Quantum Information and Computation, Vol. 8, No. 6&7 (2008) 0595–0621
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HOMOLOGICAL INVARIANTS OF STABILIZER STATES

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Received December 31, 2006
Revised December 11, 2007

We propose a new kind of invariant of multi-party stabilizer states with respect to local Clifford equivalence. These homological invariants are discrete entities defined in terms of the entanglement a state enjoys with respect to arbitrary groupings of the parties, and they may be thought of as reflecting entanglement in a qualitative way. We investigate basic properties of the invariants and link them with known results on the extraction of GHZ states.

Keywords: multipartite quantum systems, entanglement, stabilizer states, local Clifford equivalence, classification, GHZ states, cohomology of sheaves, duality
Communicated by: R Cleve & M Mosca

1 Introduction

1.1 Overview

In quantum information theory states that are jointly held by several parties play a central role, and are at the heart of the notion of entanglement. Correspondingly the problem of classifying such states has attracted a lot of interest, the correct notion of equivalence being that of local unitary (LU) equivalence of states, which allows each party to apply arbitrary local unitary operators. While classification has been successful in special cases involving few parties it is quite unreasonable to expect a meaningful answer to the classification problem in general.

Among all quantum states the so-called stabilizer states form a subclass which is easier to investigate since much of the multilinear structure of stabilizer states is reflected in the additive structure of the corresponding stabilizer group. On the other hand many important states like GHZ states can be realised as stabilizer states, and furthermore an extensive subclass of stabilizer states has been identified for which LU equivalence coincides with local Clifford (LC) equivalence [1]. On the other hand LC classification of stabilizer states just reduces to the classification of stabilizer groups. Though conceptually easier the latter problem still is far too general to have a useful answer. It therefore appears reasonable to look for and study invariants of states which are intrinsically coarse in the sense that many non-equivalent states are expected to share the same invariants — provided of course that at least some essential features of a state are still reflected in its invariants.

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In this paper we propose and construct such a set of invariants. In the case we have in mind as the most important one we consider a finite set P of parties, each of which controls a state space $\mathcal{H}_p$ with some (finite) number of qubits. To each pure stabilizer state $|\psi\rangle$ in the multi-party space $\bigotimes_{p\in P}\mathcal{H}_p$ we will assign a sequence of finite dimensional vector spaces $H^0(|\psi\rangle),\ldots,H^{|P|}(|\psi\rangle)$ which are invariant with respect to LC equivalences acting on $|\psi\rangle$, and also with respect to permutations of the parties. Our first aim in the construction of these invariants is to capture the entanglement that is present in $|\psi\rangle$ with respect to all possible subdivisions of P: this is the idea behind the cochain groups (3) below. In a second step we intentionally reduce the amount of information by passing to the cohomology groups (6) — a process whose formal properties are well understood and which in algebraic geometry, topology and other mathematical contexts has proven to preserve essential, and erase superficial information.

By its very nature the homological invariants of a state are a qualitative rather than quantitative measure of entanglement, and more specifically of entanglement expressible by linear data (rather than multilinear like the Schmidt measure introduced in [2]). Homological invariants are easily computed from their definition in terms of standard linear algebra, and we have included some of the results for small numbers of qubits. Also, the way the invariants behave under standard processes like coarsening the distribution of the quantum system among the parties, or joining quantum systems can, at least in part, be computed.

A closer investigation of homological invariants reveals the presence of a duality which eventually comes from the symplectic structure on the stabilizer groups. Interestingly, this duality seems to be an important mathematical theme underlying the work of [3] on extraction of GHZ states, where it has appeared in an ad hoc way in some of the proofs. We therefore show how to recover and complement results of the cited work, giving proofs which are conceptual in terms of the invariants.

While the direct definition of homological invariants is quite simple its properties are not easily investigated in this form. A large part of this paper therefore is concerned with identifying the invariants with various forms of so-called sheaf cohomology. For this quite general notion a powerful mathematical machinery has been developed, primarily for applications in algebraic geometry and topology. Indeed we finally prove the duality theorem, our main result, by a reduction to topological Poincaré duality.

1.2 Stabilizer states

The stabilizer formalism was introduced in [4] in the context of error-correcting quantum codes, and since has been explained and widely used in varying degrees of generality [5, 6, 7, 8, 9]. The purpose of the following very brief account is to fix the terminology and notation used below.

The notion of stabilizer state is based on the mathematical one of Heisenberg group, see [10]. Here we consider but a restricted class of such groups, built from data comprising a finite dimensional vector space G over a finite field $\mathbb{F}=\mathbb{F}_p=\mathbb{Z}/p$ of prime order, and a symplectic form $\omega\colon G\otimes G\to\mathbb{F}$ — that is, a skew-symmetric bilinear ω that induces an isomorphism $G\ni g\mapsto\omega(g,?)\in G^{\check{}}=\mathrm{Hom}_{\mathbb{F}}(G,\mathbb{F})$. The corresponding Heisenberg group is the central extension

$$1\longrightarrow S^1\longrightarrow\tilde{G}\longrightarrow G\longrightarrow 0$$

of the multiplicative group $S^1 = \{\lambda \in \mathbb{C} \,\big|\, |\lambda| = 1\}$ by the additive group G determined by

$$G \times G \ni (g,h) \longmapsto e(g,h) = \exp 2\pi i\, \frac{\omega(g,h)}{p} \in S^1$$

in the sense that $e(g,h) = \tilde{g}\,\tilde{h}\,\tilde{g}^{-1}\tilde{h}^{-1}$ is the commutator of any two group elements representing g and h. By the theorem of Stone, von Neumann, and Mackey (SNM) every Heisenberg group has a unique irreducible unitary representation space $\mathcal{H}$ on which the subgroup S^1 acts by scalars in the natural way. In the basic case where $p = 2$ and $G = \mathbb{F}_2 \times \mathbb{F}_2$ carries the canonical symplectic form this representation may be realised by sending the nonzero vectors of G to the Pauli matrices σ_x, σ_y, and σ_z with their action on one qubit. More generally, taking G as a finite orthogonal sum of l copies of $\mathbb{F}_2 \times \mathbb{F}_2$ the resulting SNM representation $\mathcal{H}$ is an l-qubit space with one set of Pauli operators acting on each qubit. Therefore physicists usually consider the Heisenberg group $\tilde{G}$ as a subgroup of the unitary group $U(\mathcal{H})$ and call it a Pauli group.

Let now $L \subset G$ be an isotropic subspace: $\omega(g,h) = 0$ for all $g, h \in L$. Then the natural projection $\tilde{G} \to G$ admits a section $s\colon L \to \tilde{G}$ over L, so that $s(L) \subset \tilde{G}$ is an abelian subgroup. Its fixed space

$$\mathcal{H}^{s(L)} = \{|\psi\rangle \in \mathcal{H} \,\big|\, g|\psi\rangle = |\psi\rangle \text{ for all } g \in s(L)\}$$

is a stabilizer state corresponding to L. It is a vector subspace of $\mathcal{H}$ of dimension $p^{\dim G - 2\dim L}$, and therefore a mixed state in general while a pure state occurs if and only if $L \subset G$ has the maximal dimension $\frac{1}{2} \cdot \dim G$: such L are called lagrangian subspaces of G.

Automorphisms of the Heisenberg group $\tilde{G}$ are by definition required to restrict to the identity on S^1, and they form a group $\operatorname{Aut}\tilde{G}$ which is an extension of the dual $\operatorname{Hom}(G, S^1) = G\check{}$ by the group $Sp(G)$ of symplectic automorphisms of G. Every element of $\operatorname{Aut}\tilde{G}$ gives rise to a unitary transformation of $\mathcal{H}$, called a Clifford automorphism. As to the classification of stabilizer states, the automorphisms from $G\check{}$ just permute the different sections s over $L \subset G$, so that the Clifford classification of states reduces to the symplectic classification of the isotropic subspaces $L \subset G$. An interesting classification problem only arises if some additional structure is given, like a finite direct decomposition of $G = \bigoplus_{p \in P} G_p$ which is orthogonal with respect to ω. Then the local symplectic group

$$\prod_{p \in P} Sp(G_p) \subset Sp(G)$$

acts on G in such a way as to restrict each party's control to the subspace owned by it, and the analogously defined local Clifford equivalence of stabilizer states in $\mathcal{H} = \bigotimes_{p \in P} \mathcal{H}_p$ corresponds to exactly this local symplectic equivalence of isotropic subspaces.

2 Results

2.1 Homological invariants

We fix a finite field $\mathbb{F}$ and a finite set P; the elements of P will be referred to as *parties.* We assume that for each party $p \in P$ a finite dimensional vector space G_p over $\mathbb{F}$, and a linear subspace

$$L \subset G := \bigoplus_{p \in P} G_p \tag{1}$$

are given. More generally we write

$$G_S := \bigoplus_{p\in S} G_p \quad \text{and} \quad L_S := L \cap G_S \tag{2}$$

for each subset $S \subset P$. Our first aim is to define homological invariants of this set of data in the most explicit way.

For each $j \in \mathbb{N}$ we form the direct sum

$$C^j(L) := \bigoplus_{|S|=j} L_S \tag{3}$$

indexed by all subsets $S \subset P$ consisting of exactly j parties. The so-called coboundary operator $\delta\colon C^j(L) \to C^{j+1}(L)$ acts by the formula

$$(\delta x)_T = \sum_{p\in T} \pm\, x_{T\setminus\{p\}} \quad \text{for } T \subset P \text{ with } |T| = j+1\,; \tag{4}$$

the signs which we have left open for the sake of simplicity will be inserted later — note that they are irrelevant in the case of qubits (rather than qudits). The composition of any two consecutive coboundary operators is $\delta \circ \delta = 0$, so that we have inclusions

$$B^j(L) := \operatorname{image}\delta \subset \ker\delta =: Z^j(L) \subset C^j(L) \tag{5}$$

for all $j > 0$. Elements of $C^j(L)$, $B^j(L)$, and $Z^j(L)$ are called cochains, coboundaries, and cocycles respectively, and we define the j-th *homological invariant* of L as the quotient space

$$H^j(L) := Z^j(L)/B^j(L) \quad (j > 0) \tag{6}$$

of cocycles modulo coboundaries.

More generally higher order invariants $H^{ij}(L)$ are obtained if in the definition of $C^j(L)$ the space L is replaced by its i-th exterior power: $C^{ij}(L) := \bigoplus_{|S|=j} \Lambda^i L_S$. Of course $H^{1j}(L)$ is just $H^j(L)$, and $H^{ij}(L)$ can be non-zero only if $i \leq \dim L$ and $j \leq |P|$. Furthermore the invariants of degree one vanish in all interesting cases:

Lemma 1. *$H^{i1}(L) = 0$ for all $i \geq 0$ unless $|P| = 1$, in which case we have $H^{i1}(L) = \Lambda^i L$.*

Proof. $Z^{i1}(L) = 0$ for $|P| \neq 1$ and $Z^{i1}(L) = \Lambda^i L$ else. □

In the case of particular interest $\mathbb{F}$ is a prime field, and for each party $p \in P$ the vector space G_p comes equipped with a symplectic form ω_p. As discussed in the introduction, these data determine a Heisenberg group which via the SNM representation acts on the corresponding Hilbert (state) space $\mathcal{H}$. Subspaces $L \subset G$ which are isotropic with respect to the induced symplectic form ω on G correspond to stabilizer states in $\mathcal{H}$, a state being a pure one, say $|\psi\rangle \in \mathcal{H}$, if and only if L is lagrangian, that is, of half the dimension of G. LC equivalence of these states is the same as local symplectic equivalence of the corresponding isotropic subspaces, which makes $H^{ij}(L) = H^{ij}(|\psi\rangle)$ an LC invariant of the state $|\psi\rangle$.

