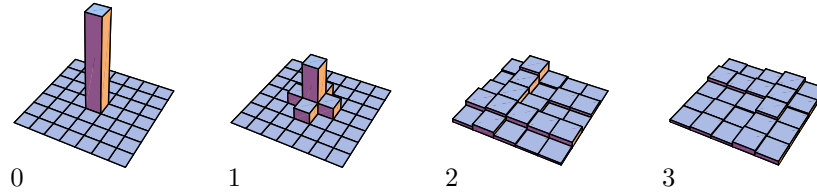


Fig. 1. The phase space distributions resulting from three applications of the Margulis expander acting on a configuration initially concentrated at the origin of a 7×7 lattice. The starting distribution can be interpreted either as a classical particle with a well-defined position on a two-dimensional lattice, or as the quantum phase space operator $A(0,0)$ (see text for definition).

They often come into play when one is concerned with a property which “typically” holds, but defies systematic understanding. A simple example is given by classical error correction codes. One can show that a randomly chosen code is extremely likely to have favorable properties, but it seems very difficult to come up with a deterministic construction of codes which are “as good as random”. Expander graphs can be explicitly constructed, but capture some aspects of generic graphs. It turns out that this property can be used to de-randomize, e.g., the construction of codes or certain probabilistic algorithms.

The formal definition is straightforward. Consider a graph G with N vertices V , each having D neighbors (we allow for multiple links and self-links). There is an obvious way to define a random walk on the graph: At each time step, a particle initially located on a vertex v will be moved to one of the D neighbors of v with equal probability. The resulting Markov process is described by an $N \times N$ doubly stochastic matrix A . The largest eigenvalue of A is $\lambda_1 = 1$, corresponding to the “totally mixed” eigenvector $(1, \dots, 1)/N$. Let λ be the absolute value of the second largest (by absolute value) eigenvalue. A small value of λ means that the Markov process is strongly mixing, i.e., converges rapidly to the totally mixed state. We call G an (N, D, λ) *expander* if it is described by these parameters. The goal is to find families of expander graphs with arbitrarily many vertices N , but constant (and small) degree D and λ .

While the notion of an expander *graph* seems hard to quantize (see, however, Ref. [3]), it makes sense to look for quantum analogues of strongly mixing Markov processes with low degree. Indeed, we call a completely positive map Λ a (N, D, λ) -*quantum expander* if Λ can be expressed in terms of D Kraus operators acting on $\mathcal{B}(\mathbb{C}^N)$ and the absolute value of its second largest singular value is bounded from above by λ (here, $\mathcal{B}(\mathcal{H})$ denotes the space of linear operators acting on a linear space \mathcal{H}). Once more: The intuition is to have a quantum channel which can be written using few Kraus operators, but which rapidly sends any input to the completely mixed state under repeated invocation [8].

Quantum expanders have been introduced independently in Ref. [2] for the purpose of constructing states of spin-chains with certain extremal entanglement and correlation properties, and in Ref. [4], where the problem was approached from a computer science perspective. Very recently, randomized [3] and explicit [2, 4, 5, 6] constructions of expanders have appeared in the literature. The basic idea of a quantum expander is implicit in earlier work [7].

2.2 Margulis expander

Margulis provided the first explicit construction of a family of expander graphs [9]. Their expansion properties can be verified by elementary (if tedious) means [1].

The vertices of Margulis' graph are given by the points of a $N \times N$ -lattice.^aWe label the axes of the lattice by the elements of $\mathbb{Z}_N = \{0, \dots, N-1\}$. Now consider the four affine transformations on \mathbb{Z}_N^2 given by

$$\begin{aligned} T_1 &: v \mapsto S_1 v, \\ T_2 &: v \mapsto S_1 v + (1, 0)^T, \\ T_3 &: v \mapsto S_2 v, \\ T_4 &: v \mapsto S_2 v - (0, 1)^T, \end{aligned} \tag{1}$$

where

$$S_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

All operations are modulo N . Let \mathcal{S} be the set of these four operations, together with their inverses. In Margulis' construction, two vertices are considered adjacent if and only if they can be mapped onto each other by an operation in \mathcal{S} .

One finds that λ is bounded above by $\sqrt{2}5/8$, independent of N [10, 1]. An instance of a random walk on the Margulis graph is visualized in Fig. 1.

