

AN EXTREMAL RESULT FOR GEOMETRIES IN THE ONE-WAY MEASUREMENT MODEL

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We present an extremal result for the class of graphs G which (together with some specified sets of input and output vertices, I and O) have a certain “flow” property introduced by Danos and Kashefi for the one-way measurement model of quantum computation. The existence of a flow for a triple (G, I, O) allows a unitary embedding to be derived from any choice of measurement bases allowed in the one-way measurement model. We prove an upper bound on the number of edges that a graph G may have, in order for a triple (G, I, O) to have a flow for some $I, O \subseteq V(G)$, in terms of the number of vertices in G and O . This implies that finding a flow for a triple (G, I, O) when $|I| = |O| = k$ (corresponding to unitary transformations in the measurement model) and $|V(G)| = n$ can be performed in time $O(k^2n)$, improving the earlier known bound of $O(km)$ given in [8], where $m = |E(G)|$.

Keywords: Measurement-based quantum computing, flows

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1 Introduction

In the one-way measurement model of quantum computation [1, 2, 3, 4], algorithms are described in part by a graph G , whose edges $E(G)$ represent entanglement operations performed on pairs of qubits which are indexed by the set of vertices $V(G)$. We distinguish two (not necessarily disjoint) sets of vertices, the *input* set $I \subseteq V(G)$ and the *output* set $O \subseteq V(G)$, which represent subsystems of qubits which are used to support the input and the output of the algorithm. The qubits of I may initially be in any state $|\psi\rangle$, while the qubits of $I^c = V(G) \setminus I$ are initially prepared in the $|+\rangle$ state. We perform controlled- Z operations on pairs of qubits which are connected by an edge in G : because these all commute and are symmetric, order and orientation is unimportant. We then measure each qubit v in $O^c = V(G) \setminus O$ in some order, each with a choice of basis $\{ |+\theta_v\rangle, |-\theta_v\rangle \}$, where $|\pm\theta_v\rangle \propto |0\rangle \pm e^{i\theta_v} |1\rangle$.^b For each

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^bThese are antipodal vectors on the Bloch sphere, which lie on the XY plane. The one-way measurement model may be generalized to allow XZ plane and YZ plane measurements as well; however, we restrict ourselves here to the original model, in which only XY plane measurements are used.

measurement, if we obtain the $|-\theta_v\rangle$ outcome, the angles θ_w for qubits w which have not yet measured may undergo a change in sign. Equivalently (albeit less efficiently), we may perform some correction operation after each measurement which yields a $|-\theta_v\rangle$ outcome in order to steer the state into what the result would have been had the $|+\theta_v\rangle$ outcome been produced.

A triple (G, I, O) is called a *geometry*: each measurement-based algorithm has an underlying geometry, and two distinct algorithms may have the same geometry. A geometry captures the discrete structure of a one-way measurement algorithm: to develop a theory of discrete structures underlying quantum algorithms, one may ask whether it is possible to determine which geometries (G, I, O) underlie particular classes of operations, e.g. unitary transformations and unitary embeddings. To this end, Danos and Kashefi [5] introduced the concept of a *flow* as a sufficient condition for a geometry to underlie a unitary embedding, independent of the measurements to be performed on each qubit of O^c . (We will refer to these as *causal flows*, in order to distinguish these from e.g. network flows.)

Definition 1. Let (G, I, O) be a geometry, and \sim be the adjacency relation in G . A causal flow on a geometry (G, I, O) is an ordered pair (f, \preceq) , with a function $f : O^c \rightarrow I^c$ and a partial order \preceq on $V(G)$, such that the relations

$$x \sim f(x) \tag{1a}$$

$$x \preceq f(x) \tag{1b}$$

$$y \sim f(x) \implies x \preceq y \tag{1c}$$

hold for all vertices $x \in O^c$ and $y \in V(G)$.

Figures 1 and 2 illustrate examples of geometries with and without causal flows.