There are two different interpretations of the tensor product of multi-party states, depending on whether the sets P and Q of parties that own the factors are considered disjoint (external product) or are required to be and remain identical (internal product). In our context we suppose given subspaces $L \subset G_P = \bigoplus G_p$ and $M \subset H_Q = \bigoplus H_q$ in the former case, and form the family of vector spaces

$$((G \oplus H)_r)_{r \in P+Q} \; ; \quad (G \oplus H)_r = \begin{cases} G_r & \text{if } r \in P, \\ H_r & \text{if } r \in Q \end{cases} \tag{7}$$

labelled by the disjoint union $P+Q$ as the new set of parties. The new subspace is $L \oplus M \subset (G \oplus H)_{P+Q}$. By contrast in the internal case $(G_p)_{p \in P}$ and $(H_p)_{p \in P}$ must be indexed by the same set of parties, and the internal direct sum $L + M$, while equal to $L \oplus M$ as an abstract space, is considered as a subspace of the total space $(G+H)_P$ associated with the family

$$\big((G+H)_p\big)_{p \in P}; \quad (G+H)_p = G_p \oplus H_p. \tag{8}$$

Lemma 2. *The first order homological invariants behave additively with respect to the internal tensor product:*

$$H^j(L+M) = H^j(L) \oplus H^j(M) \quad \text{for all } j \in \mathbb{N}. \tag{9}$$

Proof. $(L+M)_S = L_S \oplus M_S$. □

The corresponding result for the external product is less obvious and more complete. At this point it is convenient to formally extend the definition of homological invariants putting

$$H^{i0}(L) = \begin{cases} \mathbb{F} & \text{if } P = \emptyset, \\ 0 & \text{else} \end{cases} \tag{10}$$

and to combine all invariants of L into a single bi-graded vector space

$$H^{**}(L) := \bigoplus_{i,j=0}^{\infty} H^{ij}(L). \tag{11}$$

Theorem 3. *Let $(G_p)_{p \in P}$ and $(H_q)_{q \in Q}$ be families of finite dimensional vector spaces, and $L \subset G$ and $M \subset H$ two vector subspaces. There is a canonical isomorphism*

$$H^{**}(L) \otimes H^{**}(M) \xrightarrow{\simeq} H^{**}(L \oplus M) \tag{12}$$

of bi-graded vector spaces.

Corollary 4. *If the subspace $L \subset G$ is an external direct sum $L = L_1 \oplus \cdots \oplus L_r$ for some partition $P = P_1 + \cdots + P_r$ of the set of parties then*

$$H^{*j}(L) = 0 \quad \text{for } j = 0, \ldots, r-1 \tag{13}$$

and

$$H^{*r}(L) = \bigotimes_{i=1}^{r} H^{*1}(L). \tag{14}$$

Proof. This follows from Theorem 3 in view of (10). ◻

The following theorem concerning the first order invariants is our main result. We assume that for each party $p \in P$ the vector space G_p carries a bilinear form $\omega_p\colon G_p \otimes G_p \longrightarrow \mathbb{F}$. The sum of all these forms is a bilinear form ω on $G = \bigoplus_{p \in P} G_p$, so that orthogonal complements in G are defined. Quite generally, a bilinear pairing defined on $V \times W$ is called perfect if it makes V and W duals of each other.

Theorem 5. *For every subspace $L \subset G$ and all i, j with $i+j = |P|$ the forms ω_p induce bilinear pairings*

$$H^{i+1}(L) \otimes H^{j+1}(L^\perp) \xrightarrow{\hat{\omega}_i} \mathbb{F}. \tag{15}$$

If all ω_p are perfect then so are the $\hat{\omega}_i$.

Again the immediately interesting case is that of symplectic forms ω_p and a lagrangian L:

Corollary 6. *For every pure stabilizer state $|\psi\rangle$ the invariants $H^{i+1}(|\psi\rangle)$ and $H^{j+1}(|\psi\rangle)$ are duals of each other if $i+j = |P|$.*

In applications of Theorem 5 it is important to have an explicit formula for the pairings $\hat{\omega}_i$. As before we postpone consideration of signs for the sake of simplicity.

Proposition 7. *Let the ω_p be bilinear forms, and $L \subset G$ a linear subspace. For any cohomology classes $x \in H^{i+1}(L)$ and $y \in H^{j+1}(L^\perp)$ with $i + j = |P|$ there exist cochains*

$$u \in C^i(G) = \bigoplus_{|S|=i} G_S \quad \text{and} \quad v \in C^j(G) = \bigoplus_{|T|=j} G_T \tag{16}$$

with $\delta u = x$ and $\delta v = y$. For any such choice of u, v and of $p \in P$ one has

$$\hat{\omega}_i(x, y) = \sum_{\substack{S+T \;\subset\; P\backslash\{p\} \\ |S| \;=\; i-1 \\ |T| \;=\; j-1}} \pm\, \omega_p(u_{\{p\}\cup S}, v_{\{p\}\cup T}), \tag{17}$$

where ω_p is considered as a bilinear form on all G that acts as zero on the complementary orthogonal summand $G_{S\backslash\{p\}}$.

It is remarkable that the right hand side of the formula depends only on the components that u and v have in G_p. The formula immediately implies further properties of the bilinear forms $\hat{\omega}_i$ which cannot be directly read off from their definition. We state them just for the case of a base field of characteristic two in order to avoid complicated signs.

Theorem 8. *Assume that* $\operatorname{char} \mathbb{F} = 2$*, that $|P| = 2i$ is even, and that for each $p \in P$ the bilinear form ω_p on G_p is alternating. If the linear subspace $L \subset G$ is isotropic for ω then*

$$\hat{\omega}_i\colon H^{i+1}(L) \otimes H^{i+1}(L) \longrightarrow \mathbb{F} \tag{18}$$

also alternates. If each form ω_p is symplectic, then so is $\hat{\omega}_i$.

2.2 Examples and applications

In order to illustrate homological invariants we have calculated, using [11], a variety of examples including all pure stabilizer states in up to seven parties controlling a single qubit each. The base field here is $\mathbb{F}_2$, and for each $p \in P$ one has $G_p = {\mathbb{F}_2}^2 = \mathbb{F}_2 e \oplus \mathbb{F}_2 f$ with the standard symplectic form: $\omega_p(e, f) = 1$. As is well known under these assumptions each lagrangian subspace of $G = G_P$ is equivalent, under local symplectic transformations, to a (non-unique) standard form that may be described by a simple graph on the vertex set P. Though this description does not play an immediate role in the present context we make use of the classification in [12] of lagrangians in terms of graphs, retaining the labelling used in that work for the sake of convenient reference. We show the complete result for up to five parties in Table 1, and just the first order invariants in the case of six or seven parties in Table 2.

Table 1. Homological invariants of graph states with up to five parties. Graphs are labelled as in [12] and an alternative standard name for each graph or the corresponding state is shown. $l = |P|$ is the number of parties, and h^{ij} shorthand for $\dim_{\mathbb{F}_2} H^{ij}(L)$. For reasons of space the trivial information $h^{lj} = 0$ for $j < l$ and $h^{ll} = 1$ is not listed.

Graph	**Name**	l	h^{11}	h^{12}	h^{13}	h^{14}	h^{15}	h^{21}	h^{22}	h^{23}	h^{24}	h^{25}
No. 1	$\lvert A_2\rangle$	2	0	2								
No. 2	$\lvert A_3\rangle$	3	0	1	1			0	0	3		
No. 3	$\lvert \mathrm{GHZ}_4\rangle$	4	0	1	0	1		0	0	1	3	
No. 4	$\lvert \hat{A}_3\rangle$	4	0	0	2	0		0	0	1	3	
No. 5	$\lvert \mathrm{GHZ}_5\rangle$	5	0	1	0	0	1	0	0	1	0	4
No. 6	$\lvert D_5\rangle$	5	0	0	1	1	0	0	0	0	3	2
No. 7	$\lvert A_5\rangle$	5	0	0	1	1	0	0	0	0	4	1
No. 8	$\lvert \hat{A}_4\rangle$	5	0	0	1	1	0	0	0	0	5	0
Graph	**Name**	l	h^{31}	h^{32}	h^{33}	h^{34}	h^{35}	h^{41}	h^{42}	h^{43}	h^{44}	h^{45}
No. 3	$\lvert \mathrm{GHZ}_4\rangle$	4	0	0	0	4						
No. 4	$\lvert \hat{A}_3\rangle$	4	0	0	0	4						
No. 5	$\lvert \mathrm{GHZ}_5\rangle$	5	0	0	0	1	6	0	0	0	0	5
No. 6	$\lvert D_5\rangle$	5	0	0	0	1	6	0	0	0	0	5
No. 7	$\lvert A_5\rangle$	5	0	0	0	1	6	0	0	0	0	5
No. 8	$\lvert \hat{A}_4\rangle$	5	0	0	0	1	6	0	0	0	0	5

Recall that $H^{0j}(L) = 0$ for all j as soon as P is non-empty, and that the homological invariants of external tensor products of the listed states may be calculated by Theorem 3. A lagrangian which can be written as an internal direct sum of external products with more than one non-trivial factor each, is called *decomposable*, a term likewise applied to the corresponding pure stabilizer state. Thus in view of Corollary 4 and Lemma 2 we see that the first order invariants of decomposable states must vanish.

On the other hand by *reducibility* of a lagrangian (or pure stabilizer state) we mean that it can be written as an internal direct sum of two smaller lagrangians at all. Lemma 2 allows to calculate the first order invariants from those of the irreducible summands. In the case of parties controlling just one qubit every reducible state automatically is decomposable, of

Table 2. First order homological invariants of graph states with six or seven parties

Graph	**Name**	l	h^1	h^2	h^3	h^4	h^5	h^6	h^7	**Graph**	**N.**	l	h^1	h^2	h^3	h^4	h^5	h^6	h^7
No. 9	$\lvert\mathrm{GHZ}_6\rangle$	6	0	1	0	0	0	1		No. 28	$\lvert E_7\rangle$	7	0	0	0	1	1	0	0
No. 10		6	0	0	1	0	1	0		No. 29	$\lvert\hat{E}_6\rangle$	7	0	0	0	1	1	0	0
No. 11	$\lvert\hat{D}_5\rangle$	6	0	0	0	2	0	0		No. 30	$\lvert A_7\rangle$	7	0	0	0	1	1	0	0
No. 12	$\lvert D_6\rangle$	6	0	0	1	0	1	0		No. 31		7	0	0	1	0	0	1	0
No. 13	$\lvert E_6\rangle$	6	0	0	0	2	0	0		No. 32		7	0	0	0	3	3	0	0
No. 14	$\lvert A_6\rangle$	6	0	0	0	2	0	0		No. 33		7	0	0	0	1	1	0	0
No. 15		6	0	0	1	0	1	0		No. 34		7	0	0	0	1	1	0	0
No. 16		6	0	0	0	6	0	0		No. 35		7	0	0	0	1	1	0	0
No. 17		6	0	0	0	2	0	0		No. 36		7	0	0	0	3	3	0	0
No. 18	$\lvert\hat{A}_5\rangle$	6	0	0	0	4	0	0		No. 37		7	0	0	0	1	1	0	0
No. 19		6	0	0	0	12	0	0		No. 38		7	0	0	0	2	2	0	0
No. 20	$\lvert\mathrm{GHZ}_7\rangle$	7	0	1	0	0	0	0	1	No. 39		7	0	0	0	1	1	0	0
No. 21		7	0	0	1	0	0	1	0	No. 40	$\lvert\hat{A}_6\rangle$	7	0	0	0	1	1	0	0
No. 22		7	0	0	0	1	1	0	0	No. 41		7	0	0	0	2	2	0	0
No. 42		7	0	0	0	1	1	0	0	No. 23		7	0	0	1	0	0	1	0
No. 24	$\lvert\hat{D}_6\rangle$	7	0	0	1	0	0	1	0	No. 43		7	0	0	0	3	3	0	0
No. 25		7	0	0	0	1	1	0	0	No. 44		7	0	0	0	1	1	0	0
No. 26		7	0	0	0	1	1	0	0	No. 45		7	0	0	0	6	6	0	0
No. 27	$\lvert D_7\rangle$	7	0	0	0	1	1	0	0										

course. This observation suggests an alternative interpretation of the fact that $H^1(L)$ vanishes for all listed states (as it must by Lemma 1), since a non-zero $H^1(L)$ indicates the presence of a single party irreducible summand:

Proposition 9. *Let $L \subset G$ be an isotropic subspace and assume that $L' := L_{\{p\}}$ is non-zero for some party $p \in P$. Then L splits off a summand in G_p: there is an orthogonal splitting $G = G' \oplus G''$ with $0 \neq G' \subset G_p$ such that $L = L' \oplus L''$ with $L'' := L \cap G''$.*

Proof. Since $L' \subset G_p$ is isotropic we may pick a subspace $G' \subset G_p$ of dimension $2 \dim L'$ which contains L' and to which ω_p restricts as a symplectic form. We define $G'' \subset G$ as the orthogonal complement of G' in G and have $G = G' \oplus G''$ as claimed. The inclusion $L' \oplus L'' \subset L$ is clear, and equality follows from a comparison of dimensions:

$$\begin{aligned} \dim L'' &= \dim\left(L^\perp + G'\right)^\perp \\ &= \dim G - \dim L^\perp - \dim G' + \dim(L^\perp \cap G') \\ &\geq \dim L - \dim G' + \dim(L \cap G') \\ &= \dim L - \dim G' + \dim L' \end{aligned} \tag{19}$$

so that $\dim L' + \dim L'' \geq \dim L - \dim G' + 2 \dim L' = \dim L$. □

The first order invariants $H^j(L)$ for $j \geq 2$ shown in the tables nicely illustrate the duality and symmetry properties described in Corollary 6 and Theorem 8.

We do not know whether there exist irreducible stabilizer states with trivial first order invariants but such states must involve at least four parties[b]. We first make a technical but

[b]Our methods allow to raise this lower bound to five if the subtler notions of Section 3 are used.

more general statement and give a proof which is adapted from that of [3] Theorem 5.

Proposition 10. *Let $L \subset G$ be a lagrangian subspace with $L_{\{p\}} = 0$ for all $p \in P$. Assume that for some subset $P' \subset P$ we have $H^2(L_{P'}) \neq 0$ and*

$$L_{P'} + \sum_{\{Q \subset P \mid P' \not\subset Q\}} L_Q = L. \tag{20}$$

Then there exists a subspace $G' \subset G_{P'}$ such that ω restricts to a symplectic form on G' and such that $L' := L \cap G'$ is a lagrangian subspace of G' which defines a GHZ state. G and L split as internal direct sums $G = G' \oplus G''$ and $L = L' \oplus L''$ with $G'' := (G')^{\perp}$ and $L'' := L \cap G''$.

Proof. By Proposition 7 any cocycle

$$(x_{st}) \in H^2(L_{P'}) \tag{21}$$

may be represented as a coboundary $x_{st} = u_s - u_t$ with values in $G_{P'}$, and since $L_{\{p\}} = 0$ for all $p \in P$ we have a well-defined bilinear form

$$\hat{\omega}_1 \colon H^2(L_{P'}) \otimes L \longrightarrow \mathbb{F}; \quad \hat{\omega}_1(x_{st}, y) = \omega(u_s, y). \tag{22}$$

We claim that if $\hat{\omega}_1(x_{st}, y) = 0$ for all $y \in L_{P'}$ then $(x_{st}) = 0$. Indeed, if $Q \subset P$ is a subset with $P' \not\subset Q$, say with $p \in P' \backslash Q$ then trivially $\hat{\omega}_1(x_{st}, y) = \omega(u_p, y) = 0$ for all $y \in G_Q$. Thus if $\hat{\omega}_1(x_{st}, y) = 0$ holds for all $y \in L_{P'}$ then it even holds for all $y \in L$, in view of the hypothesis of the proposition. Since L is lagrangian it follows that $u_s \in L_{\{s\}}$, and therefore that u_s and a fortiori all x_{st} vanish.

Now pick any non-zero cocycle $(x_{st} = u_s - u_t)$ and let $y \in L_{P'}$ be an element such that $\hat{\omega}_1(x_{st}, y) = 1$. For each $s \in P'$ let $G'_s \subset G_s$ be the subspace spanned by u_s and the component of y in G_s: thus each G'_s is a symplectic plane. Putting $G' = \bigoplus_{s \in P'} G'_s$ the intersection $L \cap G'$ is spanned by the u_s and y, and the conclusion of the proposition is readily verified. □

Corollary 11. *Let P be a set of three parties and let $L \subset G$ be a lagrangian subspace with $H^*(L) = 0$. Then L is reducible, or $L = 0$.*

Proof. If $L_{\{p\}}$ is non-zero for some p then L splits of a summand in G_p, by Proposition 9: we thus may assume $L_{\{p\}} = 0$ for all p. If for some two-party set $P' \subset P$ we have $H^2(L_{P'}) \neq 0$ we may apply Proposition 10 since the assumption $H^3(L) = 0$ implies $\sum_{|Q|=2} L_Q = L$. Thus in that case L splits off an EPR state. There remains the case where $H^2(L_{P'}) = 0$ for all $P' \subset P$ with $|P| = 2$. Explicitly this condition means $L_{P'} = 0$ for all such P'; since on the other hand $H^3(L) = 0$ means that the coboundary homomorphism $\bigoplus_{|P'|=2} L \cap G_{P'} \to L$ is surjective we conclude that $L = 0$. □

One of the main results of [3] is a formula for for the number of GHZ states that can be extracted from a multi-party stabilizer state by LC operations. We here state and reprove this result in terms of homological invariants.

Theorem 12. *Assume $|P| \geq 2$ and let $L \subset G$ be a lagrangian subspace. The number of lagrangians of all-party GHZ states that can be split off from L is*

$$\frac{1}{2} \sum_{j \in \{2, |P|\}} \dim_{\mathbb{F}} H^j(L) \tag{23}$$

(which coincides with $\dim H^2(L) = \dim H^{|P|}(L)$ as soon as $|P| > 2$).

Proof. It suffices to show that such a lagrangian can be split off if and only if $H^2(L) \neq 0$. We put $l = |P|$. The condition is necessary by Lemma 2 since the GHZ state has the non-trivial first order invariants $H^2(L)$ and $H^l(L)$, each of dimension one unless $l = 2$ when they merge in a single space of dimension two.

Now assume that $H^2(L)$ is non-trivial. By Theorem 5 the pairing $\hat{\omega}_1 \colon H^2(L) \otimes H^l(L) \to \mathbb{F}$ is perfect, and we pick cohomology classes

$$x \in H^2(L) \quad \text{and} \quad y \in H^l(L) \tag{24}$$

with $\hat{\omega}_1(x, y) = 1$. We represent y by a vector $z = (z_p)_{p \in P} \in L$, and according to Proposition 7 find cochains

$$(u_s) \in C^1(G) \quad \text{and} \quad (v_Q) \in C^{l-1}(L) \tag{25}$$

with $\delta u = x$ and $\delta v = y$. For each $p \in P$ we then have

$$1 = \hat{\omega}_1(x, y) = \sum_{p \neq t \in P} \pm\, \omega_p(u_p, v_{P \setminus \{t\}}) = \pm\, \omega_p(u_p, \delta v) = \pm\, \omega_p(u_p, z_p). \tag{26}$$

Thus for each party p the vectors u_p and z_p span a symplectic plane $G'_p \subset G_p$, and putting $G' = \bigoplus_p G'_p$ the intersection $L \cap G'$ is spanned by the u_p and z, and is a GHZ lagrangian. □

The authors of [3] construct an example of a four-party stabilizer state $|\psi\rangle$ which is irreducible but neither a GHZ nor an $|\hat{A}_3\rangle$ state. This property can easily be read from the first order homological invariants of $|\psi\rangle$, which have dimensions $h^2 = 0$, $h^3 = 4$, and $h^4 = 0$, as follows: Two of the four parties own one qubit, and the others two qubits each. If the lagrangian of $|\psi\rangle$ were an internal direct sum, one summand would have to involve all parties since $h^3 \neq 0$, and the complementary summand would have trivial invariants. But by the classification of indecomposable four-party states in four or five qubits a state with $h^3 = 4$ does not exist.

Scanning the classification lists we found many more six- and seven-party states that lead to a four-party state with $h^3 = 4$ by suitable grouping of the parties; they include the states No. 14, 17, 18, 30, 33, 35, and 37–45. Amusingly, the six-party state No. 19 from which [3]'s example is constructed in fact gives a four-party state with $h^3 = 4$ whichever grouping is chosen, as long as two of the new parties are given two qubits each.

3 Proofs

3.1 Partition cones and sheaves

For each (finite) set P of parties we define its *partition cone* as the closed positive octant in $|P|$-dimensional euclidean space

$$X = X(P) = [0, \infty)^P \subset \mathbb{R}^P; \tag{27}$$

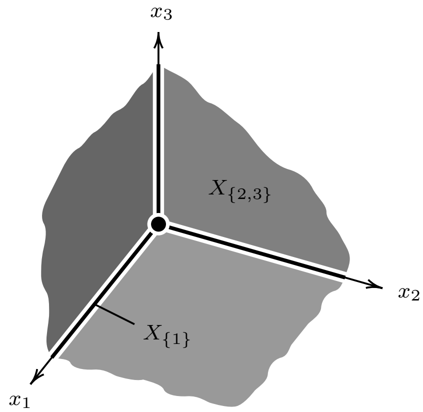

Fig. 1. The decomposition of the partition cone $X(\{1,2,3\})$ into strata

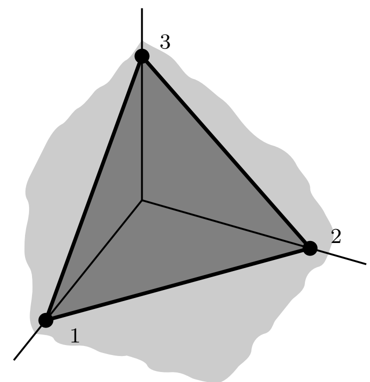

Fig. 2. The simplex $\Delta_{\{1,2,3\}}$

thus a point $x = (x_p)_{p\in P} \in X$ is a P-tuple of non-negative reals. We consider X as a topological space with a particular decomposition into the subsets

$$X_S := \bigl\{x \in X \bigm| x_p > 0 \text{ if and only if } p \in S\bigr\}, \tag{28}$$

which are indexed by subsets $S \subset P$ and called the strata of X: see Fig. 1. In particular $X_\emptyset = \{\underline{0}\}$ is the stratum that contains just the origin while X_P is the open octant in X. Note that the closure of a stratum X_S is

$$\overline{X_S} = \bigcup_{R\subset S} X_R = \{x \in X \mid x_p = 0 \text{ if } p \notin S\} \tag{29}$$

and may be identified with the partition cone $X(S)$. A useful dual notion is that of *open star* of a stratum X_S, defined as the union

$$\operatorname{st} X_S = \bigcup_{S\subset T} X_T = \{x \in X \mid x_p > 0 \text{ if } p \in S\} \tag{30}$$

or, equivalently, as the smallest union of strata that contains X_S and is open — remember that openness always is relative to X as a topological space in its own right.

