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THE COMPLEXITY OF QUANTUM SPIN SYSTEMS ON A TWO-DIMENSIONAL SQUARE LATTICE

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The problem 2-LOCAL HAMILTONIAN has been shown to be complete for the quantum computational class QMA [1]. In this paper we show that this important problem remains QMA-complete when the interactions of the 2-local Hamiltonian are between qubits on a two-dimensional (2-D) square lattice. Our results are partially derived with novel perturbation gadgets that employ mediator qubits which allow us to manipulate k-local interactions. As a side result, we obtain that quantum adiabatic computation using 2-local interactions restricted to a 2-D square lattice is equivalent to the circuit model of quantum computation. Our perturbation method also shows how any stabilizer space associated with a k-local stabilizer (for constant k) can be generated as an approximate ground-space of a 2-local Hamiltonian.

Keywords:

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1 Introduction

The novel possibilities that quantum mechanics brings to information processing have been the subject of intense study in recent years. In particular, much interest has been devoted to understanding the strengths and weaknesses of quantum computing as it pertains to important problems in computer science and physics.

An important part of this research program consists of understanding which families of quantum systems are *computationally complex*. This complexity can manifest itself in two ways. On the one hand, a positive result shows that a given family of systems is "complicated enough" to efficiently implement universal quantum computation. On the other hand, a negative result shows that certain questions about such systems are unlikely to be efficiently answerable. A proof of QMA-completeness offers compelling evidence of the negative kind while also locating the given problem in the complexity hierarchy, since QMA, –the class of decision problems that can be efficiently solved on a quantum computer with access to a quantum witness–, is analogous to the classical complexity classes NP and MA. More precisely, the class QMA is defined as

Definition 1 (QMA) *A* promise problem $L = L_{yes} \cup L_{no} \subseteq \{0,1\}^*$ *is in* QMA *if there is an efficient (of* $\mathrm{poly}(|x|)$ *size) uniform quantum circuit family* $\{V_x\}_{x\in\{0,1\}^*}$ *such that*

$$\forall\, x \in L_{yes},\ \ \exists\, |\psi_x\rangle \in \mathcal{H}^{\otimes \mathrm{poly}(|x|)},\quad \mathrm{Prob}(V_x(|\psi_x\rangle\langle\psi_x|) = 1) \geq 2/3,$$

and

$$\forall\, x \in L_{no},\ \ \forall\, |\xi\rangle \in \mathcal{H}^{\otimes \mathrm{poly}(|x|)},\quad \mathrm{Prob}(V_x(|\xi\rangle\langle\xi|) = 1) \leq 1/3.$$

The work on finding QMA-complete problems was jump-started by a 'quantum Cook-Levin Theorem' proved by Kitaev [2] (see also the survey [3]). Kitaev showed that the promise problem k-LOCAL HAMILTONIAN for $k = 5$ is QMA-complete. Before we state this problem, let us review some definitions. A Hamiltonian is a Hermitian operator. A Hamiltonian on n qubits is k-local for constant k if it can be written as $\sum_{j=1}^{r} H_j$ where each term H_j acts non-trivially on at most k qubits and thus $r \leq \mathrm{poly}(n)$. Furthermore, we require that $||H_j|| \leq \mathrm{poly}(n)$ and the entries of H_j are specified by $\mathrm{poly}(n)$ bits. The smallest eigenvalue of H, sometimes called the 'ground state energy' of H, will be denoted as $\lambda(H)$.

With these definitions in place one can define the promise problem k-LOCAL HAMILTONIAN as:

Definition 2 (k-LOCAL HAMILTONIAN) *Given is a k-local Hamiltonian H and α, β such that $\beta - \alpha \geq \frac{1}{\mathrm{poly}(n)}$. We have a promise that either $\lambda(H) \leq \alpha$ or $\lambda(H) > \beta$. The problem is to decide whether $\lambda(H) \leq \alpha$. When $\lambda(H) \leq \alpha$ we say we have a 'YES-instance'.*

Kitaev's result was strengthened in Ref. [4], which showed that 3-LOCAL HAMILTONIAN was QMA-complete. The subsequent [1] proved that also 2-LOCAL HAMILTONIAN is QMA-complete.

In another direction it was first shown by Aharonov *et al.* [5] that adiabatic quantum computation using 3-local Hamiltonians is computationally equivalent to quantum computation in the circuit model. In the adiabatic computation paradigm one starts the computation in the ground-state, i.e. the eigenstate with smallest eigenvalue, of some Hamiltonian $H(t = 0)$. The computation proceeds by slowly (at a rate at most $\mathrm{poly}(n)$) changing the parameters of the Hamiltonian $H(t)$. The adiabatic theorem (see Ref. [6] for an accessible proof thereof) states essentially that if the instantaneous Hamiltonian $H(t)$ has a sufficiently large spectral gap, – i.e. the difference between the second smallest eigenvalue and the smallest eigenvalue is $\Omega(1/\mathrm{poly}(n))$–, then the state at time t during the evolution is close to the ground-state of the instantaneous Hamiltonian $H(t)$. At the end of the computation ($t = T$), one measures the qubits in the ground-state of the final Hamiltonian $H(T)$. Ref. [1] improved on the result by Aharonov *et al.* by showing that any efficient quantum computation can be efficiently simulated by an adiabatic computation employing only 2-local Hamiltonians.

These results on the complexity of Hamiltonians can be viewed as the first (see also Ref. [7]) in a field that is still largely unexplored as compared to the classical case. The class of Hamiltonian problems is likely to be a very important class of problems in QMA. Hamiltonians govern the dynamics of quantum systems and as such contain all the physically important information about a quantum system. The problem of determining properties of the spectrum, in particular the ground state (energy) or the low-lying excitations, is a well-known problem for which a variety of methods, both numerical and analytical, (see e.g. [8, 9]) have been developed. Furthermore, finding QMA-complete problems may help us in finding new problems that are in BQP.

Let us briefly review the classical situation. In some sense the 2-LOCAL HAMILTONIAN problem is similar to the MAX-2-SAT problem [10]. But perhaps a better analogue is the set of problems defined with 'classical' Hamiltonians such as ISING SPIN GLASS:

Definition 3 (ISING SPIN GLASS) *Given is an interaction graph $G = (V, E)$ with Hamiltonian*

$$H_G = \sum_{i,j \in E} J_{ij}\, Z_i \otimes Z_j + \sum_{i \in V} \Gamma_i Z_i. \tag{1}$$

Here the couplings $J_{ij} \in \{-1, 0, 1\}$ and $\Gamma_i \in \{-1, 0, 1\}$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ is the Pauli Z operator. The problem is to decide whether $\lambda(H_G) \leq \alpha$ for a given α.

It is known that the problem ISING SPIN GLASS, which is a special case of the 2-local Hamiltonian problem, is NP-complete on a planar graph. In fact, it is even NP-complete on a planar graph when $J_{ij} = J = 1$ and $\Gamma_i = \Gamma = 1$ [11]. In this paper we prove some results on the complexity of a quantum version of this model, a quantum spin glass. Our results are based on two ideas. The first one is a small modification to the 'quantum Cook-Levin' circuit-to-5-local Hamiltonian construction that will prove QMA-completeness of a 5-local Hamiltonian on a 'spatially sparse' hypergraph (to be defined below). Such QMA-completeness result on a spatially sparse hypergraph could also have been obtained from the 6-dim particle Hamiltonian on a 2D lattice that was constructed in [5].