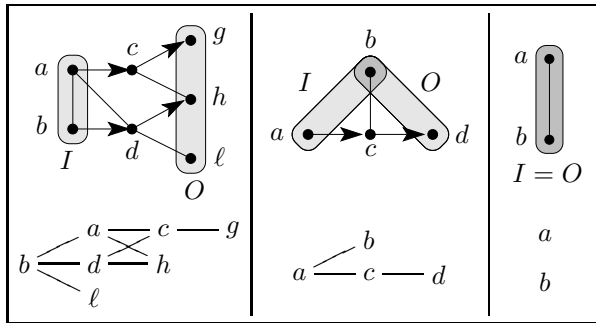


Fig. 1. Examples of geometries with causal flows. The arrows in each indicates the action of a function $f : O^c \rightarrow I^c$, along undirected edges. Compatible partial orders \preceq for each example are given by Hasse diagrams (read from left to right).

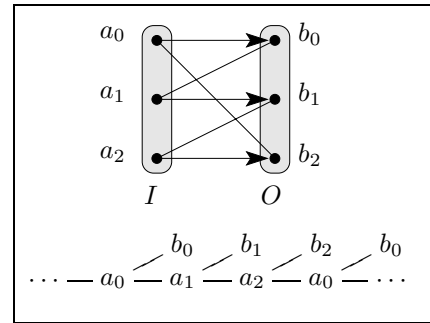


Fig. 2. A geometry with no causal flows. Illustrated here is a particular injection $f : O^c \rightarrow I^c$, and the coarsest pre-order satisfying (1b) and (1c).

For any choice of measurement operations on the qubits of O^c , a causal flow for a geometry determines an order in which the measurements may be performed, and corresponding correction operations, so that the resulting algorithm performs a unitary embedding from (the initial state of) the input subsystem I to (the final state of) the output subsystem O [5]. A causal flow then allows a partial specification of a unitary transformation in the one-way measurement model to be interpolated into a complete algorithm. The ability to find flows

for arbitrary geometries suggests that new techniques may be developed (such as suggested in [6] and [7]) to devise and analyze quantum algorithms in the one-way measurement model.

For the special case where $|I| = |O|$, corresponding e.g. to measurement patterns which implement unitary transformations, it is possible to efficiently determine whether a geometry (G, I, O) has a casual flow, and to construct one in the case that a flow exists [8]. This problem can be reduced to the Maximum Flow and Transitive Closure problems on digraphs: using standard algorithms to solve these problems, finding a causal flow for (G, I, O) when $|I| = |O| = k$ and $|E(G)| = m$ can be solved in time $O(km)$.

In this paper, we present an extremal result: in a geometry (G, I, O) which has a causal flow, with $|V(G)| = n$ and $|O| = k$, the maximum number of edges that G may have is $kn - \binom{k+1}{2}$. This allows the running time of the algorithm of [8] to be improved to $O(k^2n)$, by rejecting graphs with more than $kn - \binom{k+1}{2}$ edges as a preliminary step. It also implies that no improvement can be made in terms of a stronger upper bound on the number of edges in the geometry as a whole.

Notation and conventions.

We will represent an (undirected) edge between vertices x and y in a graph by xy , and directed edges (or *arcs*) in a directed graph (or *digraph*) by $x \rightarrow y$. Directed paths and cycles will be represented by sequences of arcs, $x \rightarrow y \rightarrow z \rightarrow \dots$. We use the convention that digraphs may contain loops on a single vertex and multiple edges between two vertices, but that graphs cannot have either.

2 Characterization of causal flows in terms of paths and digraphs

In this section, we outline the characterization of causal flows in terms of collections of paths and acyclic digraphs described in [8]. This will allow us to abstract away some details of geometries which are not essential to the analysis, and simplify the proof of the extremal result.

2.1 Vertex-disjoint paths in place of flow-functions

In a causal flow (f, \preceq) on a geometry (G, I, O) , it is easy to show that the function f must be injective: for two vertices $x, y \in O^c$, if $f(x) = f(y)$, then $x \sim f(y)$ by the relation (1a), and thus $x \preceq y$ by the relation (1c); and similarly $y \preceq x$, in which case $x = y$. Then, the orbits of the function f are a collection C of (possibly trivial) directed paths, which are vertex disjoint by the injectivity of f . More precisely, we have:

Lemma 1 ([8, Lemma 3]). *Let (f, \preceq) be a causal flow on a geometry (G, I, O) . Then there is a collection of (possibly trivial) directed paths P_1, \dots, P_k in G such that the following hold:*

- (i) *each $v \in V(G)$ is contained in exactly one path P_j ;*
- (ii) *each path P_j is either disjoint from I , or intersects I only at its initial point;*
- (iii) *each path P_j intersects O only at its final point;*
- (iv) *there is an arc $x \rightarrow y$ in some path P_j iff $y = f(x)$.*

That each path ends at a vertex in O is easy to show: because any vertex of O^c is in the domain of f , the final points of each dipaths can only be a vertex of O . Also, every vertex of O is in such a path, if only a trivial path (i.e. one of length zero, consisting of just that

vertex). Thus, there are as many paths in the collection as there are vertices of O . Note that in general, there may be more paths than there are input vertices. All of the above observations are illustrated in the examples of Figure 1, taking the arrows to represent the edges of the directed paths induced by f .