Quite generally, recall that a subset $Y \subset X$ is locally closed if and only if it can be written as the intersection of an open and a closed subset, and that this property is equivalent to local compactness of Y since X itself is locally compact. If $\mathcal{P}$ is a set of subsets of P then a union of strata $\bigcup_{S\in\mathcal{P}} X_S$ is locally closed in X if and only if $\mathcal{P}$ has the property

$$R, T \in \mathcal{P} \quad\Longrightarrow\quad S \in \mathcal{P} \text{ for all } S \subset P \text{ with } R \subset S \subset T. \tag{31}$$

Geometrically one should think of P as embedded in the partition cone $X = X(P)$, identifying $p \in P$ with the point of X with p-coordinate equal to one and the others zero. The convex hull Δ_P of P is the same as the intersection of X with the hyperplane given by $\sum_{p\in P} x_p = 1$, and clearly is a simplex of dimension $|P|-1$, see Fig. 2. The facets of Δ_P

naturally correspond to the non-empty subsets of P, and the closed facet Δ_S with label S equals $\overline{X_S} \cap \Delta_P$.

Our main tool will be certain *sheaves* on partition cones, and we therefore include a brief general explanation of this notion. A presheaf $\mathcal{F}$ on a topological space X assigns to each open subset $U \subset X$ a set $\mathcal{F}(U)$, whose elements are called sections over U, and to each inclusion of open subsets $U \subset V$ a mapping $\mathcal{F}_{UV} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction, such that $\mathcal{F}_{UU} = \mathrm{id}$ and the relation $\mathcal{F}_{UV} \circ \mathcal{F}_{VW} = \mathcal{F}_{UW}$ holds whenever $U \subset V \subset W$. Presheaves often carry additional algebraic structures, and all presheaves considered here will be presheaves of vector spaces over the fixed field $\mathbb{F}$, that is, all $\mathcal{F}(U)$ will be vector spaces and all restriction maps, linear homomorphisms.

A sheaf on X is a presheaf that satisfies two extra conditions:

- If $V \subset X$ is open, $V = \bigcup_{i \in I} U_i$ a covering of V by open sets, and $f \in \mathcal{F}(V)$ then $\mathcal{F}_{U_i V}(f) = 0$ for all $i \in I$ implies $f = 0$.
- If $V \subset X$ is open, $V = \bigcup_{i \in I} U_i$ a covering of V by open sets, and $f_i \in \mathcal{F}(U_i)$ are given such that $\mathcal{F}_{U_i \cap U_j, U_i}(f_i) = \mathcal{F}_{U_i \cap U_j, U_j}(f_j)$ for all $i, j \in I$ then there exists a section $f \in \mathcal{F}(V)$ with $\mathcal{F}_{U_i V}(f) = f_i$ for all $i \in I$.

Thus the property that characterises sheaves among presheaves is that sections over V may be described by local sections that are compatible on intersections. Given a presheaf $\mathcal{F}$ on X its so-called associated sheaf may be constructed essentially by enforcing the two extra conditions. Many sheaves are defined in this indirect way since it is easier to write down a presheaf than a sheaf.

An alternative description of sheaves is in terms of their *stalks*. The stalk of the sheaf $\mathcal{F}$ at the point $x \in X$ is defined as $\mathcal{F}(x) := \lim_{\rightarrow} \mathcal{F}(U)$ where the direct limit is taken over the system of all open neighbourhoods U of x in X.

The standard notions of linear algebra make sense for sheaves if applied to the stalks of the latter. In particular there are the notions of direct sum, tensor product, exterior power, of sub and quotient sheaves, of sheaf homomorphisms giving rise to kernel and image subsheaves, and of *exact sequences*

$$\cdots \longrightarrow \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \longrightarrow \cdots \tag{32}$$

of sheaves on X (exactness at $\mathcal{F}$ meaning that $\operatorname{image} f = \ker g$). A *short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{33}$$

is a flexible means of identifying $\mathcal{E}$ with a subsheaf of $\mathcal{F}$ and $\mathcal{G}$ with a quotient sheaf, in other words it makes $\mathcal{F}$ an extension of $\mathcal{E}$ by $\mathcal{G}$.

The simplest presheaves on a topological space X are constant and assign to each open $U \subset X$ the same vector space, say L. By abuse of language the corresponding sheaf is called constant and written L too; it assigns to an arbitrary open $U \subset X$ the cartesian product $L^{\pi(U)} = \prod_{\pi(U)} L$ where $\pi(U)$ is the set of connected components of U. In particular one has $L(U) = L$ for all non-empty connected open $U \subset X$ while $L(\emptyset) = 0$ is the zero space. Also, for every $x \in X$ the stalk $L(x)$ is just the vector space L.

More interesting sheaves are obtained if a constant sheaf L on some open subset $Y \subset X$ is extended by zero off Y. Writing the inclusion as $e: Y \hookrightarrow X$ the extended sheaf is denoted $e_!L$, and characterised by the property

$$e_!L(U) = \begin{cases} L & \text{if } U \text{ is non-empty connected and } U \subset Y \\ 0 & \text{if } U \text{ is connected and } U \not\subset Y. \end{cases} \tag{34}$$

Note that the same procedure works more generally to define $e_!\mathcal{F}$ for an arbitrary sheaf $\mathcal{F}$ on Y. There is a second and quite different way to produce a sheaf on X from a given sheaf $\mathcal{F}$ on a subspace $Y \subset X$, called the direct image $e_*\mathcal{F}$ and defined by

$$e_*\mathcal{F}(U) = \mathcal{F}(U \cap Y) \quad \text{for all open } U \subset X. \tag{35}$$

While this makes sense for arbitrary subspaces Y we will usually apply it to closed ones.

In the converse direction, given a subspace $e: Y \hookrightarrow X$ every sheaf $\mathcal{G}$ on X has a unique restriction to a sheaf $\mathcal{G}|Y = e^{-1}\mathcal{G}$ on Y such that $e^{-1}\mathcal{G}(x) = \mathcal{G}(x)$ holds for all $x \in Y$.

We will find it useful to call *elementary sheaves* on a partition cone X those of the form $e_!L$ where $e: Y \hookrightarrow X$ is the inclusion of an open union of strata, and *quasi-elementary* sheaves those that can be obtained by successive extension of elementary sheaves in finitely many steps.

Suppose now data given as in Section 2, comprising a set P of parties and for each party $p \in P$ a finite dimensional vector space G_p over $\mathbb{F}$. The assignment

$$U \longmapsto L_{\{p \in P \,|\, x_p > 0 \text{ for all } x \in U\}} \tag{36}$$

for open subsets U of the partition cone $X = X(P)$, together with the inclusion

$$L_{\{p \in P \,|\, x_p > 0 \text{ for all } x \in V\}} \hookrightarrow L_{\{p \in P \,|\, x_p > 0 \text{ for all } x \in U\}} \tag{37}$$

as the restriction homomorphism in case $U \subset V$, defines a presheaf on X. The associated sheaf $\mathcal{F}L$ is called the *partition sheaf*. Its stalk at an arbitrary point $x \in X_S \subset X$ is

$$\mathcal{F}L(x) = L_{\{p \in P \,|\, x_p > 0\}} = L_S \tag{38}$$

and may be identified with the space of sections $\mathcal{F}L(\operatorname{st} X_S)$. Note in particular the extreme cases $\mathcal{F}L(X) = \mathcal{F}L(\operatorname{st} X_\emptyset) = 0$ and $\mathcal{F}L(X_P) = \mathcal{F}L(\operatorname{st} X_P) = L$.

If $L' \subset L$ is a linear subspace then $\mathcal{F}L' \subset \mathcal{F}L$ is a subsheaf. We also may apply exterior powers to $\mathcal{F}L$, and clearly have $\Lambda^i\mathcal{F}L(U) = \Lambda^i L \cap \Lambda^i \mathcal{F}G(U)$ for all open $U \subset X$ and all $i \in \mathbb{N}$.

Example. If there is just one party then $X = [0, \infty)$ is the closed ray and $\mathcal{F}L = e_!L$ where $e: (0, \infty) \hookrightarrow [0, \infty)$ is the inclusion.

In general partition sheaves are no longer elementary but still quasi-elementary, a fact that we need to record in more generality:

Proposition 13. *For all subspaces $L' \subset L \subset G$ and all $i \in \mathbb{N}$ the quotient sheaf $\Lambda^i\mathcal{F}L/\Lambda^i\mathcal{F}L'$ is a quasi-elementary sheaf.*

Proof. We argue by induction on the dimension of L/L'. The assertion being trivial in case $L' = L$ we assume that $L' \subset L$ is a proper subspace. Since extensions of quasi-elementary sheaves inherit that property by definition we may further suppose that $\dim L = \dim L' + 1$. Define the sheaf $\mathcal{C}$ as the cokernel in the short exact sequence

$$0 \longrightarrow \mathcal{F}L' \longrightarrow \mathcal{F}L \longrightarrow \mathcal{C} \longrightarrow 0. \tag{39}$$

For any vector $g \in G$ we let $\operatorname{supp} g := \{p \in P \mid g_p \neq 0\}$ be the *support* of g. We put

$$W = \bigcup_{g \in L \setminus L'} \{x \in X \mid x_p > 0 \text{ for all } p \in \operatorname{supp} g\} = \bigcup_{g \in L \setminus L'} \operatorname{st} X_{\operatorname{supp} g} \tag{40}$$

and claim that

$$\mathcal{C} \simeq e_!(L/L') \tag{41}$$

where $e\colon W \hookrightarrow X$ is the open inclusion.

It suffices to study sections in $\mathcal{C}(U)$ with $U = \operatorname{st} X_S$ for some $S \subset P$. If $U \subset W$ then there is some $g \in L \setminus L'$ with $U = \operatorname{st} X_S \subset \operatorname{st} X_{\operatorname{supp} g}$, so that $\operatorname{supp} g \subset S$. Therefore we have $g \in L_S = \mathcal{F}L(U)$, hence

$$\mathcal{F}L(U) = \mathcal{F}L'(U) \oplus \langle g \rangle \tag{42}$$

in this case.

If on the other hand $U \not\subset W$ then we pick some $x \in U \setminus W$. Thus we have $x \in \operatorname{st} X_S$ but for all $g \in L \setminus L'$ necessarily $x \notin \operatorname{st} X_{\operatorname{supp} g}$ and therefore $\operatorname{supp} g \not\subset S$. We conclude

$$\mathcal{F}L(U) = \mathcal{F}L'(U) \tag{43}$$

in this second case.

We thereby have shown that the canonical sheaf homomorphism $\mathcal{F}L/\mathcal{F}L' \to L/L'$ induces an isomorphism (41) as claimed. The conclusion of the proposition now follows from the identity

$$\Lambda^i \mathcal{F}L / \Lambda^i \mathcal{F}L' = \Lambda^{i-1} \mathcal{F}L' \otimes \mathcal{F}L/\mathcal{F}L', \tag{44}$$

the fact that the tensor product of elementary sheaves is elementary, and induction on i. □

3.2 Cohomology

Let $\mathcal{F}$ be a sheaf of vector spaces on a topological space X. Associated with any such sheaf is the sequence $H^0(X;\mathcal{F})$, $H^1(X;\mathcal{F})$ ... of cohomology groups (traditionally referred to as groups even if they are, in fact, vector spaces over the same field as $\mathcal{F}$). The quite technical definition is treated in standard texts of sheaf theory and algebraic geometry, for example [13, 14, 15, 16]. It is often convenient to combine all groups from the sequence into the single graded vector space

$$H^*(X;\mathcal{F}) := \bigoplus_{j=0}^{\infty} H^j(X;\mathcal{F}). \tag{45}$$

Apart from $H^0(X;\mathcal{F}) = \mathcal{F}(X)$, the space of global sections of $\mathcal{F}$, these groups have no simple direct interpretation and in no case of interest it is possible to compute them directly.