2.3 Discrete phase space methods

The present section can be approached from a purely mathematical, or from a physically-oriented point of view. To make the argument more accessible, we will briefly outline both approaches before going into details.

Mathematically, one starts by noting that the operations T_i used in the construction of the Margulis graph (Eq. (1)) are affine transformations on \mathbb{Z}_N^2 , where in addition the linear part S_i has unit-determinant. The functions of this kind form a finite group, which we will refer to as G_N . Two facts will be established below. Firstly,

- there is a (projective) unitary representation

$$T \mapsto U_T$$

of G_N on \mathbb{C}^N .

This representation facilitates the quantization of the expander. Indeed, the quantum Margulis expander will be defined as the c.p. map Λ_N , which applies one of the unitaries $U_T, T \in \mathcal{S}$ at random. To prove that this construction defines an expander, we will need a second fact:

- Let N be odd. There are N^2 hermitian operators $A(a) \in \mathcal{B}(\mathbb{C}^N)$, labeled by the points $a \in \mathbb{Z}_N^2$, such that

1. The operators form an orthonormal basis with respect to the Hilbert-Schmidt inner product:

$$\frac{1}{N} \operatorname{tr} (A(v) A(w)) = \delta_{v,w}. \tag{2}$$

^aNote a slight inconsistency in our notation: the number of vertices in this case is N^2 , not N .

2. The basis thus defined is compatible with the unitary representation of G_N in that

$$U_T A(v) U_T^\dagger = A(T(v)), \tag{3}$$

for $T \in G_N, v \in \mathbb{Z}_N^2$.

In order to analyze the action of Λ_N on a density operator ρ , we will use Eq. (2) to expand ρ in terms of the $A(a)$'s and then Eq. (3) to reduce the problem to the classical case (see Section 3).

Physically speaking, we will employ a phase space description of the quantum system. Recall that the term *phase space* originates in classical mechanics. Here, the state of a single particle in one spatial dimension is completely specified by two real parameters: its position and its momentum. The two-dimensional real vector space spanned by the position and the momentum axes is referred to as the particle's phase space. Likewise, the state of a single quantum system can be specified by a quasi-probability distribution on phase space, namely the particle's Wigner function. The Wigner function shares many properties of classical probability distributions, except for the fact that it can take negative values (see Ref. [11, 12, 13] for an analysis of quantum states which exhibit only positive values).

In the context of continuous-variable systems, affine volume-preserving transformations of the phase space are known as *canonical transformations*. Let ρ be a density matrix and denote by $W_\rho(v)$ the associated Wigner function. It is well-known [21] that for every canonical transformation T , there is a unitary operator U_T which implements the map T in the sense that

$$W_{U_T \rho U_T^\dagger}(v) = W_\rho(T(v)).$$

As detailed below, a similar relation holds for finite-dimensional quantum systems, associated with discrete phase spaces. Indeed, the Wigner function of a density operator ρ turns out to be nothing but the collection of expansion coefficients of ρ with respect to the basis given in Eq. (2); canonical transformations are elements of G_N ; and the correspondence $T \mapsto U_T$ is just the representation mentioned in the first paragraph of this section.

So in this physical language, the basic realization is that the building blocks of the Margulis scheme (Eq. (1)) are canonical transformations of a discrete phase space.

To make all this more precise, let N be odd,^b $\mathcal{H} = \mathbb{C}^N$ and assume that some basis $\{|0\rangle, \dots, |N-1\rangle\}$ in \mathcal{H} has been chosen. Let $\omega = e^{\frac{2\pi}{N}i}$ be an N th root of unity. We define the *shift* and *boost* operators as the generalizations of the X and Z Pauli matrices by

$$x(q)|k\rangle = |k+q\rangle, \quad z(p)|k\rangle = \omega^{pk}|k\rangle \tag{4}$$

(arithmetic is modulo N). The *Weyl operators* are

$$w(p, q) = \omega^{-2^{-1}pq} z(p)x(q), \tag{5}$$

where $2^{-1} = (N+1)/2$ is the multiplicative inverse of 2 modulo N . For vectors $a = (p, q) \in \mathbb{Z}_N^2$, we write $w(a)$ for $w(p, q)$. Let

$$A(0, 0) : |x\rangle \mapsto |-x\rangle \tag{6}$$

^bWe restrict attention to odd dimensions, as the theory of discrete Wigner functions is much more well-behaved in this case.