Secondly, we introduce a set of mediator qubit gadgets[a] to manipulate k-local interactions. These gadgets can be used to reduce any k-local interaction for constant k to a 2-local interaction. Then we use the gadgets to reduce a 2-local Hamiltonian on a spatially sparse graph to a 2-local Hamiltonian on a planar graph, or alternatively to a 2-local Hamiltonian on a 2D lattice. The general technique is based on the idea of perturbation gadgets introduced in Ref. [1]. However the gadgets that we introduce here are more general and more powerful than the one in Ref. [1].

Before we state the results, let us give a few more useful definitions. With a 2-local Hamiltonian H_G acting on n qubits we can associate an *interaction graph* $G = (V, E)$ with $|V| = n$. For every edge in $e \in E$ between vertices a and b there is a nonzero 2-local term H_e on qubits a and b such that H_e is not 1-local nor proportional to the identity operator I. We can write $H_G = \sum_{e \in E} H_e + \sum_{v \in V} H_v$ where H_v is a potential 1-local term on the vertex v. Similarly, with a k-local Hamiltonian one can associate an interaction hypergraph in which the k-local terms correspond to hyper-edges in which k vertices are involved. We also use the following definition of a spatially sparse hypergraph. A spatially sparse interaction (hyper)graph G is defined as a (hyper)graph in which (i) every vertex participates in $O(1)$ hyper-edges, (ii) there is a straight-line drawing in the plane such that every hyper-edge overlaps with $O(1)$ other hyper-edges and the surface covered by every hyper-edge is $O(1)$.

A Pauli edge of an interaction graph G is an edge between vertices a and b associated with an operator $\alpha_{ab} P_a \otimes P_b$ where P_a, P_b are Pauli matrices $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and α_{ab} is some real number. For an interaction graph in which every edge is a *Pauli edge*, the degree of a vertex is called its *Pauli degree*. For such a graph, the X- (resp. Y-, resp. Z-) degree of a vertex a is the number of edges with endpoint a for which $P_a = X$ (resp. $P_a = Y$, resp. $P_a = Z$).

We will prove the following results. First we show that

[a] These gadgets are inspired by the idea of superexchange between particles with spin. Loosely speaking, superexchange is the creation of an effective spin 'exchange' interaction due to a mediating particle, first calculated by H.A. Kramers in 1934 [12].

Theorem 4 2-LOCAL HAMILTONIAN *on a planar graph with maximum Pauli degree equal to 3 is* QMA*-complete.*

With only a little more work, we prove that

Theorem 5 2-LOCAL HAMILTONIAN *with Pauli interactions on a subgraph of the 2-D square lattice is* QMA*-complete.*

Lastly, we answer an open problem in Ref. [5] (see Section 5 for a more detailed statement of the result), namely that:

Theorem 6 *Universal quantum computation can be efficiently simulated by a quantum adiabatic evolution of qubits interacting on a 2-D square lattice.*

We believe that our Theorem 5 is in some sense the strongest result that one can expect for qubits, since we consider it unlikely that 2-LOCAL HAMILTONIAN restricted to a linear chain of qubits is QMA-complete. A recent surprising result in this respect is that 2-LOCAL HAMILTONIAN on a one-dimensional lattice with 12-dimensional *qudits* is QMA-complete [13]. With regards to Theorem 6, one should note that Aharonov *et al.* [5] had already proven that interactions of six-dimensional particles on a two-dimensional square lattice suffice for universal quantum adiabatic computation. Our improvement to qubits on a two-dimensional lattice is an application of our perturbation gadgets to [5]'s 6-dim particle construction.

We would like to draw attention to the power of the perturbative method and in particular to the gadgets that we develop in this paper. There are a variety of interesting states that can be defined as the ground-states or ground-spaces of k-local Hamiltonians. Prime examples are the stabilizer states where the Hamiltonian equals $H = I - \sum_i S_i$ and $S = \{S_i\}$ is a set of commuting stabilizer operators. The ground-space is formed by all states with $+1$ eigenvalue with respect to the stabilizer S and this space is separated by a constant gap from the rest of the spectrum. An example is the cluster state [14], the toric code space [15] or any stabilizer code space. Typically, the stabilizer operators S_i are k-local with $k > 2$ which seems to preclude the generation of such ground-space as the ground-space of a *natural* Hamiltonian, see the arguments in Ref. [16]. The perturbative gadgets introduced in this paper show how to generate a 2-local Hamiltonian which has a ground-space with is *approximately* a product of a trivial ancilla-qubit space times the ground-space of the desired k-local Hamiltonian. Thus the use of ancillas and the use of approximation get us past the constraints derived in [16]. If the original k-local Hamiltonian has some restricted spatial structure, one can show that the resulting 2-local Hamiltonian can be defined on a planar graph or, if desired, on a 2-D lattice.

In the Appendix of this paper we prove a stronger perturbation theorem than what has been shown in [1]. The results in the Appendix show that under the appropriate conditions the perturbative method does not only reproduce the eigenvalues of the target Hamiltonian, but also the eigenstates, possibly restricted to the low-lying levels of the target Hamiltonian. We believe that these results may have applications beyond reductions in QMA and the adiabatic universality results in Section 5.

This paper is organized as follows. In Section 2 we show how to modify Kitaev's original 5-local Hamiltonian construction [2] to a 5-local Hamiltonian with interactions restricted to a spatially sparse hypergraph. In Section 3 we introduce our perturbation gadgets and in Section 3.1 we show how to go from a 5-local to a 2-local Hamiltonian using our basic mediator qubit gadget. In Section 3.2 we use new variants of the basic gadget to further reduce the 2-local Hamiltonian on a spatially sparse hypergraph to a 2-local Hamiltonian

on a planar graph of Pauli degree at most 3, Theorem 4. With a bit more work we reduce it to a 2-local Hamiltonian on a 2-D square lattice, Theorem 5. Finally, Section 5 presents the proof that adiabatic quantum computation using 2-local Hamiltonians on a 2D lattice is computationally universal (Theorem 6).

2 A Spatially Sparse 5-local Hamiltonian Problem

We start by modifying the proof that 5-LOCAL HAMILTONIAN is QMA-complete in Ref. [2] (see also [3]). The essential insight is (1) to modify any quantum circuit to one in which any qubit is used a constant number of times and (2) make sure that the program to execute the gates in the correct time sequence is spatially local. We note that some of the ideas in this section are quite similar to those behind the adiabatic 2D-lattice Hamiltonian construction with 6-dim particles in Ref. [5].

Let a quantum circuit use N qubits where n qubits are input qubits and the other $N-n$ qubits are ancilla qubits. We first modify this circuit such that gates are executed in $R = \text{poly}(N)$ 'rounds' where in every round only 1 (non-trivial) gate is performed [b] After a round, the N qubits are swapped to a next row of N qubits and then the next gate in the original circuit is executed. The total number of qubits in this modified circuit is $M = RN$. The rows of N qubits for different rounds R are depicted in Fig. 1. Let us specify an order in which the swap and gate operations are executed. In the first round $R = 1$ we start by applying gates, I and the non-trivial gate, with the qubit on the left in Figure 1. After this round, the swapping starts with the qubit on the right. Then again the $R = 2$ gate-round starts with qubits on the left etc. If we label the gates (including I) with a time-index depending on when they are executed, then it is clear that in this model time changes in a spatially local fashion.

We also note that in our construction, each physical qubit enters a gate at most 3 times, twice in a swap gate, and once in a I gate or a nontrivial gate.

In the class QMA the verifier Arthur uses a verifying quantum circuit V_x for an instance x. We will use the fact that we can always replace such verifying quantum circuit by a modified verifying circuit with the properties that we derived above.