Nevertheless in many instances cohomology of sheaves is highly computable by methods that rely on the formal properties of cohomology rather than one particular definition. Among the most prominent of these properties is the fact that every homomorphism $\mathcal{F} \to \mathcal{G}$ of sheaves over X induces linear mappings $H^j(X;\mathcal{F}) \to H^j(X;\mathcal{G})$ in a functorial way (that is, compatible with composition of maps), and that every short exact sequence (33) gives rise to a long exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(X;\mathcal{E}) \longrightarrow H^0(X;\mathcal{F}) \longrightarrow H^0(X;\mathcal{G}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^1(X;\mathcal{E}) \longrightarrow H^1(X;\mathcal{F}) \longrightarrow H^1(X;\mathcal{G}) \xrightarrow{\delta} \cdots \end{aligned} \tag{46}$$

where $\delta\colon H^j(X;\mathcal{G}) \to H^{j+1}(X;\mathcal{E})$ is the so-called (Bockstein) coboundary homomorphism.

Cohomology of a sheaf that lives on a subspace $e\colon Y \hookrightarrow X$ is related to cohomology on X via $H^j(Y;\mathcal{F}) = H^j(X;e_*\mathcal{F})$. A sheaf $\mathcal{F}$ on X is called acyclic if $H^j(X;\mathcal{F}) = 0$ for all $j > 0$.

While abstract sheaf cohomology is generally accepted to be the universally correct concept there are many special cases where it allows a more concrete interpretation. A case in point is that of a constant sheaf L on a (reasonable) space X; it is well-known [17, 14] that the sheaf cohomology $H^*(X;L)$ can be identified with topological (singular) cohomology of X with coefficients in the field L. In a similar vein we now will identify the homology invariants introduced in Section 2 with the cohomology of the corresponding partition sheaf over a suitable subspace of the partition cone. The link is provided by the following construction.

Let $\mathcal{F}$ be a sheaf on a topological space X, and let $\mathfrak{U} = (U_s)_{s\in S}$ be an open covering of X indexed by a strictly ordered set S. The so-called Čech complex is the sequence

$$0 \longrightarrow C^0(\mathfrak{U};\mathcal{F}) \xrightarrow{\delta} C^1(\mathfrak{U};\mathcal{F}) \xrightarrow{\delta} \cdots \tag{47}$$

of vector spaces

$$C^j(\mathfrak{U};\mathcal{F}) = \prod_{s_0<\cdots<s_j} \mathcal{F}(U_{s_0} \cap \cdots \cap U_{s_j}) \tag{48}$$

and linear maps $\delta\colon C^j(\mathfrak{U};\mathcal{F}) \to C^{j+1}(\mathfrak{U};\mathcal{F})$ which are called coboundary operators and act by

$$(\delta x)_{s_0\ldots s_{j+1}} = \sum_{\alpha=0}^{j+1} (-1)^\alpha \mathcal{F}_{U_{s_0}\cap\cdots\cap U_{s_{j+1}},\,U_{s_0}\cap\cdots\cap\widehat{U_{s_\alpha}}\cap\cdots\cap U_{s_{j+1}}} \left(x_{s_0\ldots\widehat{s_\alpha}\ldots s_{j+1}}\right) \tag{49}$$

(by convention terms covered by a hat are to be omitted). The Čech complex indeed is a complex in the sense that the composition of any two consecutive coboundary operators is $\delta\circ\delta = 0$, so that for each j we have inclusions

$$B^j(\mathfrak{U};\mathcal{F}) := \operatorname{image}\delta \subset \ker\delta =: Z^j(\mathfrak{U};\mathcal{F}) \subset C^j(\mathfrak{U};\mathcal{F}). \tag{50}$$

Elements of $C^j(\mathfrak{U};\mathcal{F})$, $B^j(\mathfrak{U};\mathcal{F})$, and $Z^j(\mathfrak{U};\mathcal{F})$ are called (Čech) cochains, coboundaries, and cocycles respectively, and the quotient space

$$H^j(\mathfrak{U};\mathcal{F}) := Z^j(\mathfrak{U};\mathcal{F})/B^j(\mathfrak{U};\mathcal{F}) \tag{51}$$

of cocycles modulo coboundaries is the j-th Čech cohomology of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$. It is easily seen to be independent of the way $\mathfrak{U}$ is ordered.

Lemma 14. *Let $W = X(P)\backslash\{\underline{0}\}$ be the complement of the origin in a partition cone, and let $\mathfrak{U} = \{\mathrm{st}\, X_{\{p\}} \,|\, p \in P\}$ be the covering of W by the open stars of one-dimensional strata. Then for every partition sheaf $\mathcal{F}L$ and each degree j the Čech cohomology $H^j(\mathfrak{U}; \mathcal{F}L)$ coincides with the invariant $H^{j+1}(L)$ of (6), and similarly*

$$H^j(\mathfrak{U}; \Lambda^i \mathcal{F}L) = H^{i,j+1}(L) \quad \textit{for all } i \textit{ and } j. \tag{52}$$

Proof. The definition of $C^j(\mathfrak{U}; \mathcal{F}L)$ becomes

$$C^j(\mathfrak{U}; \mathcal{F}L) = \bigoplus_{s_0 < \cdots < s_j} \mathcal{F}L(\mathrm{st}\, X_{\{s_0,\ldots,s_j\}}) = \bigoplus_{s_0 < \cdots < s_j} L_{\{s_0,\ldots,s_j\}} = \bigoplus_{|S|=j+1} L_S \tag{53}$$

which is $C^{j+1}(L)$ of (3), and as the restriction homomorphisms of a partition sheaf are just inclusions the formula for δ reduces to

$$(\delta x)_{s_0 \ldots s_{j+1}} = \sum_{\alpha=0}^{j+1} (-1)^\alpha x_{s_0 \ldots \widehat{s_\alpha} \ldots s_{j+1}}, \tag{54}$$

which is (4) together with the specification of the sign left open there. □

As the next step we establish the connection with sheaf cohomology.

Proposition 15. *Let $\mathcal{F}$ a quasi-elementary sheaf on a partition cone X, and suppose that the subspace $Y = \bigcup_{S \in \mathcal{P}} X_S$ is a locally closed union of strata. Then the subsets $\mathrm{st}\, X_S \cap Y$, where S runs through the minimal sets in $\mathcal{P}$, form an open covering $\mathfrak{U}$ of Y such that the Čech cohomology $H^j(\mathfrak{U}; \mathcal{F})$ coincides with sheaf cohomology $H^j(Y; \mathcal{F})$. In particular the invariant $H^{i,j+1}(L)$ of (52) coincides with $H^j(W; \Lambda^i \mathcal{F}L)$.*

Proof. We need but assemble known general results. First note that any intersection of covering sets from $\mathfrak{U}$ has the form

$$\mathrm{st}\, X_{S_0} \cap \cdots \cap \mathrm{st}\, X_{S_i} \cap Y = \mathrm{st}\, X_{S_0 \cup \cdots \cup S_i} \cap Y \tag{55}$$

and therefore is the open star in Y of some stratum, or empty. On the other hand the definition of elementary sheaves implies at once that they belong to a more general class of sheaves called constructible [18, 19]. Such sheaves are known to be acyclic on the open star of a stratum, see [18] Proposition 8.1.4. We thus have

$$H^j(U_0 \cap \cdots \cap U_i; \mathcal{F}) = 0 \quad \text{if } j > 0 \tag{56}$$

for all $U_0, \ldots, U_i \in \mathfrak{U}$, and this means that $\mathfrak{U}$ is a so-called Leray covering (in other words, sufficiently simple so that intersections of the individual covering sets do not carry any cohomology of positive degree). The conclusion now follows by [13] Théorème II.5.4.1. □

Corollary 16. *If $Y \subset X$ is a closed union of strata then every quasi-elementary sheaf $\mathcal{F}$ is acyclic on Y:*

$$H^j(Y; \mathcal{F}) = 0 \quad \textit{if } j > 0. \tag{57}$$

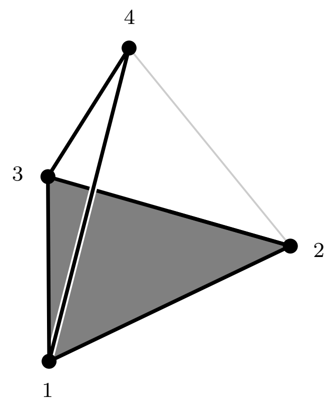

Fig. 3. A four party example — the polyhedron $\Gamma_Y = \Delta_{\{1,2,3\}} \cup \Delta_{\{1,4\}} \cup \Delta_{\{3,4\}}$ is associated to $Y = \overline{X_{\{1,2,3\}}} \cup \overline{X_{\{1,4\}}} \cup \overline{X_{\{3,4\}}}$

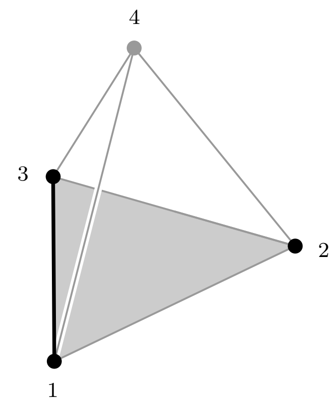

Fig. 4. $(\Gamma_Y)\check{} = \Delta_{\{1,3\}} \cup \Delta_{\{2\}}$ is the dual of the polyhedron Γ_Y of Fig. 3

Proof. In this case the covering $\mathfrak{U}$ of Y consists of the single set $Y = \operatorname{st} X_\emptyset \cap Y$, so that the Čech complex has no terms of positive degree. □

While general sheaf cohomology is conceived as a means to study the structure of the sheaf rather than that of the underlying space, cohomology of quasi-elementary sheaves still is fairly close to topology. We illustrate this by a completely topological description of cohomology in the even more special case of an elementary sheaf.

Remember that Δ_P denotes the simplex spanned by P as a subset of $X = X(P)$. More generally, for each closed union of strata $Y \subset X$ the intersection

$$\Gamma_Y := Y \cap \Delta_P \tag{58}$$

is a subpolyhedron (simplicial subcomplex) of Δ_P which we call the *associated polyhedron* of Y, see Fig. 3.

Proposition 17. *Let X be a partition cone, $Y \subset X$ and $Z \subset X$ closed unions of strata, and L a vector space. Denote by $e\colon X \backslash Z \hookrightarrow X$ the inclusion; then there is a canonical isomorphism*

$$H^*(Y \backslash \{\underline{0}\}; e_! L) \simeq H^*(\Gamma_Y, \Gamma_{Y\cap Z}; L) \tag{59}$$

between sheaf cohomology and cohomology of the topological pair with coefficient group L.