be the *parity operator* and denote by $A(p, q)$ its translated version,

$$A(p, q) = w(p, q) A(0, 0) w(p, q)^\dagger. \tag{7}$$

We will refer to the $A(p, q)$'s as *phase space operators*. One can check by direct calculation that Eq. (2) holds. The *Wigner function* of an operator ρ is the collection of the expansion coefficients of ρ with respect to the basis formed by the phase space operators. Formally:

$$W_\rho(p, q) = \frac{1}{N} \text{tr} (A(p, q) \rho). \tag{8}$$

There are two symmetries associated with a phase space: translations and volume-preserving linear operations. We shortly look at both in turn. Firstly, it is simple to verify that for $a, b \in \mathbb{Z}_N^2$

$$w(a) A(b) w(a)^\dagger = A(a + b). \tag{9}$$

Hence, Weyl operators implement translations on phase space. Secondly, let S be a unit-determinant matrix with entries in \mathbb{Z}_N . It turns out [17, 15, 12] that there exists a unitary operator $\mu(S)$ such that, for all $a \in \mathbb{Z}_N^2$ the relation

$$\mu(S) A(a) \mu(S)^\dagger = A(S a) \tag{10}$$

holds^c

It follows immediately that for every affine transformation T of the type given in Eq. (1), there exists a unitary operator U_T such that

$$W_{U_T \rho U_T^\dagger}(a) = W_\rho(T^{-1}(a)). \tag{11}$$

Hence, one can unitarily implement the building blocks of Margulis' random walk.

3 A quantum Margulis expander

With these preparations, it is obvious how to proceed. Define the completely positive map Λ_N by

$$\Lambda_N(\rho) = \frac{1}{|\mathcal{S}|} \sum_{T \in \mathcal{S}} U_T \rho U_T^\dagger, \tag{12}$$

where we have used the notation defined in Eq. (11) above. One immediately gets:

Observation 1 (Quantum Margulis expander) *For odd N , the map Λ_N (Eq. (12)) acts on Wigner functions in the same way the Margulis expander acts on classical probability distributions. In particular, its degree and its spectrum are identical to the ones of the Margulis random walk. The Wigner functions of Λ 's eigen-operators are the eigen-distributions of the classical random walk.*

Proof. Let $\Lambda_N^{(C)}$ be the stochastic matrix associated with the random walk on the classical Margulis graph. For $v \in \mathbb{Z}_N^2$, let $e(v)$ be the function on \mathbb{Z}_N^2 , which takes the value 1 at v and 0 else. Clearly, the set $\{e(v)\}_{v \in \mathbb{Z}_N^2}$ spans the space of all functions on the lattice. Also,

$$\Lambda_N^{(C)}(e(v)) = \frac{1}{|\mathcal{S}|} \sum_{T \in \mathcal{S}} e(T(v)).$$

^cThe operator $\mu(S)$ is the *metaplectic representation* of the symplectic matrix S . In quantum information theory, the set $\{w(a) \mu(S) : a \in \mathbb{Z}_N^2, \det(S) = 1\}$ is called the *Clifford group* [14], which must to be confused with the Clifford group appearing in the context of Fermions or representation theory of $SO(n)$.

Using Eqs. (11, 12), we get for the quantum version

$$\Lambda_N(A(v)) = \frac{1}{|S|} \sum_{T \in S} A(T(v)).$$

Hence the action of the classical and the quantum expanders are identical on a basis. The claims follow. \square

4 Efficient implementation

Consider a quantum expander which acts on a tensor-product Hilbert space $(\mathbb{C}^d)^{\otimes n} \simeq \mathbb{C}^N$ for $N = d^n$. The expander is *efficient* if it can be realized using $\text{poly}(n)$ single-qudit or two-qudit quantum gates. So far, only two efficient constructions have been published [5, 6]. The Margulis expander adds to this list.

Theorem 2 (Efficient implementation) *The quantized Margulis expander acts efficiently on $(\mathbb{C}^d)^{\otimes n}$.*

To establish the claim, we need to clarify how we introduce a tensor product structure in \mathbb{C}^N . Every $0 \leq j \leq N - 1$ can be expressed in a d -adic expansion as $j = j_1 \dots j_n$ for $0 \leq j_l \leq d$. More precisely, $j = \sum_{l=1}^n j_l d^{n-l}$. The tensor product structure is now given by $|j\rangle = |j_1\rangle \otimes \dots \otimes |j_n\rangle$.