Given any instance x of a promise problem $L \in QMA$ and the verification circuit V_x, we will construct a 5-local Hamiltonian $H^{(5)}$ such that

- if on some input $|\xi, 0\rangle$ V_x accepts with probability more than $1-\epsilon$ (x is a YES-instance), then $H^{(5)}$ has an eigenvalue less than $\frac{\epsilon}{p_1(n)}$ for some polynomial $p_1(n)$.
- if V_x accepts with probability less than ϵ then all eigenvalues of $H^{(5)}$ are larger than $\frac{1-\epsilon-\sqrt{\epsilon}}{p_2(n)}$ for some polynomial $p_2(n)$.

Thus we can map each promise problem in QMA onto a 5-local Hamiltonian problem where the specific restricted form of Arthur's verifying circuit will lead to restrictions on the interactions in the 5-local Hamiltonian, that is, the interaction hypergraph will be spatially sparse. In particular, when $\epsilon = O(2^{-n})$ for a n qubit proof from Merlin, we obtain a Hamiltonian which obeys the promise in Definition 2. Note that Definition 1 uses $\epsilon = 1/3$ but it has been shown, see e.g. [17], that one can make the error $\epsilon = O(2^{-n})$ for a n qubit proof input.

[b] One could do more gates per round, but this construction is perhaps more easily explained.

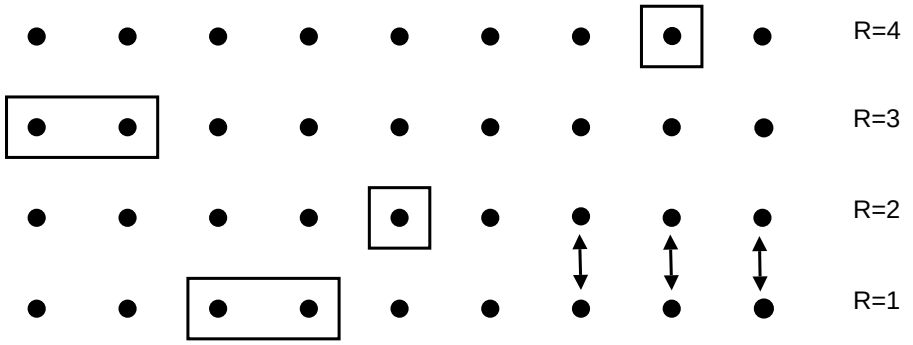

Fig. 1. Two-dimensional spatial layout of the qubits in a quantum circuit for $R = 4$. A qubit is indicated by a •. One and two-qubit gates are indicated by boxes. After the gate is executed in row R, those qubits are swapped with the qubits above them in row $R + 1$. The order in which the swap and gate operations are executed can be represented by a (time)cursor that snakes over the circuit as follows. We start with the qubit on the left in row $R = 1$. Identity gates are applied on qubits in this row except for the one non-trivial gate. We end up at the right and then start swapping the qubits in row 1 with those in row 2, starting with the qubit on the right. By doing this we end up at the left. Now we perform a round of gate-applications (going right) on the qubits in row $R = 2$. We end up at the right and go left while swapping the qubits in rows $R = 2$ and $R = 3$. We continue until all necessary gates are executed and the computational qubits are sitting in the last row.

Thus, these arguments will prove that the 5-local Hamiltonian problem on a so-called spatially sparse hypergraph is QMA-hard. Since it is also known that 5-LOCAL HAMILTONIAN is in QMA [2], this proves the QMA-completeness of 5-LOCAL HAMILTONIAN on a spatially sparse hypergraph.

Let us now look at the details of mapping a QMA circuit onto a Hamiltonian problem. Our construction is a small modification from the standard construction by Kitaev [2]. We define a set of clock-qubits. We use $T = (2R - 1)N$ clock-qubits labeled as $c_1 \ldots, c_T$. Time t will be represented as the state $|1^t 0^{T-t}\rangle_{c_1 \ldots c_T}$ as in Ref. [2]. Let $U_1 \ldots U_T$ be the sequence of operations on the computational qubits of the quantum circuit V, one operation for every clock-qubit $c_1, \ldots, c_T$. The set of operations includes the actual gates, the I operations when only time advances and the swap gates. Let $\mathrm{Q_{in}}$ be the set of n qubits that contain the input $|\xi\rangle$. Let q_{out} be the final qubit that is measured in the quantum circuit V_x. The 5-local Hamiltonian $H^{(5)}$ that we associate with this circuit is as follows. $H^{(5)} = H_{\mathrm{in}} + H_{\mathrm{out}} + H_{\mathrm{clock}} + \frac{1}{2}\sum_{t=0}^{T} H_{\mathrm{evolv}}(t)$ where

$$\begin{aligned} H_{\mathrm{in}} &= \sum_{q \notin Q_{\mathrm{in}}} |1\rangle\langle 1|_q \otimes |100\rangle\langle 100|_{c_{t_q-1}, c_{t_q}, c_{t_q+1}}, \\ H_{\mathrm{out}} &= |0\rangle\langle 0|_{q_{\mathrm{out}}} \otimes |1\rangle\langle 1|_{c_T}, \\ H_{\mathrm{clock}} &= \sum_{t=1}^{T-1} |01\rangle\langle 01|_{c_t, c_{t+1}}. \end{aligned} \tag{2}$$

and

$$\begin{aligned}
H_{\mathrm{evolv}}(1) &= |00\rangle\langle 00|_{c_1,c_2} + |10\rangle\langle 10|_{c_1,c_2} \\
&\quad -U_1 \otimes |10\rangle\langle 00|_{c_1,c_2} - U_1^\dagger \otimes |00\rangle\langle 10|_{c_1,c_2}, \\
H_{\mathrm{evolv}}(t) &= |100\rangle\langle 100|_{c_{t-1},c_t,c_{t+1}} + |110\rangle\langle 110|_{c_{t-1},c_t,c_{t+1}} \\
&\quad -U_t \otimes |110\rangle\langle 100|_{c_{t-1},c_t,c_{t+1}} - U_t^\dagger \otimes |100\rangle\langle 110|_{c_{t-1},c_t,c_{t+1}}, \quad 1 < t < T \\
H_{\mathrm{evolv}}(T) &= |10\rangle\langle 10|_{c_{T-1},c_T} + |11\rangle\langle 11|_{c_{T-1},c_T} \\
&\quad -U_T \otimes |11\rangle\langle 10|_{c_{T-1},c_T} - U_T^\dagger \otimes |10\rangle\langle 11|_{c_{T-1},c_T}. \qquad (3)
\end{aligned}$$

H_{in} is the only term that is different from the 5-local Hamiltonian considered in Ref. [2]; it uses the definition of a set of special times t_q. Before we define these times, let us look more closely at the interactions in the Hamiltonian and how the qubits can be laid out so that each qubit only interacts with a set of qubits in its neighborhood. The precise form of this neighborhood is irrelevant, we only require that the interaction hyper-graph of this Hamiltonian spatially sparse, as defined in the Introduction.

Given the lay-out of the computational (non-clock) qubits in Figure 1 we can 'drape a string' of clock qubits over the line following the sequence of computational steps. This ensures that the terms in H_{evolv} involve qubits that are in each other's local neighborhood. We can also ensure this locality property of H_{out} by choosing the output qubit q_{out} to be the last qubit on the right in the final row. Now let us consider H_{in}. For every qubit in the layout in Figure 1 there is a time in which the running cursor which snakes over the circuit first arrives at this qubit. For the qubits in $R = 1$, this is when the cursor comes from the left doing the I operations or the non-trivial gate. For the qubits in the other rows $R > 1$, it is when the cursor, coming from the right, starts swapping the qubit with the previous row $R-1$. These cursor actions are represented in H_{evolv}. For a qubit q we define the clock-qubit c_{t_q} as the clock-qubit whose bit is flipped in the interaction representing the *earliest* gate (the action of the cursor) on the qubit q in H_{evolv}. Then it is clear that the clock-qubit c_{t_q} is local to the qubit q and therefore H_{in} again represents an interaction between qubits that are in each other's local neighborhood. It is also clear that the role of H_{in} is to make sure that the state of the qubits is set to 0 before the gates actually act on these qubits. Note that we set the state of all qubits (except those in Q_{in}) to zero, also the ones in the later rows that are merely used as dummy qubits to be used in swaps. This is not absolutely necessary but merely convenient.