Proof. We apply Proposition 15. Every intersection of covering sets has the form $\operatorname{st} X_S \cap Y$ for some non-empty subset $S \subset P$, and the corresponding direct summand in the Čech complex is either L, if $\operatorname{st} X_S \cap Y \neq \emptyset$ but $\operatorname{st} X_S \cap Y \cap Z = \emptyset$, or else the zero group. The condition is met if and only if Δ_S is a simplex of Γ_Y but not of $\Gamma_{Y\cap Z}$, and this sets up an isomorphism with the relative simplicial cochain complex $C^*(\Gamma_Y)/C^*(\Gamma_{Y\cap Z})$ of the polyhedral pair $(\Gamma_Y, \Gamma_{Y\cap Z})$. The proposition follows. □

3.3 Local cohomology

In order to investigate the multiplicative properties of homological invariants yet another re-interpretation is required. Once more let X be a partition cone, $Y \subset X$ a closed union of strata, and $\mathcal{F}$ a quasi-elementary sheaf. In principle two refined notions of cohomology play a role — so-called local cohomology $H^j_{\{\underline{0}\}}(Y;\mathcal{F})$ and cohomology with compact supports $H^j_c(Y;\mathcal{F})$, both obtained from a variation of the standard definition [13, 14, 16]. Nevertheless, due to the cone-like structure of $\mathcal{F}$ they coincide here, and we will write cohomology with compact supports throughout while using some features of the local version. These include the fact that it is truly local in the sense that the inclusion $e\colon U \hookrightarrow Y$ of an arbitrary neighbourhood U of the origin in Y induces isomorphisms

$$H^j_{\{\underline{0}\}}(Y;\mathcal{F}) \simeq H^j_{\{\underline{0}\}}(U;\mathcal{F}) \tag{60}$$

for all j. Another general feature is a long exact sequence which relates local cohomology to ordinary one [15]. In our situation it yields:

Proposition 18. *Let X be a partition cone, $Y \subset X$ a closed union of strata, $W = Y \setminus \{\underline{0}\}$, and $\mathcal{F}$ a quasi-elementary sheaf. Then there is an exact sequence*

$$0 \longrightarrow H^0_c(Y;\mathcal{F}) \longrightarrow H^0(Y;\mathcal{F}) \longrightarrow H^0(W;\mathcal{F}) \xrightarrow{\delta} H^1_c(Y;\mathcal{F}) \longrightarrow 0, \tag{61}$$

and for every $j > 1$ the coboundary homomorphism

$$H^{j-1}(W;\mathcal{F}) \xrightarrow{\delta} H^j_c(Y;\mathcal{F}) \tag{62}$$

is isomorphic. Thus the homological invariant $H^j(L)$ of (6) is identified with $H^j_c(Y;\mathcal{F}L)$, and more generally, $H^{ij}(L)$ with $H^j_c(Y;\Lambda^i\mathcal{F}L)$ for all i and j.

Proof. The exact sequence just mentioned runs

$$\cdots \longrightarrow H^{j-1}_{\{\underline{0}\}}(Y) \longrightarrow H^{j-1}(Y) \longrightarrow H^{j-1}(Y \setminus \{\underline{0}\}) \xrightarrow{\delta} H^j_{\{\underline{0}\}}(Y) \longrightarrow \cdots \tag{63}$$

where we plug in $H^j(Y;\mathcal{F}) = 0$ for $j > 0$ from Corollary 16. Thus every third term in the sequence vanishes, and the remaining pieces are (61) and

$$0 \longrightarrow H^{j-1}(W;\mathcal{F}) \xrightarrow{\delta} H^j_c(Y;\mathcal{F}) \longrightarrow 0 \tag{64}$$

for all $j > 1$, the latter expressing isomorphy of δ.

Putting $Y = X$ and $\mathcal{F} = \Lambda^i\mathcal{F}L$ we read off the conclusion on homological invariants for $j > 1$, and also for $j = 1$ since $H^0(Y;\Lambda^i\mathcal{F}L) = 0$ apart from the trivial case $P = \emptyset$, $i = 0$. This one, as well as that of $j = 0$ are easily verified by hand. □

We now turn to tensor products. The external tensor product of two sheaves $\mathcal{F}$ on X, and $\mathcal{G}$ on Y is the sheaf $\mathcal{F}\hat{\otimes}\mathcal{G}$ on $X \times Y$ with stalks $\mathcal{F}\hat{\otimes}\mathcal{G}\,(x,y) = \mathcal{F}(x)\otimes\mathcal{G}(y)$. In case $X = Y$ we may likewise consider the internal tensor product $\mathcal{F}\otimes\mathcal{G}$ on X with stalks $\mathcal{F}\otimes\mathcal{G}\,(x) = \mathcal{F}(x)\otimes\mathcal{G}(x)$. In either case, if $\mathcal{F}$ and $\mathcal{G}$ happen to be sheaves of algebras over $\mathbb{F}$ (that is, support $\mathbb{F}$-bilinear ring multiplications) then so does their tensor product.

Proposition 19. *Let $X = X(P)$ and $Y = X(Q)$ be partition cones, and assume that $\mathcal{F}$ and $\mathcal{G}$ are quasi-elementary sheaves on X respectively Y. Then the external tensor product $\mathcal{F} \hat{\otimes} \mathcal{G}$ is a quasi-elementary sheaf on $X \times Y = X(P+Q)$. The cohomology cross product*

$$H_c^*(X;\mathcal{F}) \otimes H_c^*(Y;\mathcal{G}) \xrightarrow{\times} H_c^*(X \times Y; \mathcal{F} \hat{\otimes} \mathcal{G}) \tag{65}$$

is an isomorphism which preserves the $\mathbb{F}$-algebra structures given by the cup product.

Proof. It is clear that the (internal or external) tensor product of elementary sheaves is elementary. The analogous statement for quasi-elementary sheaves follows from this since tensor products over a field preserve exact sequences. The final and main assertion is a special case of the so-called Künneth formula, confer [18] Exercise II.18 or [19] Corollary 2.0.4. □

Directly from the definition of partition sheaves we obtain the canonical graded isomorphism

$$\Lambda^*\mathcal{F}(L \oplus M) = \Lambda^*\mathcal{F}L \hat{\otimes} \Lambda^*\mathcal{F}M \tag{66}$$

of sheaves of $\mathbb{F}$-algebras. Substituting this identity in (65) proves Theorem 3.

3.4 Duality

Let a partition cone $X = X(P)$ be given. To prepare the ground for the duality pairing of Theorem 5 we consider $X \times X$ as the partition cone of the set $P+P$, the disjoint union of two copies of P. The corresponding simplex $\Delta_{P+P} = \Delta_P * \Delta_P$ is the so-called join [17, 20] of two copies of Δ_P. Geometrically speaking, the join of two simplicial complexes Γ and Θ puts a closed interval (more precisely a 1-simplex) between each point of Γ and each point of Θ, so that $\Gamma * \Theta$ has three types of simplexes: those of type $\Delta_{S+\emptyset} \subset \Gamma$, or $\Delta_{\emptyset+T} \subset \Theta$, and thirdly simplexes $\Delta_{S+T} = \Delta_S * \Delta_T$ built of some $\Delta_S \subset \Gamma$ and some $\Delta_T \subset \Theta$ as complementary facets, see Fig. 5.

Consider the closed subspace $Z := \bigcup_{S\cap T=\emptyset} X_S \times X_T \subset X \times X$. Its topology is completely determined by the polyhedron

$$\Sigma := \Gamma_Z = Z \cap \Delta_{P+P} = Z \cap (\Delta_P * \Delta_P) \tag{67}$$

whose simplexes correspond to subsets $\emptyset \neq S+T \subset P+P$ with $S \cap T = \emptyset$.

Lemma 20. *Σ is a topological sphere of dimension $|P|-1$. If an orientation of Δ_P is chosen, say by ordering $P = \{0, 1, \ldots, l-1\}$ then Σ is oriented by its ordered faces*

$$(-1)^r(-1)^{\pi(s,t)}[s_1 \ldots s_r; t_1 \ldots t_{l-r}] \tag{68}$$

with vertex set $\{s_1, \ldots, s_r\} + \{t_1, \ldots, t_{l-r}\} \subset P+P$. Here $\pi(s,t)$ stands for the permutation that puts the entries of $(s_1, \ldots, s_r, t_1, \ldots, t_{l-r})$ in increasing order, see Fig. 6.

Proof. We argue by induction on $|P| \geq 0$. For $P = \emptyset$ clearly $\Sigma = \emptyset$ is a (-1)-sphere. For arbitrary P let $\tilde{Z} \subset \tilde{X}$ denote the subset of the partition cone $\tilde{X} = [0, \infty) \times X$ associated to the set of parties $\{p\} + P$. The simplexes of $\tilde{\Sigma} := \Gamma_{\tilde{Z}}$ are

$$\Delta_{\{p\}+\emptyset}, \quad \Delta_{(\{p\}\cup S)+T}, \quad \Delta_{S+T}, \quad \Delta_{S+(\{p\}\cup T)}, \quad \text{and} \quad \Delta_{\emptyset+\{p\}} \tag{69}$$

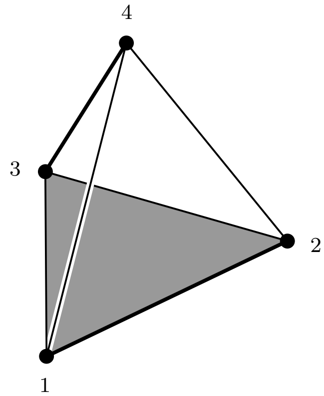

Fig. 5. The join $\Delta_{\{1,2\}} * \Delta_{\{3,4\}}$ is $\Delta_{\{1,2,3,4\}}$

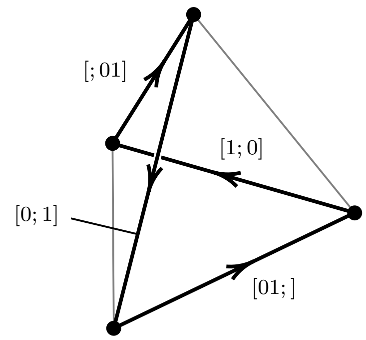

Fig. 6. The oriented sphere Σ for $P = \{0, 1\}$

with S and T subject to the condition that Δ_{S+T} be a simplex of Σ. Therefore $\tilde{\Sigma}$ is the unreduced suspension of Σ — intuitively, a double cone over Σ, see [17, 20]. Since Σ is a sphere by induction hypothesis, $\tilde{\Sigma}$ too is a sphere of the correct dimension.

The statement on orientation follows by direct calculation. □

We let

$$e\colon (X\times X)\setminus Z = \bigcup_{p\in P} \operatorname{st} X_p \times \operatorname{st} X_p \hookrightarrow X \times X \tag{70}$$

be the inclusion of the complement of Z and observe:

Corollary 21. *Assuming $P \neq \emptyset$ we have isomorphisms*

$$\begin{aligned} H^j_c(X\times X; e_!\mathbb{F}) &= H^{j-1}((X\times X)\setminus\{\underline{0}\}; e_!\mathbb{F}) = H^{j-1}(\Delta_{P+P}, \Sigma; \mathbb{F}) \\ &\simeq \begin{cases} \mathbb{F} & \text{if } j-1 = |P|, \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{71}$$

which are canonical apart from the last one which depends on the choice of an orientation for Δ_P. If $p \in P$ is an arbitrarily chosen party then this isomorphism may be realised as evaluation on the relative fundamental class

$$\Delta_p = \bigcup_{\emptyset\neq S+T \subset P+P,\ S\cap T\subset\{p\}} \Delta_{S+T} \tag{72}$$

where the top dimensional simplex $\Delta_{\{p,s_1,\ldots,s_{r-1}\}+\{p,t_1,\ldots,t_{|P|-r}\}}$ is given the orientation of $(-1)^p(-1)^{\pi(s,t)}[p\, s_1 \ldots s_{r-1}; p\, t_1 \ldots t_{|P|-r}]$.