For the question of whether a geometry on n vertices and with m edges has a causal flow, we may consider any graph G with these properties, under the constraint that it admits some family of vertex-disjoint (directed) paths P_1, \dots, P_k covering the entire graph. Without loss of generality, we may then let O be the final points of those paths, and I be an arbitrary subset of the initial points of those paths. We may then consider whether the function f mapping each (non-terminal) vertex to its' successor in its' respective path is consistent with some partial order \preceq to form a causal flow (f, \preceq) .

2.2 An acyclic digraph construction for the partial order

Given a candidate function f , there is a natural choice of binary relation to consider in order to determine whether f is part of a causal flow:

Definition 2. Let (G, I, O) be a geometry, and $f : O^c \rightarrow I^c$ be an injective function. The natural pre-order^c \preceq for f is the transitive closure on $V(G)$ of the conditions

$$x \preceq x \tag{2a}$$

$$x \preceq f(x) \tag{2b}$$

$$y \sim f(x) \implies x \preceq y \tag{2c}$$

for all $x, y \in V(G)$.

By definition, the natural pre-order \preceq for a function f is the coarsest reflexive and transitive binary relation which satisfies the causal flow relations (1b) and (1c): thus, if f is a part of any causal flow, then (f, \preceq) in particular will be a causal flow.

Given a collection of vertex-disjoint dipaths P_1, \dots, P_k characterizing a function f , rather than consider the natural pre-order \preceq of f explicitly, we may characterize \preceq in terms of a digraph containing the paths P_1, \dots, P_k . Following [8], given an injective function f from such a family of paths, we may construct the *influencing digraph* \mathcal{J}_f on the vertices $V(G)$, where $x \rightarrow y$ is an arc of \mathcal{J}_f if one of $y = x$, $y = f(x)$, or $y \sim f(x)$ holds in G . (Equivalently: we have $x \rightarrow y$ an arc of \mathcal{J}_f for distinct x and y either if y is the successor of x in a path P_i , or if y is adjacent in G to the successor of x in a path P_i .) The latter two conditions correspond to the relations (2b) and (2c): then, for the natural pre-order \preceq for f , we have $x \preceq y$ whenever there is a directed path from x to y in \mathcal{J}_f .

In order for \preceq to be a partial order, it must be anti-symmetric, which is equivalent to \mathcal{J}_f having no circuits (except for circuits which repeatedly visit the same vertex by traversing the loop at that vertex). We may simplify this by considering the digraph obtained by deleting the loops from \mathcal{J}_f . Expressed in terms of families of dipaths P_1, \dots, P_k , this digraph is:

Definition 3. Let G be a graph with vertex-disjoint dipaths P_1, \dots, P_k covering $V(G)$. Then $D(G, P_1, \dots, P_k)$ is the digraph such that $x \rightarrow y$ is an arc of D if and only if either there is a path P_j which contains an arc $x \rightarrow y$, or there is a vertex $z \in V(G)$ and a path P_j such that $x \rightarrow z$ is an arc of P_j and $y \sim z$ in G .

^cA pre-order is a binary relation which is reflexive and transitive, but not necessarily antisymmetric.

Then, the injective function f induced by the family of dipaths P_1, \dots, P_k is consistent with a causal flow (again taking the inputs and outputs to be endpoints of the paths P_j) if and only if the digraph $D(G, P_1, \dots, P_k)$ is acyclic.