These arguments show that the interaction hypergraph of the Hamiltonian is spatially sparse. Note also that given a quantum circuit with N qubits one can efficiently construct the interaction hypergraph of the corresponding Hamiltonian and draw this hypergraph in the plane where hyperedges involving 5 qubits are represented as five-sided polygons.

The proof of the following Lemma is analogous to the proof of Theorem 14.3 in [2].

Lemma 1 *Let* $|\psi\rangle = \sqrt{\frac{1}{T+1}} \sum_{t=0}^{T} |\xi_t\rangle_{q_1 \ldots q_M} |1^t 0^{T-t}\rangle_{c_1 \ldots c_T}$ *where* $|\xi_t\rangle = U_t |\xi_{t-1}\rangle$ *for all* $1 \leq t \leq T$ *and* $|\xi_0\rangle = |\xi\rangle|0^{M-n}\rangle$ *for some state* $|\xi\rangle$ *of the input qubits. If Arthur's verifying quantum circuit* V_x *accepts with probability more than* $1-\epsilon$ *on some input* $|\xi, 00\ldots0\rangle$ *then* $\langle\psi|H^{(5)}|\psi\rangle < \frac{\epsilon}{T+1}$. *If* V_x *accepts with probability less than* ϵ *on all inputs* $|\xi, 0\rangle$ *then all eigenvalues of* $H^{(5)}$ *are larger than or equal to* $\frac{c(1-\epsilon-\sqrt{\epsilon})}{T^3}$ *for some constant* c.

Proof. Consider first $\langle\psi|H^{(5)}|\psi\rangle$. We only need to check that $\langle\psi|H_{\rm in}|\psi\rangle = 0$ since this term is different than the one in Ref. [2]. We note that $H_{\rm in}|\psi\rangle \propto \sum_{q\notin Q_{\rm in}} |1\rangle\langle 1|_q|\xi_{t_q-1}, 1^{t_q-1}0^{T-(t_q-1)}\rangle = 0$ since in $|\psi\rangle$ all computational qubits are set to 0 before they are being acted upon, i.e. qubit q is the state 0 at all times $t < t_q$. Thus $|\psi\rangle$ has zero eigenvalue with respect to all terms in $H^{(5)}$ except $H_{\rm out}$. If V_x accepts with probability more than $1-\epsilon$, this implies that $\langle\psi|H^{(5)}|\psi\rangle = \langle\psi|H_{\rm out}|\psi\rangle < \frac{\epsilon}{T+1}$. The second part of the proof is to show that if V_x accepts with small probability, the eigenvalues of H are bounded from below. Again the proof is identical in structure to the proof in [2] except for $H_{\rm in}$. We first note that $H^{(5)}$ preserves the space of 'legal' clock-states $\mathcal{S}$, i.e. clock-states of the form $|1^t0^{T-t}\rangle$ and thus we can consider the minimum eigenvalue problem of $H^{(5)}$ on $\mathcal{S}$ and $\mathcal{S}^\perp$ separately. On $\mathcal{S}^\perp$ this minimum eigenvalue is 1 since at least one of the constraints of $H_{\rm clock}$ is not satisfied. Now we consider $H^{(5)}|_{\mathcal{S}}$ which we can express using the definition $|t\rangle \equiv |1^t0^{T-t}\rangle$. We have $H_{\rm in}|_{\mathcal{S}} = \sum_{q\notin Q_{\rm in}} |1\rangle\langle 1|_q \otimes |t_q-1\rangle\langle t_q-1|$. As in the standard proof we perform a rotation W to a more convenient basis where $W = \sum_{t=0}^T U_t \dots U_1 \otimes |t\rangle\langle t|$. Let

$$H_2 \equiv W^\dagger H_{\rm evolv}|_{\mathcal{S}} W = I \otimes E, \tag{4}$$

where E is defined below Eq. (14.9) in [2]. Let

$$H_1 \equiv W^\dagger (H_{\rm in} + H_{\rm out})|_{\mathcal{S}} W = \sum_{q\notin Q_{\rm in}} |1\rangle\langle 1|_q \otimes |t_q-1\rangle\langle t_q-1| + U^\dagger|0\rangle\langle 0|_{q_{\rm out}} U \otimes |T\rangle\langle T|, \tag{5}$$

where $U = U_T \dots U_1$. Note that $H_{\rm in}|_{\mathcal{S}}$ is unchanged by the rotation W since there are no gates acting on a qubit q prior to the time t_q. Now we would like to use Lemma 14.4 in Ref. [2] and lower-bound the smallest eigenvalue of $H_1 + H_2$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the non-empty null-spaces of H_1 and H_2. Lemma 14.4 states that for such $H_1 \geq 0$ and $H_2 \geq 0$ we can bound $H_1 + H_2 \geq 2v\sin^2(\theta/2)$ where v is the smallest non-zero eigenvalue of H_1 and H_2 and $\cos^2\theta = \max_{\eta\in\mathcal{L}_2} \langle\eta|\mathbf{P}_{\mathcal{L}_1}|\eta\rangle$ where $\mathbf{P}_{\mathcal{L}_1}$ is the projector on $\mathcal{L}_1$. The minimum of the smallest non-zero eigenvalue of H_1 and H_2 is as in Ref. [2], namely $v \geq cT^{-2}$.

Now we show that, as in [2], one can bound $\sin^2\theta \geq \frac{1-\epsilon-\sqrt{\epsilon}}{T+1}$. Putting these results together shows that the minimum eigenvalue of $H^{(5)}$ is at least $\frac{c(1-\epsilon-\sqrt{\epsilon})}{T^3}$ for some constant c, as claimed. As in Ref. [2] any state in $\mathcal{L}_2$ is of the form $|\xi\rangle \otimes \frac{1}{\sqrt{T+1}} \sum_{t=0}^T |t\rangle$ where $|\xi\rangle$ is arbitrary. We can also write $\mathbf{P}_{\mathcal{L}_1} = \sum_{t=0}^T P_t \otimes |t\rangle\langle t|$ where $P_T = U^\dagger|1\rangle\langle 1|_{q_{\rm out}} U$, and $P_t = \Pi_{q\notin Q_{\rm in}|t_q=t+1}|0\rangle\langle 0|_q \otimes I_{{\rm else},t}$ where $I_{{\rm else},t}$ is the I operator on all computational qubits for which $t_q \neq t+1$. At some times P_t may just be I on all qubits. Here $\Pi_{q\notin Q_{\rm in}|t_q=t+1}$ is tensor product of $|0\rangle\langle 0|$ for all qubits q for which $t_q = t+1$. Thus we need to bound

$$\cos^2\theta = \frac{1}{T+1} \max_\xi \langle\xi| \sum_t P_t |\xi\rangle. \tag{6}$$

All P_t for $t < T$ commute and their common eigenspace is the space where all qubits $q \notin Q_{\rm in}$ are set to $|00\dots0\rangle$. We can write any $|\xi\rangle$ as $|\xi\rangle = \alpha|00\dots0, \psi_0\rangle + |\beta\rangle$ where ψ_0 is a state for all qubits in $Q_{\rm in}$ and $|\beta\rangle$ is a state with norm $1-|\alpha|^2$ in which at least one of the k non-input qubits is not in $|0\rangle$. Thus we have

$$\begin{aligned} \cos^2\theta \leq \frac{1}{T+1} \quad [\quad & |\alpha|^2 T + |\alpha|^2 \langle 0,\psi_0|P_T|0,\psi_0\rangle + 2|\alpha|\,|\langle 0,\psi_0|P_T|\beta\rangle| \\ + \quad & (T-1)\langle\beta|\beta\rangle + \langle\beta|P_T|\beta\rangle]. \end{aligned} \tag{7}$$

Given the acceptance probability of the circuit V_x we can bound $\langle 0, \psi_0 | P_T | 0, \psi_0 \rangle < \epsilon$. We also bound $\langle \beta | P_T | \beta \rangle \leq \langle \beta | \beta \rangle$. This gives

$$\cos^2 \theta \leq \frac{1}{T+1} \left(T + |\alpha|^2 \epsilon + 2|\alpha| \sqrt{\epsilon} \sqrt{1 - |\alpha|^2} \right) \leq 1 - \frac{1 - \epsilon - \sqrt{\epsilon}}{T+1}. \tag{8}$$

□.