Lemma 1 (Efficient constituents) *Let $N = d^n$. The following operators act efficiently on \mathbb{C}^N :*

1. The quantum Fourier transform

$$F : |j\rangle \mapsto N^{-1/2} \sum_{k=0}^{N-1} \exp\left(i \frac{2\pi}{N} jk\right) |k\rangle.$$

2. The Weyl operators $w(1, 0)$ and $w(0, 1)$.
3. The operators $\mu(T_1)$ and $\mu(T_2)$.

Proof. The first statement is well-known. See Chapter 5 in Ref. [18] for the qubit version, which can easily be adapted to general d . Next, consider $w(1, 0) = z(1)$. We have

$$\begin{aligned} z(1)|j\rangle &= \exp\left(i \frac{2\pi}{d^n} j\right) |j_1, \dots, j_n\rangle \\ &= \exp\left(i 2\pi \sum_{l=1}^n j_l d^{-l}\right) |j_1, \dots, j_n\rangle \\ &= \bigotimes_l \exp\left(i 2\pi j_l d^{-l}\right) |j_l\rangle. \end{aligned}$$

Hence $z(1)$ is actually local. One confirms that $x(1) = F z(1) F^\dagger$ and thus $x(1)$ is efficient.

To conclude the proof, we need to borrow three statements from the theory of metaplectic representations. Firstly, μ is a projective representation,^d i.e.,

$$\mu(ST) = e^{i\phi(S,T)} \mu(S)\mu(T)$$

^dActually, μ is even a *faithful* representation, but that fact is irrelevant for our purposes.

for some phase $\phi(S, T)$ [17, 15, 12]. Secondly,

$$F = \mu \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right),$$

and thirdly,

$$U_{\pm} = \mu \left(\begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \right)$$

is given by

$$U_{\pm}|j\rangle = \exp(i2\pi/N(\mp j^2))|j\rangle.$$

The last two statements can be found in Theorem 4.1 of Ref. [15] (strictly speaking only for the case of prime N , but the proofs work for any odd value) or in Lemma 2 to Lemma 4 of Ref. [17]. The claim becomes easy to verify:

$$\begin{aligned} U_{\pm}|j\rangle &= \exp \left(i2\pi(\mp \sum_{l,l'=1}^n j_l j_{l'} d^{n-l-l'}) \right) |j\rangle \\ &= \prod_{l,l'} R(l, l')|j\rangle, \end{aligned}$$

where we have introduced the diagonal two-qudit unitary

$$R(l, l')|j_l, j_{l'}\rangle = \exp(i2\pi(\mp j_l j_{l'} d^{n-l-l'}))|j_l, j_{l'}\rangle.$$

Thus U_{\pm} – and therefore in particular $\mu(T_1)$ – are efficient. Finally,

$$T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1},$$

which implies that $\mu(T_2) \propto F U_+ F^{-1}$ is efficient. \square

The proof of Theorem 2 is now immediate, as all the U_T 's which appear in the construction of Λ can be implemented by combining the unitaries treated in the above lemma and their inverses.

5 Continuous variable systems

The quantum phase space terminology of Section 2.3 has originally been introduced in the context of continuous variable systems (see e.g. Ref. [19]). In particular, if we re-interpret the affine transformations \mathcal{S} given in Eq. (1) as operations on \mathbb{R}^2 , we immediately obtain a completely positive map Λ_{∞} acting on the infinite-dimensional Hilbert space of a single mode. Does it constitute a quantum expander? After reviewing some definitions in Section 5.1, we will give an affirmative answer in Section 5.2. The action of expanders on second moments is discussed in Section 5.3.