Proof. Combine Propositions 18, 17, Lemma 20, and calculate for the fundamental class. □

Let now a collection $(G_p)_{p\in P}$ of finite dimensional vector spaces over $\mathbb{F}$ be given, as well as a bilinear form

$$\omega_p\colon G_p \otimes G_p \longrightarrow \mathbb{F} \tag{73}$$

for each $p \in P$. Thus $\omega := \sum_{p\in P} \omega_p$ is a bilinear form on $G = \bigoplus_{p\in P} G_p$, and this direct sum decomposition is orthogonal with respect to ω.

Furthermore let $L \subset G$ and $M \subset G$ be linear subspaces that are orthogonal in the sense that $\omega(L \otimes M) = 0$. Writing $\mathcal{G} = \mathcal{F}G$ we have a homomorphism of sheaves on $X \times X$

$$\mathcal{G}/\mathcal{F}L \;\hat{\otimes}\; \mathcal{F}M \xrightarrow{\hat{\omega}} e_!\mathbb{F} \tag{74}$$

which on the presheaf level, say over the open set $\operatorname{st} X_S \times \operatorname{st} X_T$ is given by

$$\begin{array}{ccccc} G_S/L_S & \otimes & M_T & \longrightarrow & \mathbb{F} \\ \overline{x} & \otimes & y & \mapsto & \omega(x,y). \end{array} \tag{75}$$

Note that $\omega(x, y)$ can be non-zero only if the supports of the vectors x and y intersect, which implies $p \in S \cap T$ for some $p \in P$ and therefore $Z \cap (\operatorname{st} X_S \times \operatorname{st} X_T) = \emptyset$: thus $\hat{\omega}$ indeed takes values in $e_!\mathbb{F}$. The bilinear pairing of sheaves $\hat{\omega}\colon \mathcal{G}/\mathcal{F}L \;\hat{\otimes}\; \mathcal{F}M \to e_!\mathbb{F}$ induces one in cohomology

$$H_c^*(X \times X; \mathcal{G}/\mathcal{F}L \,\hat{\otimes}\, \mathcal{F}M) \longrightarrow H_c^*(X \times X; e_!\mathbb{F}). \tag{76}$$

Corollary 4 shows that $H_c^*(X;\mathcal{G}) = 0$ if $|P| \geq 2$, and the exact cohomology sequence of $0 \to \mathcal{F}L \to \mathcal{G} \to \mathcal{G}/\mathcal{F}L \to 0$ thus yields an isomorphism $H_c^*(X;\mathcal{F}L) \simeq H_c^*(X;\mathcal{G}/\mathcal{F}L)$ for all non-empty P. Using Proposition 19 we obtain an isomorphism

$$H_c^*(X;\mathcal{F}L)\otimes H_c^*(X;\mathcal{F}M) \simeq H_c^*(X;\mathcal{G}/\mathcal{F}L)\otimes H_c^*(X;\mathcal{F}M) \simeq H_c^*(X\times X;\mathcal{G}/\mathcal{F}L \,\hat{\otimes}\, \mathcal{F}M) \tag{77}$$

while from Corollary 21 we have

$$H_c^*(X \times X; e_!\mathbb{F}) \simeq H^*(\Delta_{P+P}, \Sigma; e_!\mathbb{F}) \simeq H^{|P|}(\Delta_{P+P}, \Sigma; e_!\mathbb{F}) \simeq \mathbb{F}. \tag{78}$$

Composing (77), (76), and (78) we obtain an equivalent pairing

$$H_c^*(X;\mathcal{F}L) \otimes H_c^*(X;\mathcal{F}M) \longrightarrow \mathbb{F} \tag{79}$$

which we refer to as the *pairing induced by* ω and likewise write $\hat{\omega}$. By construction it is homogeneous of degree $|P| + 2$, that is, it non-trivially pairs $H_c^i(X;\mathcal{F}L)$ with $H_c^j(X;\mathcal{F}M)$ only if $i + j = |P| + 2$ — let us write $\hat{\omega}_i$ for this part as in (15).

Turning now to the proof of Theorem 5 we suppose that each ω_p is a perfect pairing: the claim is that the choice $M = L^\perp$ makes $\hat{\omega}$ perfect too. Undoing one step of the construction we shall rather work with the equivalent pairing $\hat{\omega}\colon H_c^*(X;\mathcal{G}/\mathcal{F}L) \otimes H_c^*(X;\mathcal{F}L^\perp) \to \mathbb{F}$ and switch to a slightly more general setting. Let $K \subset G$ be a second subspace, with $L \subset K$; then

$$\omega\colon K/L \otimes L^\perp/K^\perp \longrightarrow \mathbb{F} \tag{80}$$

is perfect, and $\hat{\omega}$ induces a bilinear form

$$H_c^*(X;\mathcal{F}K/\mathcal{F}L) \otimes H_c^*(X;\mathcal{F}L^\perp/\mathcal{F}K^\perp) \longrightarrow \mathbb{F} \tag{81}$$

as before; note that $K = G$ is the original case. If on the other hand we can prove that (81) is perfect under the condition $\dim K = \dim L + 1$ then the general case will follow: Given a proper subspace $L \subset G$, such a K may be picked arbitrarily, and in the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_c^*(\mathcal{F}K/\mathcal{F}L) & \longrightarrow & H_c^*(\mathcal{G}/\mathcal{F}L) & \longrightarrow & H_c^*(\mathcal{G}/\mathcal{F}K) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\hat\omega} & & \downarrow{\scriptstyle\hat\omega} & & \downarrow{\scriptstyle\hat\omega} & & \\ \cdots & \to & \operatorname{Hom}\big(H_c^*(\mathcal{F}L^\perp/\mathcal{F}K^\perp),\mathbb{F}\big) & \to & \operatorname{Hom}\big(H_c^*(\mathcal{F}L^\perp),\mathbb{F}\big) & \to & \operatorname{Hom}\big(H_c^*(\mathcal{F}K^\perp),\mathbb{F}\big) & \to & \cdots \end{array} \tag{82}$$

with exact rows we may suppose by induction that the left hand and middle vertical arrows are isomorphisms. The so-called five lemma then implies that the remaining vertical also is isomorphic. — Proposition 17 and the proof of Proposition 13 suggest the point of this reduction: $H_c^*(\mathcal{F}K/\mathcal{F}L)$ and $H_c^*(\mathcal{F}L^\perp/\mathcal{F}K^\perp)$ now have a purely topological meaning.

Topological digression. Fix a finite set $P \neq \emptyset$, and consider a proper subpolyhedron Γ of the simplex Δ_P. Then

$$\Gamma\check{} := \bigcup_{S\neq P,\ \Delta_{P\setminus S}\not\subset\Gamma} \Delta_S \tag{83}$$

is another proper subpolyhedron called the *dual polyhedron* of Γ in Δ_P, see Fig. 4 for an example. The dual is at once seen to obey the rules

- $\Gamma_1 \subset \Gamma_2 \;\Rightarrow\; \Gamma_1\check{} \supset \Gamma_2\check{}$,
- $(\Gamma_1 \cup \Gamma_2)\check{} = \Gamma_1\check{} \cap \Gamma_2\check{}$, $\quad (\Gamma_1 \cap \Gamma_2)\check{} = \Gamma_1\check{} \cup \Gamma_2\check{}$,
- $\Gamma\check{}\check{} = \Gamma$.

From Lemma 20 we already know that $\Sigma = \bigcup_{S+T\neq\emptyset,\, S\cap T=\emptyset} \Delta_S * \Delta_T \subset \Delta_P * \Delta_P$ is a simplicial sphere of dimension $|P|-1$.

Lemma 22. *Let $\Gamma \subset \Delta_P$ be a proper subpolyhedron. Then $\Sigma \subset \Gamma\check{} * \Delta_P \cup \Delta_P * \Gamma$.*

Proof. Let $S+T \subset P+P$ a non-empty subset with $S \cap T = \emptyset$. If $S = \emptyset$ or $T = \emptyset$ then $\Delta_S * \Delta_T$ certainly is a simplex of $\Gamma\check{} * \Delta_P$ respectively $\Delta_P * \Gamma$. Thus assume $S \neq \emptyset \neq T$, and furthermore that $\Delta_T \not\subset \Gamma$. But then $\emptyset \neq T \subset P\setminus S$ implies $\Delta_{P\setminus S} \not\subset \Gamma$, so that Δ_S is a simplex of $\Gamma\check{}$, and therefore $\Delta_S * \Delta_T$ one of $\Gamma\check{} * \Delta_P$. ☐

Comparing Figs. 3 and 4 one may see the lemma at work. It now allows to write down the topological duality pairing

$$\begin{aligned} H^*(\Delta_P,\Gamma\check{}) \otimes H^*(\Delta_P,\Gamma) &\xrightarrow{\times} H^*(\Delta_P\times\Delta_P, \Gamma\check{}\times\Delta_P \cup \Delta_P\times\Gamma) \\ &\xrightarrow{\delta_{\mathrm{MV}}} H^*(\Delta_P*\Delta_P, \Gamma\check{}*\Delta_P \cup \Delta_P*\Gamma) \\ &\longrightarrow H^*(\Delta_P*\Delta_P,\Sigma) \;\simeq\; \mathbb{F} \end{aligned} \tag{84}$$

where $\times$ is the cohomology cross product, and δ_{MV} indicates the Mayer-Vietoris coboundary isomorphism obtained from the decomposition

$$\Delta_P * \Delta_P = (\Delta_P * \Delta_P)\setminus(\emptyset * \Delta_P) \;\cup\; (\Delta_P * \Delta_P)\setminus(\Delta_P * \emptyset). \tag{85}$$

This duality pairing has degree $|P|$ and depends on the choice of an orientation for Δ_P as usual.

Lemma 23. *This pairing is perfect.*

Proof. This is a version of Poincaré duality, and we indicate a proof but for the sake of convenience. The statement is true if $\Gamma = \emptyset$ since in that case the pairing becomes

$$\begin{aligned} H^*(\Delta_P, \partial\Delta_P) \otimes H^*(\Delta_P) &\xrightarrow{\times} H^*(\Delta_P \times \Delta_P, \partial\Delta_P \times \Delta_P) \\ &\xrightarrow{\delta} H^*(\Delta_P * \Delta_P, \partial\Delta_P * \Delta_P \cup \Delta_P * \emptyset) \\ &\longrightarrow H^*(\Delta_P * \Delta_P, \Sigma) \simeq \mathbb{F} \end{aligned} \tag{86}$$

where each arrow is an isomorphism. The statement also holds trivially in the case where $\Gamma = \{p\}$ consists of a single vertex, since then $\Gamma^{\check{}}$ comprises all proper simplexes apart from the face opposite p, so that both Γ and $\Gamma^{\check{}}$ are contractible.

The general case now follows by induction on the number of simplexes in Γ, using Mayer-Vietoris sequences and naturality of the duality pairing. □

We return to the proof of Theorem 5, now assuming that $L \subset K$ is a subspace of codimension one. Composing the isomorphisms (62), (41), and (59) we obtain

$$H_c^*(X; \mathcal{F}K/\mathcal{F}L) \simeq H^*(W; \mathcal{F}K/\mathcal{F}L) \simeq H^*(W; e_! K/L) \simeq H^*(\Delta_P, \Gamma; K/L) \tag{87}$$

where $W = X \setminus \{\underline{0}\}$, where $e\colon \bigcup_{\emptyset \neq S \subset P,\, K \cap G_S \not\subset L} X_S \hookrightarrow W$ is the open inclusion, and the polyhedron $\Gamma = \bigcup_{\emptyset \neq S \subset P,\, K \cap G_S \subset L} \Delta_S$ is associated to the complement. We have an analogous isomorphism

$$H_c^*(X; \mathcal{F}L^\perp/\mathcal{F}K^\perp) \simeq H^*(\Delta_P, \Gamma'; L^\perp/K^\perp) \tag{88}$$

for the polyhedron $\Gamma' = \bigcup_{\emptyset \neq S \subset P,\, L^\perp \cap G_S \subset K^\perp} \Delta_S$.