3 Perturbation Theory

In this section we introduce the perturbation method. Our main new idea is the use of mediator qubits that perturbatively generate interactions. The mediator qubits are weakly coupled to the other qubits and to lowest order in the perturbation this coupling generates an interaction between the other qubits, see Section 3.1. We will show as a first step how this can be used to reduce any k-local Hamiltonian problem to a 3-local Hamiltonian problem. We can then use the perturbation gadget in [1] to reduce a 3-local to a 2-local Hamiltonian (we also sketch an alternative mediator qubit method). To reduce a 2-local Hamiltonian to a 2-local Hamiltonian on a 2D lattice or a planar graph, we need a few other applications of our mediator qubit gadgets which will be introduced in Section 3.2.

In Ref. [1] the authors reduce the problem 3-LOCAL HAMILTONIAN to 2-LOCAL HAMILTONIAN by introducing a perturbation gadget. The idea is to approximate $\lambda(H_{\text{target}})$ of a desired (3-local) Hamiltonian H_{target} by $\lambda(\tilde{H})$ of a 2-local Hamiltonian $\tilde{H}$ where $\lambda(\tilde{H})$ is calculated using perturbation theory. One sets $\tilde{H} = H + V$ where H is the 'unperturbed' Hamiltonian which has a large spectral gap Δ and V is a *small* perturbation operator. We will choose H such that it has a degenerate ground-space associated with eigenvalue 0 and the eigenvalues of the 'excited' eigenstates are at least Δ. The effect of the perturbation V is to lift the degeneracy in the ground-space and create the target Hamiltonian in this space.

More accurately, we have a Hilbert space $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ where $\mathcal{L}_-$ is the ground-space of H. Let $\Pi_\pm$ be the projectors on $\mathcal{L}_\pm$. For some operator X we define $X_{++} = \Pi_+ X \Pi_+, X_{-+} = \Pi_- X \Pi_+, X_{+-} = \Pi_+ X \Pi_-, X_{--} = \Pi_- X \Pi_-$ and $X_+ \equiv X_{++}$. In order to calculate the perturbed eigenvalues, one introduces the self-energy operator $\Sigma_-(z)$ for real-valued z

$$\Sigma_-(z) = H_- + V_{--} + V_{-+} G_+ (I_+ - V_{++} G_+)^{-1} V_{+-}, \tag{9}$$

where we can perturbatively expand

$$(I_+ - V_{++} G_+)^{-1} = I_+ + V_{++} G_+ + V_{++} G_+ V_{++} G_+ + \ldots. \tag{10}$$

Here G_+, called the unperturbed Green's function (or resolvent) in the physics literature, is defined by

$$G_+^{-1} = z I_+ - H_+. \tag{11}$$

In Ref. [1] the following theorem is proved (here we state the case where the ground-space of H has eigenvalue 0 and H has a spectral gap Δ above the ground-space):

Theorem 7 *([1]) Let $||V|| \leq \Delta/2$ where Δ is the spectral gap of H and $\lambda(H) = 0$. Let $\tilde{H}|_{<\Delta/2}$ be the restriction of $\tilde{H} = H + V$ to the space of eigenstates with eigenvalues less than*

$\Delta/2$. *Let there be an effective Hamiltonian* $H_{\rm eff}$ *with* $\mathrm{Spec}(H_{\rm eff}) \subseteq [a,b]$. *If the self-energy* $\Sigma_-(z)$ *for all* $z \in [a-\epsilon, b+\epsilon]$ *where* $a < b < \Delta/2 - \epsilon$ *for some* $\epsilon > 0$, *has the property that*

$$||\Sigma_-(z) - H_{\rm eff}|| \leq \epsilon, \tag{12}$$

then each eigenvalue $\tilde{\lambda}_j$ *of* $\tilde{H}|_{<\Delta/2}$ *is* ϵ*-close to the* j*th eigenvalue of* $H_{\rm eff}$. *In particular*

$$|\lambda(H_{\rm eff}) - \lambda(\tilde{H})| \leq \epsilon. \tag{13}$$

This theorem can be generalized to Theorem A.1 proved in the Appendix. Theorem A.1 shows that under appropriate conditions, the effective Hamiltonian is approximately identical to $\tilde{H}$ restricted to its low-lying eigenspaces. With the same technique we also prove Lemma A.1 in the Appendix which shows that the ground-space of a target Hamiltonian can be generated perturbatively (under the assumption that the target Hamiltonian has a $1/\mathrm{poly}(n)$ gap). Lemma A.1 was also proved in [1] in the special case that the ground-space is non-degenerate.

3.1 Mediator Qubit Gadgets

In the following explanation of the gadgets we will refer to $H_{\rm target}$ as the desired Hamiltonian that we want to generate perturbatively and the effective Hamiltonian is $H_{\rm eff} = H_{\rm target} \otimes |00\ldots\rangle\langle 00\ldots|$, i.e. the ancillary 'mediator' qubits are in their ground-state $|00\ldots0\rangle$.

The gadgets that we introduce below to accomplish the reduction are what we call mediator qubit gadgets and seem to be useful in general to manipulate k-local interactions. The idea is that we replace a direct interaction between two groups of $\lceil k/2 \rceil$ qubits with *indirect* interactions through a mediator qubit. In the ground-state of the unperturbed Hamiltonian H the mediator qubit is in state $|0\rangle$. The perturbation V is chosen such that interaction with the other qubits can flip the mediator qubit. The perturbative corrections to the self-energy, up to second order in the perturbation, involve the process of flipping the mediator qubit by interaction with a group of qubits a and flipping the mediator qubit back to $|0\rangle$ by a second interaction with a group of qubits b. If $a = b$ we potentially obtain some $\lceil k/2 \rceil$-local terms. For $a \neq b$ we obtain an effective k-local interaction involving groups a and b. This gadget could also be used with three or more groups of qubits (or higher dimensional quantum systems); in this case interactions would be generated between all groups of qubits. An example of such application is the Cross gadget, explained in Section 3.2.

Fig. 2. Subdivision gadget. A k-local interaction is reduced to $\lceil k/2 \rceil + 1$-local interactions using a mediator qubit vertex w. The operators A, B, X next to the edges indicate which operators correspond to the edges.

Subdivision Gadget. Assume that a k-local operator associated with (hyper)edge ab is of the form $A \otimes B$ and let $r = \max(||A||, ||B||)$. The hyper-edge ab is part of a larger

(hyper)graph and a corresponding Hamiltonian. Let all other terms in the Hamiltonian be H_{else}. We can write the Hamiltonian as

$$H_{\text{target}} = (H_{\text{else}} + A^2/2 + B^2/2) - (-A+B)^2/2 \equiv H'_{\text{else}} - (-A+B)^2/2, \tag{14}$$

so that H'_{else} contains some additional $\lceil k/2 \rceil$-local terms as compared to H_{else}. W.l.o.g. we assume that $\max(||H'_{\text{else}}||, r) \geq 1$.