3 Analysis of the extremal problem

The extremal problem to be solved is: given that a geometry (G, I, O) has a causal flow, what is the maximum number of edges that G may have, given that $|V(G)| = n$ and $|O| = k$, in terms of n and k ? Using the characterizations of the previous section of causal flows, and the graphs of geometries which have causal flows, we may rephrase this problem as follows:

Problem. *Let n, k be integers where $n \geq k$. Let G be a graph on n vertices which includes k vertex-disjoint (directed) paths P_1, \dots, P_k that cover $V(G)$, and let $D(G, P_1, \dots, P_k)$ be the digraph given in Definition 3. What is the maximum number of edges $\Gamma(n, k)$ that G may have, under the constraint that $D(G, P_1, \dots, P_k)$ is acyclic?*

Theorem 1. $\Gamma(n, k) = kn - \binom{k+1}{2}$ for all integers $n \geq k \geq 1$.

In this section, we prove this Theorem by bounding the number of edges between any two paths P_i and P_j , and then provide a construction which saturates this bound.

3.1 Upper bound

To provide an upper bound on $\Gamma(n, k)$, we make the following observations. Let G and P_1, \dots, P_k be as described in the problem above, and let $D = D(G, P_1, \dots, P_k)$.

Observation 1. Consider any one of the paths $P_i = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{n_i}$. If D is acyclic, then $v_a v_b \in E(G)$ for $a < b$ if and only if $b = a + 1$. (Otherwise, D would contain the directed cycle $v_a \rightarrow v_{a+1} \rightarrow \dots \rightarrow v_{b-1} \rightarrow v_a$, contrary to hypothesis.)

Observation 2. Consider any two distinct paths $P_i = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{n_i}$ and $P_j = w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_{n_j}$. If D is acyclic, then there cannot be two edges $v_a w_b, v_c w_d \in E(G)$ where $a < c$ and $b > d$. (Otherwise, D would contain the directed cycle $v_a \rightarrow \dots \rightarrow v_{c-1} \rightarrow w_d \rightarrow \dots \rightarrow w_{b-1} \rightarrow v_a$, contrary to hypothesis.)

The first observation indicates that aside from the edges contained in the paths P_i themselves, the only edges G may have are between pairs of paths, which we will call *connecting edges*. The second observation imposes a constraint on the connecting edges that may exist between any two paths. We may use these to obtain:

Lemma 2. $\Gamma(n, k) \leq kn - \binom{k+1}{2}$ for all integers $n \geq k \geq 1$.

Proof. Consider a graph G and dipaths P_1, \dots, P_k as above, where each path P_i has n_i vertices, such that $D(G, P_1, \dots, P_k)$ is acyclic. We will proceed by bounding the number of connecting edges in G that may exist between each pair of paths P_i and P_j .

Define a function λ from the connecting edges of G to the integers as follows. For any connecting edge $v_a w_b$, where v_a is the a^{th} vertex of some path P_i and w_b be the b^{th} vertex of some path P_j , let $\lambda(v_a w_b) = a + b$. Consider two distinct connecting edges $v_a w_b, v_c w_d \in E(G)$ between the same two paths P_i and P_j . By Observation 2, if $a < c$, then $b \leq d$, and if $a > c$, then $b \geq d$. In any case, we have $\lambda(v_a w_b) = a + b \neq c + d = \lambda(v_c w_d)$: that is, each connecting edge between P_i and P_j has a different image in the function λ . Because $2 \leq \lambda(e) \leq n_i + n_j$

for a connecting edge e between P_i and P_j , there are at most $n_i + n_j - 1$ connecting edges between P_i and P_j .

Applying this to all pairs of paths P_i and P_j , the number of connecting edges in G is then bounded above by

$$\begin{aligned} \sum_{1 \leq i < j \leq k} (n_i + n_j - 1) &= \frac{1}{2} \left[\sum_{i=1}^k \sum_{j=1}^k (n_i + n_j - 1) - \sum_{i=1}^k (2n_i - 1) \right] \\ &= \frac{1}{2} \left[\sum_{i=1}^k (kn_i + n - k) - \sum_{i=1}^k (2n_i - 1) \right] \\ &= kn - n - \frac{1}{2}(k^2 - k). \end{aligned} \tag{3}$$

As the number of edges in the paths P_i themselves is $n - k$, the total number of edges G may have is at most $kn - k - \frac{1}{2}(k^2 - k) = kn - \binom{k+1}{2}$. \square .