5.1 Continuous phase space methods

In the continuous case, the phase space is given by \mathbb{R}^2 . Let X and P be the canonical position and momentum operators. The Weyl operators [19, 20, 21] are now

$$w(p, q) = \exp(iqP - ipX). \tag{13}$$

As in Eq. (6), the parity operator $A(0, 0)$ acts on state vectors $\psi \in L^2(\mathbb{R})$ as

$$(A(0, 0)\psi)(x) = \psi(-x).$$

We define the phase space operators $A(p, q)$ for $(p, q) \in \mathbb{R}^2$ exactly as in Eq. (7). The Wigner function becomes

$$W_\rho(p, q) = \pi^{-1} \text{tr}(A(p, q) \rho)$$

c.f. Eq. (8). The obvious equivalents of Eqs. (9,10) hold for $a \in \mathbb{R}^2$ and $S \in \text{Sp}(2, \mathbb{R})$, the group of unit-determinant transformations of the two-dimensional real plane. Hence it is plain how to interpret Eq. (11) and finally how to turn Eq. (12) into a definition of Λ_∞ , the infinite-dimensional quantum Margulis map.

5.2 A continuous quantum Margulis expander

A slight technical problem arises when transferring the definition of an expander to the infinite-dimensional case: both the invariant distribution $f(v) = 1$ of a classical expander and the invariant operator $\mathbb{1}$ of a quantum expander map are not normalizable. Hence, if we define e.g. the action of a completely positive map Λ on the set of trace-class operators $\mathcal{T}^1(\mathcal{H})$, the would-be eigenvector with eigenvalue 1 is not even in the domain of definition. In the light of this problem, we switch to the following definition of a quantum expander, which is compatible with the notion used up to now.

Definition 1 *Let $N \leq \infty$ and set $\mathcal{H} = \mathbb{C}^N$. A completely positive map Λ is an (N, D, λ) -quantum expander if, for all traceless operators $X \in \mathcal{T}^1(\mathcal{H})$,*

$$\|\Lambda(X)\|_2 \leq \lambda \|X\|_2.$$

The definition above is best understood in terms of the Heisenberg picture:

$$|\text{tr}(\Lambda^n(\rho) X)| = |\text{tr}(\rho (\Lambda^\dagger)^n(X))| \leq \lambda^n$$

for all normalized ($\|X\|_2 = 1$), traceless observables X . Thus the state becomes “featureless” exponentially fast when being acted on by Λ . Let λ_M be the second largest eigenvalue of the finite Margulis expanders. Then:

Observation 3 (Continuous quantum expander) *The infinite-dimensional quantum Margulis map Λ_∞ is an $(\infty, 8, \lambda_M)$ -quantum expander.*

Note that by the previous section, we know there are $(N, 8, \lambda_M)$ quantum expanders for arbitrarily high N . A priori, however, this does not imply the existence of a solution for $N = \infty$.

Once more, by switching to the phase-space picture, the proof of Observation 3 can be formulated completely in classical terms. The intuition behind the argument is simple to state. Take an element T of \mathcal{S} , e.g.

$$T : v \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} v. \tag{14}$$

The inverse is given by

$$T^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

regardless of whether the matrix is interpreted as acting on $\mathbb{R}^2, \mathbb{Z}^2$ or \mathbb{Z}_N^2 . As the same is true for all other elements of \mathcal{S} , the action of the classical Margulis map “looks similar” on continuous, infinite discrete and on finite phase spaces – at least as long as it acts on distributions which are concentrated close to the origin, so that the cyclic boundary conditions of \mathbb{Z}_N^2 do not come into play. Using this insight, the following lemma reduces the continuous to the finite case.

Lemma 2 *Let $f \in C_0^0(\mathbb{R}^2)$ be a continuous function with compact support, such that*

$$\int_{\mathbb{R}^2} f(v)dv = 0. \tag{16}$$

Let $A : L^1(\mathbb{R}^2) \rightarrow L^1(\mathbb{R}^2)$ be the classical Margulis map acting on distributions on \mathbb{R}^2 . Then

$$\|A(f)\|_2 \leq \lambda_M \|f\|_2. \tag{17}$$

Proof. We discretize the problem by partitioning \mathbb{R}^2 into a net of squares with side length δ . More specifically, for $(x, y) \in \mathbb{Z}^2$, let

$$Q_\delta(x, y) = [(x - 1/2)\delta, (x + 1/2)\delta] \times [(y - 1/2)\delta, (y + 1/2)\delta]$$

be the square with edge length δ centered around $(x\delta, y\delta) \in \mathbb{R}^2$. The discretized version of f is $f_\delta : \mathbb{Z}^2 \rightarrow \mathbb{C}$ defined by

$$f_\delta(x, y) = \frac{1}{\delta^2} \int_{Q_\delta(x, y)} f(v)dv.$$

Note that $\sum_{x, y} f_\delta(x, y) = 0$. On \mathbb{Z}^2 , we use the δ -dependent norm