The terms in the gadget Hamiltonian $\tilde{H} = H + V$ are the following

$$H = \Delta|1\rangle\langle 1|_w, \;\; V = H'_{\text{else}} + \sqrt{\Delta/2}\,(-A+B) \otimes X_w. \tag{15}$$

The operator X_w is the Pauli X operator acting on qubit w. The degenerate ground-space $\mathcal{L}_-$ of H has the mediator qubit in the state $|0\rangle$. We have the following: $H_- = 0$, $G_+(z) = \frac{|1\rangle\langle 1|_w}{z-\Delta}$, $V_{--} = H'_{\text{else}} \otimes |0\rangle\langle 0|_w$ and

$$V_{+-} = \sqrt{\Delta/2}(-A+B) \otimes |1\rangle\langle 0|_w. \tag{16}$$

Thus the self-energy $\Sigma_-(z)$ equals

$$\Sigma_-(z) = \left(H'_{\text{else}} + \frac{\Delta}{2(z-\Delta)}\,(-A+B)^2 \right) \otimes |0\rangle\langle 0|_w + O\left(\frac{||V||^3}{(z-\Delta)^2} \right). \tag{17}$$

We can expand the self-energy around $z = 0$ and identify $H_{\text{eff}} = H_{\text{target}} \otimes |0\rangle\langle 0|$. This gives

$$||\Sigma_-(z) - H_{\text{eff}}|| = O\left(\frac{|z|r^2}{\Delta^2} \right) + O\left(\frac{||V||^3}{\Delta^2} \right) + O\left(\frac{|z|||V||^3}{\Delta^3} \right). \tag{18}$$

In order for Theorem 7 to apply the following must hold: (1) for $z \in [-\|H_{\text{eff}}\| - \epsilon, \|H_{\text{eff}}\| + \epsilon]$, $\Sigma_-(z)$ should be ϵ-close to H_{eff} and (2) $||V|| \leq \Delta/2$. Let us consider how to choose Δ such that these conditions are fulfilled. We can bound $||V|| \leq ||H'_{\text{else}}|| + \sqrt{2\Delta}r \leq \sqrt{\Delta}\left(||H'_{\text{else}}|| + \sqrt{2}r\right)$. We will choose Δ such that $|z| < \Delta$. Then, using the bound on $||V||$ gives

$$||\Sigma_-(z) - H_{\text{eff}}|| \leq O\left(\frac{r^2}{\Delta} \right) + O\left(\frac{(||H'_{\text{else}}|| + \sqrt{2}r)^3}{\Delta^{1/2}} \right). \tag{19}$$

Let us choose

$$\Delta = \left(||H'_{\text{else}}|| + C_2 r\right)^6 / \epsilon^2, \tag{20}$$

for some constant $C_2 \geq \sqrt{2}$. This choice lets us bound the last term in Eq. (19) by $O(\epsilon)$. Since $\Delta^{-1} \leq \frac{\epsilon^2}{C_2 r^6}$, we can bound the first term in Eq. (19) by $O(\epsilon^2)$. Let us verify the second condition $||V|| \leq \Delta/2$ with this choice of Δ. We have indeed

$$\frac{||V||}{\Delta} \leq \frac{\epsilon}{(||H'_{\text{else}}|| + \sqrt{2}r)^2} \leq \epsilon. \tag{21}$$

Consider the conditions on $|z|$, i.e. $z \in [-\|H_{\text{eff}}\| - \epsilon, \|H_{\text{eff}}\| + \epsilon]$ and $|z| < \Delta$. Since $||H_{\text{eff}}|| \leq ||H'_{\text{else}}|| + 2r^2$, we can consider the interval $|z| \leq ||H'_{\text{else}}|| + 2r^2 + \epsilon$. For sufficiently small ϵ we have (using $\max(||H'_{\text{else}}||, r) \geq 1$)

$$\frac{|z|}{\Delta} = \frac{\epsilon^2(||H'_{\text{else}}|| + 2r^2 + \epsilon)}{(||H'_{\text{else}}|| + C_2 r)^6} \leq O(\epsilon^2) < 1. \tag{22}$$

Thus for the choice of Δ as in Eq. (19) we have $\Sigma_-(z) = H_{\rm target} \otimes |0\rangle\langle 0|_w + O(\epsilon)$. ¿From Theorem 7 it follows that $|\lambda(H_{\rm eff}) - \lambda(\tilde{H})| = O(\epsilon)$. When $||H'_{\rm else}||$, r and $1/\epsilon$ are polynomial in n (n is the number of qubits of $H_{\rm target}$), it is clear from Eq. (20), that the norm of the gadget Hamiltonian $\tilde{H}$ which uses Δ is polynomially larger than the norm of the effective Hamiltonian. This implies that the gadget can only be used a *constant* number of times *in series* if norms have to remain polynomial.

We will use this type of gadget in parallel, that is, in many places in an interaction graph at once. Let us explain how this happens in detail and argue that the local gadgets operate independently, i.e. there are no cross-gadget contributions to 2nd order in the perturbation. Let $H_{\rm target} = H_{\rm else} - \sum_{i=1}^k H^i_{\rm target}$ where $H^i_{\rm target} = (-A_i + B_i)^2/2$ for some operators A_i and B_i. $H_{\rm else}$ contains all interactions that are not generated perturbatively in addition to the compensating terms $A_i^2/2$ etc., similar as above. We introduce k mediator qubits $w_1 \dots w_k$ and choose $\tilde{H} = \sum_i H_i + V$ where $H_i = \Delta|1\rangle\langle 1|_{w_i}$ and $V = H_{\rm else} + \sqrt{\Delta/2}\sum_i (-A_i + B_i) \otimes X_{w_i}$.

The degenerate ground-space $\mathcal{L}_-$ of H has all mediator qubits $w_1 \dots w_k$ in the state $|0\rangle$. Let $h(x)$ be the Hamming weight of a bit-string $x \in \{0,1\}^k$ of the qubits $w_1 \dots w_k$. We have the following: $G_+ = \sum_{x \neq 00\dots0} \frac{|x\rangle\langle x|}{z - h(x)\Delta}$, $V_{--} = H_{\rm else} \otimes |00\dots0\rangle\langle 00\dots0|$ and

$$V_{+-} = \sqrt{\Delta/2} \sum_i (-A_i + B_i)|00\dots1_i\dots0\rangle\langle 00\dots0|, \tag{23}$$

where $|00\dots1_i\dots0\rangle$ has qubit w_i in the state $|1\rangle$. To second order in the perturbation V, there are no cross-gadget terms in $\Sigma_-(z)$. Thus the self-energy $\Sigma_-(z)$ to second order equals

$$\Sigma_-(z) = \left(H_{\rm else} + \frac{\Delta}{2(z-\Delta)} \sum_i (-A_i + B_i)^2 \right) \otimes |00\dots0\rangle\langle 00\dots0| + O\left(\frac{||V||^3}{(z-\Delta)^2} \right). \tag{24}$$

Choosing $\Delta = {\rm poly}(n)/\epsilon^2$ for some sufficiently large ${\rm poly}(n)$ gives

$$\Sigma_-(z) = H_{\rm target} \otimes |00\dots0\rangle\langle 00\dots0| + O(\epsilon). \tag{25}$$

We need to use the parallel application of this gadget twice in order to reduce the ground-state energy problem of our 5-local Hamiltonian to that of a 3-local Hamiltonian; one application results in a 4-local Hamiltonian, another one reduces it to 3. Similarly, any k-local Hamiltonian for constant k can be reduced to a 3-local Hamiltonian by these means. A 3-to-2-local reduction can be carried out using the gadget in [1]. However an alternative construction exists which we now explain.