3.2 Lower bound

Consider the following construction for any n and k . Let n_1, n_2, \dots, n_k be an integer partition of n such that $n_1 \leq n_2 \leq \dots \leq n_k$. For each $1 \leq i \leq k$, let $P_i = v_{i,1} v_{i,2} \dots v_{i,n_i}$. Define $G(n_1, \dots, n_k)$ to be the graph containing these paths, as well as the following connecting edges for each $1 \leq i < j \leq k$:

- (i) If $n_i > 1$, then for each $1 \leq a < n_i$, we include the edge $v_{i,a} v_{j,a}$;
- (ii) If $n_j > 1$, then for each $1 \leq a < n_i$, we include the edge $v_{i,a+1} v_{j,a}$;
- (iii) For each $n_i \leq a \leq n_j$, we include the edge $v_{i,n_i} v_{j,a}$.

An example of this construction for $k = 3$ and an integer partition of $n = 23$ is illustrated in Figure 3.

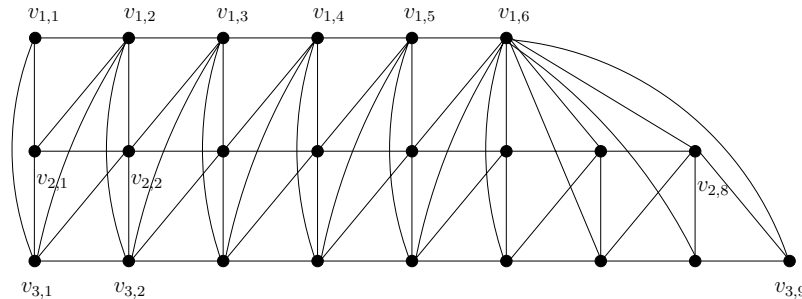


Fig. 3. The graph $G(n_1, n_2, n_3)$ for $n_1 = 6, n_2 = 8, n_3 = 9$.

We will describe (i) – (iii) as the *types* of connecting edges in $G = G(n_1, \dots, n_k)$. Each connecting edge in G induces up to two arcs in the associated digraph $D = D(G, P_1, \dots, P_k)$. For each $1 \leq i < j \leq k$, the arcs of D induced by connecting edges between the paths P_i and P_j are of six different types, labelled here from (a) to (f), which we group together by the type of the connecting edge which induce them:

- (i) $\left\{ \begin{array}{ll} \text{(a)} & v_{i,a-1} \rightarrow v_{j,a} \quad \text{for } 1 < a \leq n_i \text{ (if } n_i > 1), \text{ and} \\ \text{(b)} & v_{j,a-1} \rightarrow v_{i,a} \quad \text{for } 1 < a \leq n_i \text{ (if } n_j > 1); \end{array} \right.$
- (ii) $\left\{ \begin{array}{ll} \text{(c)} & v_{i,a} \rightarrow v_{j,a} \quad \text{for } 1 \leq a < n_i - 1 \text{ (if } n_i > 1), \text{ and} \\ \text{(d)} & v_{j,a-1} \rightarrow v_{i,a+1} \quad \text{for } 1 < a \leq n_i - 1 \text{ (if } n_i > 1); \end{array} \right.$
- (iii) $\left\{ \begin{array}{ll} \text{(e)} & v_{i,n_i-1} \rightarrow v_{j,a} \quad \text{for } n_i \leq a \leq n_j, \text{ and} \\ \text{(f)} & v_{j,a-1} \rightarrow v_{i,n_i} \quad \text{for } \max\{n_i, 2\} \leq a \leq n_j \text{ (if } n_j > 1). \end{array} \right.$

In addition to these arcs induced by connecting edges, D also contains arcs $v_{i,a} \rightarrow v_{i,a+1}$ from orienting the paths P_i themselves.

Lemma 3. *The digraph $D = D(G, P_1, \dots, P_k)$ described above is acyclic.*

Proof. The arcs in D produced by the rules (a) – (e) are either of the form $v_{i,a} \rightarrow v_{j,b}$ with $a < b$ and no constraints on i and j , or $v_{i,a} \rightarrow v_{j,a}$ with $i < j$. In either case, if an arc $v_{i,a} \rightarrow v_{j,b}$ is of one of the types (a) – (e), we have $(a, i) < (b, j)$ in the lexicographic ordering on ordered pairs of integers. The same also holds for the arcs $v_{i,a} \rightarrow v_{i,a+1}$ of the paths P_i . Then, if there are arcs in $D(G, P_1, \dots, P_k)$ for $v_{j,b} \rightarrow v_{i,a}$ where $(b, j) > (a, i)$, they must arise from the rule (f), in which case $a = n_i$.