$$\|f_\delta\|_2 = \left(\delta^2 \sum_{x, y} |f_\delta(x, y)|^2 \right)^{1/2}$$

(the factor δ^2 corresponds, of course, to the volume of the squares $Q_\delta(x, y)$). Now, let T be one of the affine transformations in \mathcal{S} . We can interpret T as an operation on \mathbb{Z}^2 and define its action on f_δ accordingly by

$$(T(f_\delta))(x, y) = f_\delta(T^{-1}(x, y)).$$

For small enough δ , the approximation is going to be arbitrarily good: using the uniform continuity of f , and the fact that all $T \in \mathcal{S}$ are continuous and volume-preserving, one finds that for every $\varepsilon > 0$, there is a $\delta > 0$ such that simultaneously

$$|\|f_\delta\|_2 - \|f\|_2| < \varepsilon/2, \tag{18}$$

$$|\|A(f_\delta)\|_2 - \|A(f)\|_2| < \varepsilon/2. \tag{19}$$

As the support of f is compact, there is an $R \in \mathbb{N}$ such that $f_\delta(x, y)$ and $A(f_\delta)(x, y)$ are equal to zero whenever $|x| \geq R$ or $|y| \geq R$. This enables us to pass from \mathbb{Z}^2 to the finite lattice \mathbb{Z}_N^2 for $N > 2R$. Indeed, when we re-interpret f_δ as a function $\mathbb{Z}_N^2 \rightarrow \mathbb{C}$ and the $T \in \mathcal{S}$

as affine transformations on \mathbb{Z}_N^2 , the values of $\|f_\delta\|_2$ and $\|A(f_\delta)\|_2$ remain unchanged. But we know that A is an $(N, 8, \lambda_M)$ -expander for every finite N . Hence

$$\|A(f_\delta)\|_2 \leq \lambda_M \|f_\delta\|_2,$$

implying (by Eqs. (18,19))

$$\|A(f)\|_2 \leq \lambda_M \|f\|_2 - \varepsilon.$$

This proves the claim, as the right hand side can be chosen to be arbitrarily small. \square

Proof (of Observation 3). Once again, the quantum Margulis map Λ_∞ acts on the Wigner function W_X of any operator X in the same way the classical Margulis scheme acts on distributions on \mathbb{R}^2 . Now, $X \in \mathcal{T}^1(\mathcal{H})$ implies $W_X \in L^2(\mathbb{R}^2)$. Because $C_0^0(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$ and Λ_∞ is continuous, Lemma 2 suffices to establish the claim. \square

5.3 Action on second moments

In physics, one often measures the concentration of a phase space distribution by its second moments with respect to canonical coordinates. Thus, it may be interesting to look for signatures of the strong mixing properties of a quantum expander in its action on second moments.

More precisely, first moments are the expectation values of the position and momentum operators $(\langle X \rangle, \langle P \rangle)^T$ (where $\langle A \rangle = \text{tr}(\rho A)$ for an operator A). The second moments are defined as the entries of the *covariance matrix*:

$$\gamma = 2 \text{Re} \begin{bmatrix} \langle X^2 \rangle - \langle X \rangle^2 & \langle XP \rangle - \langle X \rangle \langle P \rangle \\ \langle PX \rangle - \langle X \rangle \langle P \rangle & \langle P^2 \rangle - \langle P \rangle^2 \end{bmatrix}.$$

As the action of the continuous quantum expander in state space is defined via the metaplectic representation, the change in second moments can be computed explicitly. In particular, any $S \in \text{Sp}(2, \mathbb{R})$ gives rise to a congruence $\gamma \mapsto S\gamma S^T$ for second moments. More generally, it is not difficult to see that for arbitrary convex combinations of states subject to affine transformations, the output's first and second moments depend only on the same moments of the input.