3-TO-2-LOCAL GADGET. Assume that we have a target Hamiltonian $H_{\rm target} = A \otimes B \otimes C + H_{\rm else}$. The idea is to generate the 3-local term $A \otimes B \otimes C$ by using perturbative effects up to third order. As before one introduces a mediator qubit w whose ground-state is $|0\rangle$ for the unperturbed operator. And, as before, we have perturbations proportional to $A \otimes X_w$ and $B \otimes X_w$ which can flip the mediator qubit. We also have a perturbation V which contains a term proportional to $C \otimes |1\rangle\langle 1|_w$ which implies that there is an interaction with C *if* the mediator qubit is 'excited'. Thus, the second-order perturbative corrections give us terms proportional to $A \otimes B$ whereas third-order corrections gives us the desired $A \otimes B \otimes C$ (and some additional 2-local terms). More precisely, let $H_{\rm target} = H_{\rm else} + A \otimes B \otimes C$. Let $r = \max(||A||, ||B||, ||C||)$. We choose $H = \Delta|1\rangle\langle 1|_w$ and

$$V = H_{\rm else} + V_{\rm extra} - \Delta^{2/3} C \otimes |1\rangle\langle 1|_w + \Delta^{2/3}(-A + B) \otimes X_w/\sqrt{2} \tag{26}$$

where the additional 2-local compensating term is $V_{\text{extra}} = \Delta^{1/3}(-A+B)^2/2+(A^2+B^2)\otimes C/2$. One can show that

$$\Sigma_-(z) = [H_{\text{else}} + A \otimes B \otimes C] \otimes |0\rangle\langle 0|_w + O(|z|\Delta^{-2/3}) + O(\Delta^{-1/3}). \tag{27}$$

For sufficiently large Δ and $|z| \leq ||H_{\text{else}}|| + O(r^3) + \epsilon$ we make $\Sigma_-(z)$ sufficiently close to $H_{\text{target}} \otimes |0\rangle\langle 0|$.

The important conclusion of this section is that one can derive a 2-local Hamiltonian on a spatially sparse graph for which the ground-state energy problem is QMA-complete. The interaction graph is restricted because the perturbation gadgets preserve the spatial restrictions of the original hypergraph of the 5-local Hamiltonian.

3.2 More Mediator Qubit Gadgetry

For our next round of reductions we need to describe some different uses of the subdivision gadget acting on 2-local interactions. In the following we will assume that every edge in the interaction graph is a Pauli edge. It may thus be that the interaction graph contains other edges between the same vertices, each edge associated with a different product of Paulis. The Pauli degree of a vertex is then the number of Pauli edges that are incident on this vertex.

The Cross Gadget. For the Cross Gadget we assume that we have a graph G which, when embedded in the plane, contains two crossing edges such as in Fig. 3. Assume that the operator on edge ad is $\alpha_{ad}P_a \otimes P_d$ and on edge bc we have $\alpha_{bc}P_b \otimes P_c$. Our desired Hamiltonian is

$$H_{\text{target}} = H_{\text{else}} - (-\alpha_{ad}P_a - \alpha_{bc}P_b + P_c + P_d)^2/2. \tag{28}$$

It is clear that the last term in this Hamiltonian generates the desired crossing edges $\alpha_{ad}P_a \otimes P_d$ and $\alpha_{bc}P_b \otimes P_c$ in addition to other operators on the edges ab, bd, cd and ac. Thus H_{else} is a sum of all other operators associated with the original graph G and a set of operators on the edges around the cross, see Figure 3, which are meant to cancel the extra operators generated by the last term in H_{target}. As before we set $\tilde{H} = H + V$ with

$$H = \Delta|1\rangle\langle 1|_w, \;\; V = H_{\text{else}} + \sqrt{\Delta/2}\,(-\alpha_{ad}P_a - \alpha_{bc}P_b + P_c + P_d) \otimes X_w, \tag{29}$$

and the analysis follows as for the subdivision gadget. Note that if there are no edges ab, bd, cd, or ac in H_{target}, there will be such edges in $\tilde{H}$, as indicated in Fig. 3.

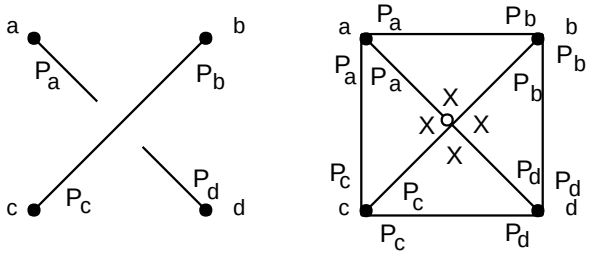

Fig. 3. Cross gadget. A crossing between two edges is removed by placing a mediator qubit in the middle. Additional edges ab, ac, bd and cd are created.

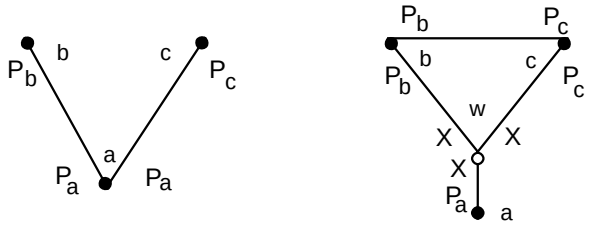

Fig. 4. Fork gadget. Two edges of the same type at vertex a are merged by the placement of a mediator qubit w. The additional edge bc is created.

The Fork Gadget. For the Fork gadget we have a subgraph as in Fig. 4 where the operator on edge ab is $\alpha_{ab}P_a \otimes P_b$ and on edge ac it is $\alpha_{ac}P_a \otimes P_c$. The Fork gadget merges the 2 edges coming from vertex a at the cost of creating an additional edge between b and c. Our desired Hamiltonian is

$$H_{\text{target}} = H_{\text{else}} - (P_a - \alpha_{ab}P_b - \alpha_{ac}P_c)^2/2, \tag{30}$$

where H_{else} contains all other terms not involving edge ab and ac. We take

$$H = \Delta|1\rangle\langle 1|_w, \;\; V = H_{\text{else}} + \sqrt{\Delta/2}\,(P_a - \alpha_{ab}P_b - \alpha_{ac}P_c) \otimes X_w, \tag{31}$$

and the analysis follows as before.

The Triangle Gadget The Fork gadget can also be used in order to reduce the degree of a vertex, see Fig. 5; this is achieved by applying the Fork gadget together with the subdivision gadget *in series*. We first apply a subdivision gadget on the edges ab and ac. Then we apply the Fork gadget on vertex a, thus generating the inner triangle in Fig. 5.

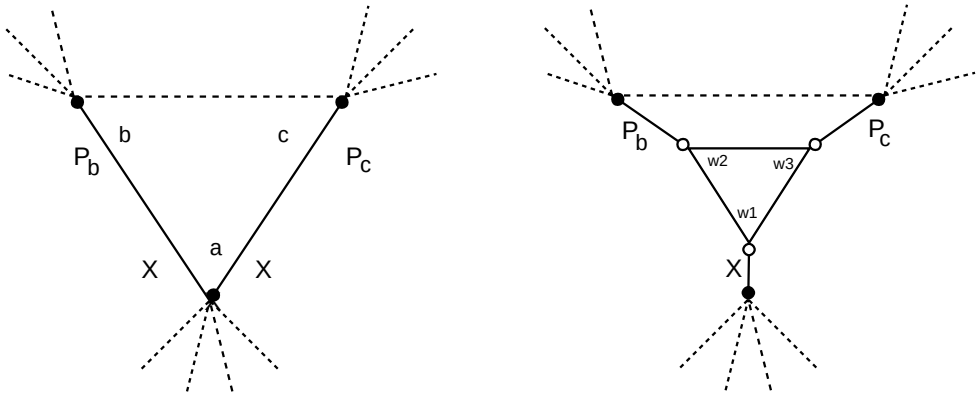

Fig. 5. Triangle Gadget. We first subdivide edges ab and ac and then apply the Fork gadget on vertex a. This give rise to a 'mediator triangle' such that vertices b and c have the same degree as before and vertex a has reduced its degree by 1.