Lemma 24. *For every subset $S \subset P$ the following three statements are equivalent.*

(i) $L^\perp \cap G_S \subset K^\perp$,
(ii) $K + G_{P \setminus S} = L + G_{P \setminus S}$,
(iii) $K \cap G_{P \setminus S} \not\subset L$.

In particular $\Gamma' = \Gamma^{\check{}}$ is the dual polyhedron.

Proof. (i) ⇔ (ii): property (ii) means $K \subset L + G_{P \setminus S}$, and since ω is perfect we may translate this to

$$L^\perp \cap G_S = (L + G_{P \setminus S})^\perp \subset K^\perp, \tag{89}$$

which is just (i).

(ii) ⇒ (iii): pick any $g \in K \setminus L$. By assumption we may write

$$g = f + h \quad \text{with } f \in L,\ h \in G_{P \setminus S}. \tag{90}$$

Then $g - f \in K \cap G_{P \setminus S} \setminus L$, which shows (ii).

(iii) ⇒(ii): choose any $g \in K \cap G_{P \setminus S} \setminus L$, then

$$K = L + \langle g \rangle \subset L + G_{P \setminus S}, \tag{91}$$

and (iii) follows.

We finally verify that the condition for a simplex $\Delta_S \subset \Delta_P$ to belong to $\Gamma^{\check{}}$ may be rephrased

$$\begin{array}{lclcl} \Delta_S \subset \Gamma^{\check{}} & \iff & P \backslash S \neq \emptyset \text{ and } K \cap G_{P\backslash S} \not\subset L & \iff & P \backslash S \neq \emptyset \text{ and } L^\perp \cap G_S \subset K^\perp \\ & \iff & P\backslash S \neq \emptyset \text{ and } \Delta_S \subset \Gamma' & \iff & \Delta_S \subset \Gamma', \end{array}$$

using that $\Delta_P \not\subset \Gamma'$ since $L^\perp \neq K^\perp$ anyway. ◻

We can now easily complete the proof of Theorem 5. The pairing in question is

$$H_c^*(X; \mathcal{F}K/\mathcal{F}L) \otimes H_c^*(X; \mathcal{F}L^\perp/\mathcal{F}K^\perp) \xrightarrow{\hat{\omega}} \mathbb{F} \tag{92}$$

and we have just seen how to translate sheaf cohomology to topological cohomology, which results in a pairing

$$H^*(\Delta_P, \Gamma; K/L) \otimes H^*(\Delta_P, \Gamma^{\check{}}; L^\perp/K^\perp) \longrightarrow \mathbb{F}. \tag{93}$$

This is easily seen to coincide up to a degree dependent sign with the Poincaré duality pairing, hence is perfect by Lemma 23.

In order to describe the duality pairing in terms of Čech cohomology let us return to the more general situation where the ω_p are arbitrary bilinear forms, and the linear subspaces L and M of G orthogonal with respect to ω. As before we write $\mathcal{G} = \mathcal{F}G$ and keep the notations $W = X \backslash \{\underline{0}\}$ as well as Σ of (67) and e of (70). From the diagram

$$\begin{array}{ccc}
H^*(W;\mathcal{F}L)\otimes H^*(W;\mathcal{F}M) & \xrightarrow[\simeq]{\delta\otimes\delta} & H_c^*(X;\mathcal{F}L)\otimes H_c^*(X;\mathcal{F}M) \\
{\scriptstyle \delta\otimes\mathrm{id}}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\uparrow{\scriptstyle \delta\otimes\mathrm{id}} \\
H^*(W;\mathcal{G}/\mathcal{F}L)\otimes H^*(W;\mathcal{F}M) & \xrightarrow[\simeq]{\delta\otimes\delta} & H_c^*(X;\mathcal{G}/\mathcal{F}L)\otimes H_c^*(X;\mathcal{F}M) \\
{\scriptstyle\times}\downarrow & & \\
H^*(W\times W;\mathcal{G}/\mathcal{F}L\,\hat{\otimes}\,\mathcal{F}M) & & \downarrow{\scriptstyle\times} \\
{\scriptstyle\delta_{\mathrm{MV}}}\downarrow & & \\
H^*((X\times X)\backslash\{\underline{0}\};\mathcal{G}/\mathcal{F}L\,\hat{\otimes}\,\mathcal{F}M) & \xrightarrow[\simeq]{\delta} & H_c^*(X\times X;\mathcal{G}/\mathcal{F}L\,\hat{\otimes}\,\mathcal{F}M) \\
{\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\
H^*((X\times X)\backslash\{\underline{0}\};k_!\mathbb{F}) & \xrightarrow[\simeq]{\delta} & H_c^*(X\times X;k_!\mathbb{F}) \\
\| & & \downarrow \\
H^*(\Delta_{P+P},\Sigma;\mathbb{F}) & \longrightarrow & \mathbb{F}
\end{array}$$

which commutes up to a degree dependent sign we read the following rule.

Take two cohomology classes $x \in H^i(W;\mathcal{F}L)$ and $y \in H^j(W;\mathcal{F}M)$ with $i+j=l:=|P|$, and represent them by Čech cocycles

$$(x_{s_0 s_1 \ldots s_i}) \text{ and } (y_{t_0 t_1 \ldots t_j}), \tag{94}$$

each index $s = s_0 s_1 \ldots s_i$ and $t = t_0 t_1 \ldots t_j$ representing an ordered simplex of dimension i and j respectively. Write both x and y as coboundaries $x = \delta u$ and $y = \delta v$ with values in $\mathcal{G}$, that is, explicitly:

$$\begin{aligned} x_{s_0 \ldots s_i} &= \textstyle\sum_{\alpha=0}^{i} (-1)^\alpha u_{s_0 \ldots \widehat{s_\alpha} \ldots s_i} \\ y_{t_0 \ldots t_j} &= \textstyle\sum_{\beta=0}^{j} (-1)^\beta v_{t_0 \ldots \widehat{t_\beta} \ldots t_j} \end{aligned} \tag{95}$$

We fix an arbitrary $p \in P$ and realise the isomorphism $H^l(\Delta_{P+P}, \Sigma; \mathbb{F}) \simeq \mathbb{F}$ as evaluation on the oriented polyhedron Δ_p of (72). Then up to a fixed sign the value $\hat{\omega}_i(x,y)$ of the duality pairing is

$$\sum_{\substack{S+T = P\backslash\{p\} \\ |S| = i-1 \\ |T| = j}} (-1)^p (-1)^{\pi(s,t)} \omega(u_{ps_1 \ldots s_{i-1}}, y_{pt_1 \ldots t_j}) \tag{96}$$

where it is understood that for each set S an oriented simplex $[s_1 \ldots s_{i-1}]$ with support S is chosen, and similarly for T. Substituting the expression of y as a coboundary and noting that $u_{ps_1 \ldots s_{i-1}}$ and $v_{t_1 \ldots t_j}$ have disjoint supports as vectors in $G = G_P$ we obtain

$$\sum_{S+T=P\backslash\{p\}} (-1)^p (-1)^{\pi(s,t)} \cdot \sum_{\beta=1}^{j} (-1)^\beta \omega(u_{ps_1 \ldots s_{i-1}}, v_{pt_1 \ldots \widehat{t_\beta} \ldots t_j}). \tag{97}$$

Replacing t_β by r, and T by the smaller set $T\backslash\{t_\beta\}$ we rewrite this expression as

$$\sum_{\substack{S+T \subset P\backslash\{p\} \\ |S| = i-1 \\ |T| = j-1}} (-1)^p (-1)^{\pi(s,t)} \epsilon \cdot \omega(u_{ps_1 \ldots s_{i-1}}, v_{pt_1 \ldots t_{j-1}}) \tag{98}$$

with $\epsilon = (-1)^\beta (-1)^{\pi(p,r)} (-1)^{i+r-\beta}$ if r is the unique element of $P\backslash(S+T+\{p\})$. We finally obtain the following statement which includes and refines Proposition 7.

Proposition 25. *Let $L, M \subset G$ be linear subspaces with $\omega(L \otimes M) = 0$, and consider $x \in H^{i+1}(L)$ and $y \in H^{j+1}(M)$ with $i+j=|P|$. Pick $\mathcal{G}$-valued cochains u, v with $\delta u = x$ and $\delta v = y$. Then*

$$\hat{\omega}_i(x,y) = \sum_{\substack{S+T \subset P\backslash\{p\} \\ |S| = i-1 \\ |T| = j-1}} (-1)^{p+i+r} (-1)^{\pi(p,r)} (-1)^{\pi(s,t)} \cdot \omega_p(u_{ps_1 \ldots s_{i-1}}, v_{pt_1 \ldots t_{j-1}}) \tag{99}$$

if $r \in P$ denotes the unique element not in $S+T+\{p\}$.

An essential aspect of this formula is that it treats both factors in $\hat{\omega}(x,y)$ on an equal footing. Thus $\hat{\omega}$ might as well have been defined using $\mathcal{F}L$ and $\mathcal{G}/\mathcal{F}M$ in the same way as we have used $\mathcal{G}/\mathcal{F}L$ and $\mathcal{F}M$. Apart from being interesting in its own right this fact immediately implies Theorem 8.

4 Conclusion

We have introduced homological invariants of a system of vector spaces which is partitioned over a finite set of parties, and thereby LC invariants of multi-party stabilizer states. We have investigated some basic properties of these invariants, in particular their duality. In the simplest cases we have explicitly calculated the invariants, and we have shown their connection with known results on the extraction of GHZ states from stabilizer states.

We believe that the potential of homological invariants in fact reaches much further, and wish to suggest several ways in which we believe the present work can be continued.

As mentioned in the introduction LC equivalence of stabilizer states seems to be not that much stricter than LU equivalence, the conceptually more important notion[c]. While homological invariants refer to LC equivalence by definition it is nevertheless possible that they in fact are LU invariants. The most satisfactory positive answer to that question would, of course, involve an extension of our construction from stabilizer to all quantum states.

Homological invariants do not, nor are intended to separate all the different LC classes of states. Yet it may be true, for an arbitrary number of parties, that the first order invariant $H^*(|\psi\rangle)$ vanishes *only* if $|\psi\rangle$ is decomposable. Quite generally it would be worthwhile to obtain a global view of the homological classification of states, including asymptotic information on the size of the equivalence classes for large sets of parties, or with respect to large (but finite) field extensions. For instance this might give a way to re-interpret and extend the results of [22] which show that stabilizer states from which many GHZ states can be extracted — that is, those with large $H^2(|\psi\rangle)$ — are exceptional rather than the rule.

Our main result, the duality theorem, has so far been used in quite a limited way, relating H^2 and $H^{|P|}$. It would be reasonable to expect that a more systematic application will lead to an improved qualitative understanding of multi-party entanglement of stabilizer states, completing the picture for up to four or five parties at least. Systematic use of homological invariants probably will take advantage of the fact that rather than being mere numbers they are algebraic structures with a well-defined functorial behaviour. This pertains even more to the higher order invariants $H^{ij}(L)$ — so far unused at all —which combine to form the algebra $H^*(L) = \bigoplus_{ij} H^{ij}(L)$ and thus carry a multiplication as an additional structure.

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[c] On the other hand these two notions are not identical, as has been recently shown [21].

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