Under the Margulis random walk, one obtains for the first moments

$$\langle X \rangle \mapsto \frac{1}{|\mathcal{S}|} \sum_{T \in \mathcal{S}} x_T, \quad \langle P \rangle \mapsto \frac{1}{|\mathcal{S}|} \sum_{T \in \mathcal{S}} p_T$$

with $(x_T, p_T)^T = T(\langle X \rangle, \langle P \rangle)^T$. For the second moments:

$$\gamma \mapsto f(\gamma) := \sum_{i=1}^2 \left(T_i \gamma T_i^T + T_i^{-1} \gamma (T_i^{-1})^T \right) + 2G, \tag{20}$$

where the matrix G is given by

$$G = \begin{bmatrix} \sum_T \frac{x_T^2}{|\mathcal{S}|} - \left(\sum_T \frac{x_T}{|\mathcal{S}|} \right)^2 & \sum_T \frac{x_T p_T}{|\mathcal{S}|} - \sum_{T, T'} \frac{x_T p_{T'}}{|\mathcal{S}|^2}, \\ \sum_T \frac{x_T p_T}{|\mathcal{S}|} - \sum_{T, T'} \frac{x_T p_{T'}}{|\mathcal{S}|^2} & \sum_T \frac{p_T^2}{|\mathcal{S}|} - \left(\sum_T \frac{p_T}{|\mathcal{S}|} \right)^2 \end{bmatrix}.$$

The latter matrix is evidently positive [22]. To show that the main diagonal entries of $f^{(n)}(\gamma)$ diverge exponentially in the number n of applications of the map f , it is hence sufficient to consider the map

$$\gamma \mapsto g(\gamma) = \sum_{i=1}^2 \left(T_i \gamma T_i^T + T_i^{-1} \gamma (T_i^{-1})^T \right),$$

since

$$f^{(n)}(\gamma) \geq g^{(n)}(\gamma).$$

A simple calculation yields

$$\gamma = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mapsto g(\gamma) = \begin{bmatrix} a+2c & b \\ b & c+2a \end{bmatrix}.$$

Let $\gamma^{(n)} = g^{(n)}(\gamma)$ be the covariance matrix after n iterations of g and define $\alpha = (a+c)/2$, and $\beta = (a-c)/2$ to simplify notation. Then

$$\gamma^{(n)} = \begin{bmatrix} 3^n \alpha + (-1)^n \beta & b \\ b & 3^n \alpha - (-1)^n \beta \end{bmatrix}.$$

This means that

$$\frac{1}{n} \log_3(\gamma^{(n)}) \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (n \rightarrow \infty).$$

Thus, the elements of the main diagonal – and therefore also $\text{tr}(f^{(n)}(\gamma))$, $\det(f^{(n)}(\gamma))$, and $\text{spec}(f^{(n)}(\gamma))$ – diverge exponentially in the number n of iterations.

6 Summary and Outlook

Employing phase space methods, we were able to quantize a well-established combinatorial structure with almost no technical effort. Until now, discrete Wigner functions have been studied mainly for their mathematical appeal. As far as we know, the present work is the first instance where a problem not related to the phase space formalism itself has been solved using the properties of discrete Wigner functions.

The unitaries which appear in the construction of expanders have randomization properties which are in some sense extremal. It would be interesting to see whether connections to other extremal sets of unitaries – e.g., *unitary designs* [23, 24] – can be found. Also, more practical applications may be anticipated, e.g., when one aims at initializing quantum systems in the maximally mixed state with few (i.e. D) operations, under repeated invocation of the same completely positive map Λ . From a physics perspective, the existence of quantum expanders gives rise to the quite counterintuitive observation that small environments can lead to strongly mixing and decohering dynamics: One may think of a controlled-U operation, the unitaries being those from a quantum expander, where the control has the dimension of a mere $\log_2 D$. Lastly, the programme may improve the understanding of iterated *randomization procedures*, as the one discussed in Ref. [25].

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8. It follows directly from the definition of an (N, D, λ) -quantum expander that

$$\|\Lambda_N(\rho) - \mathbb{1}/N\|_2 \leq \lambda \|\rho - \mathbb{1}/N\|_2, \quad (21)$$

that is, each invocation of the quantum expander contracts the 2-norm distance to the maximally mixed state in \mathbb{C}^N by at least λ .

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22. This can be shown by writing G as $G = AA^T$ with $A \in \mathbb{R}^{2,|S|}$ with entries

$$A_{1,T} = \frac{x_T}{|S|^{1/2}} - \sum_{T'} \frac{x'_T}{|S|^{1/2}}, \quad (22)$$

$$A_{2,T} = \frac{p_T}{|S|^{1/2}} - \sum_{T'} \frac{p'_T}{|S|^{1/2}}. \quad (23)$$

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