Note that none of the rules (a) – (f) produce arcs which leave the final vertex v_{i,n_i} of any path P_i , so there are no non-trivial walks in D which leave such a vertex. Then, it is easy then to show by induction that if there is a directed walk in D between distinct vertices $v_{i,a}$ and $v_{j,b}$, either $(a, i) < (b, j)$ in the lexicographic order, or $b = n_j$.

Let $v_{i,a}$ and $v_{j,b}$ be two vertices, with a directed walk W from $v_{i,a}$ to $v_{j,b}$. Because of the existence of W , we know that $a \neq n_i$; then, there is a directed walk from $v_{j,b}$ to $v_{i,a}$ only if $(b, j) < (a, i)$. We would then have $b = n_j$, in which case there are no directed walks from $v_{j,b}$ to any other vertices in D . So, for any two distinct vertices $v_{i,a}$ and $v_{j,b}$, there cannot be a directed walk from $v_{i,a}$ to $v_{j,b}$ and also from $v_{j,b}$ to $v_{i,a}$, in which case D is acyclic \square .

As well as giving rise to an acyclic digraph $D(G, P_1, \dots, P_k)$, we also have:

Lemma 4. $|E(G(n_1, \dots, n_k))| = kn - \binom{k+1}{2}$, for any $n \geq k \geq 1$ and integer partition $n_1 \leq \dots \leq n_k$ of n .

Proof. Between any pair of paths P_i and P_j in $G(n_1, \dots, n_k)$, there are $n_i - 1$ connecting edges of type (i), $n_i - 1$ connecting edges of type (ii), and $n_j - n_i + 1$ connecting edges of type (iii). There are then $n_i + n_j - 1$ connecting edges between P_i and P_j . This saturates the upper bound for connecting edges between pairs of paths in Lemma 2: summed over all pairs of paths, and including the edges in the paths P_i , the total number of edges in $G(n_1, \dots, n_k)$ is then $kn - \binom{k+1}{2}$ \square .

Theorem 1 then follows from Lemmas 2 and 4. Together with the characterizations described in Section 2, we then have the result:

Theorem 2. *If a geometry (G, I, O) has a causal flow, then $|E(G)| \leq kn - \binom{k+1}{2}$, where $n = |V(G)|$ and $k = |O|$; and the geometry (G, I, O) given by $G = G(n_1, \dots, n_k)$, with $I = \{v_{i,1}\}_{i=1}^k$ and $O = \{v_{i,n_i}\}_{i=1}^k$, saturates this bound for any partition $n_1 \leq \dots \leq n_k$ of n .*

4 Remarks and Open Problems

This paper addresses an open problem of [9], which asked whether a construction similar to that of Section 3.2 had the maximum possible number of edges for a geometry having a causal flow, on n vertices and k output vertices.

This extremal result allows us to derive an improved upper bound on the time complexity given by [8] for recognizing geometries (G, I, O) with flows for the special case $|I| = |O|$: by adding a preliminary step where $|E(G)|$ is compared to $\Gamma(n, k)$, we can quickly eliminate geometries with too many edges, and perform the rest of the algorithm of [9] for geometries with $|E(G)| \leq \Gamma(n, k)$. This yields a running time of $O(k^2n)$ for finding causal flows (or determining that none exist) in the case $|I| = |O| = k$. As well, although there is no known efficient algorithm for determining whether a geometry (G, I, O) has a causal flow in the case $|O| > |I|$, comparing $|E(G)|$ to $\Gamma(n, k)$ provides a simple check which can show that some geometries cannot have a flow without having to analyze their structure.

Can these techniques can be generalized beyond the special case of causal flows? The presence of a causal flow is a sufficient *but not a necessary* condition for a geometry (G, I, O) to underlie a unitary embedding in the one-way measurement model [5, 10]: a more general class of geometries which underlie unitary embeddings are ones with “generalized flows”, as defined in [10]. Obtaining bounds on the number of edges in a geometry with a generalized flow (with or without imposing constraints such as, e.g. that all measurement operators are XY plane measurements) would be a step towards more general algorithms for determining when geometries underlie unitary operations in the one-way measurement model.

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