4 2-LOCAL HAMILTONIAN on a 2-D Square Lattice

With these tools in place, we are ready to state the reduction which we obtain by applying the gadgets in the previous section. Together with our previously argued 5-local to 2-local reduction, this Lemma implies Theorem 4.

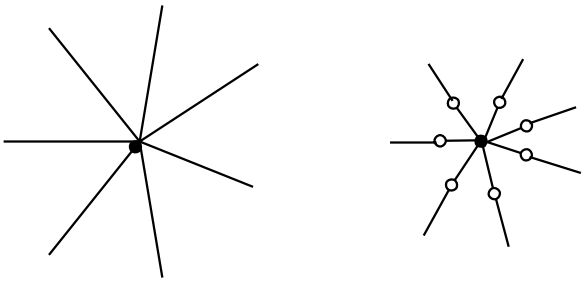

Fig. 6. Localizing a vertex.

Lemma 2 *Let H_G be a 2-local Hamiltonian related to a spatially sparse graph $G = (V, E)$ with $|V| = n$ and where $H_G = \sum_{e\in E} H_e + \sum_{v\in V} H_v$ such that $\|H_e\| \leq \text{poly}(n)$ and $\|H_v\| \leq \text{poly}(n)$. For any $\epsilon > 0$ there exists a graph G^{sim} which is planar with maximum Pauli degree at most 3 and a polynomially bounded 2-local Hamiltonian $H_{G^{\text{sim}}}$ such that*

$$|\lambda(H_G) - \lambda(H_{G^{\text{sim}}})| = O(\epsilon). \tag{32}$$

Moreover, there is a planar straight-line drawing of G^{sim} such that all edges in G^{sim} have length $O(1)$, and all angles between adjacent edges are $\Omega(1)$.

Proof.

- We use the subdivision gadget in order to localize each vertex with Pauli degree more than 3, see Fig. 6. Then we are ready to reduce the Pauli degree (which is some constant) of these vertices.

- Consider the set of vertices with Pauli degree more than 3. We are going to apply the Triangle gadget to all these high degree vertices in the following way. We first apply the subdivision gadget to all edges that we intend to merge using the Fork gadget; we can do this in one parallel application. We do this so that the triangle gadgets that we will apply in parallel never act on the same edges. Then, for a vertex with X-degree d_x, Y-degree d_y, Z-degree d_z we do the following. We pair the X-edges and apply to each pairing a Fork gadget. This means we have reduced the X-degree to $\lceil d_x/2 \rceil$. In parallel we pair the Y-edges and the Z-edges using the Fork gadget, halving their degrees. We do this single perturbative step in parallel for all high-degree vertices in the graph. We repeat this Triangle gadget process $O(1)$ number of times (since the maximum degree initially was $O(1)$) until the total Pauli degree of every vertex is at most 3. Since the initial degree of every vertex was $O(1)$, the number of additional crossings that we generate per edge is constant.

- Next, we reduce the number of crossings per edge, by subdividing each edge a constant number of times, see Fig. 7. Every subdivision is done in parallel on all edges of the graph that need subdividing.

- Then we use the subdivision gadget to *localize* each crossing, see Fig. 8. We apply the subdivision gadget in parallel on every crossing in the graph and we repeat the process 4 times so that for all crossing edges ab, cd, the quadrilateral $acbd$ contains only these points and the crossing edges.

- We apply the Cross gadget, see Fig. 3, in parallel to every localized crossing in order to remove the crossing. Note that due to the localization step the cross-gadget only involves mediator qubit vertices with degree at most 2. Thus the cross-gadget generates additional 2-local terms around the square, but the total Pauli degree of the resulting vertices is at most 4. Note that these vertices with degree 4 are all mediator qubits which only have non-zero X-degree (and zero Y- and Z-degree).

- On all mediator qubits with X-degree 4 we apply the Triangle gadget reducing the degree to 3. Since the triangle gadget generates mediator qubits with X-degree 3 we cannot do any further reductions.

Thus in this final Hamiltonian there are no vertices with Pauli degree more than 3 and the graph is planar. Theorem 7 is used in every gadget application to give the final result, Eq. (32). Note that by this reduction all original system qubits have Pauli degree at most 3 by having X-degree, Y-degree and Z-degree ranging from 0 to 1. The mediator qubits have X-degree ranging from 2 to 3 and 0 Y- and Z-degree. □.

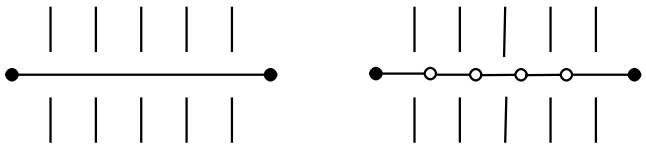

Fig. 7. An edge that crosses C other edges is subdivided $\lceil \log C \rceil$ times by inserting a mediator qubit.

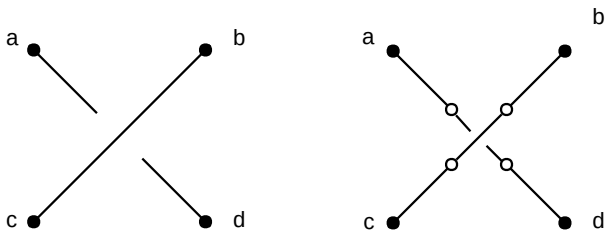

Fig. 8. Localizing a crossing by applying the subdivision gadget four times.

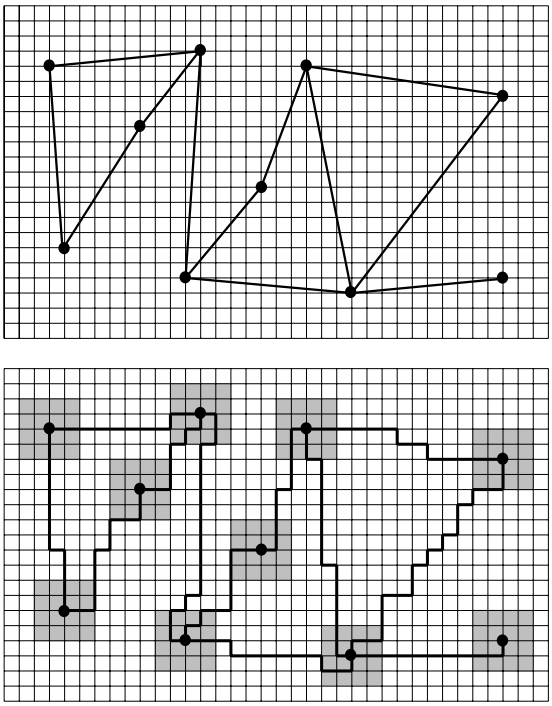

Fig. 9. A planar graph of maximal degree ≤ 3 and its representation in the lattice. In the gray squares, the paths are rerouted to avoid crossings.

4.1 Representation on a 2-D Square Lattice

Any planar graph $G = (V, E)$ with maximal degree 3 in which the (straight-line) edges have length $O(1)$ and adjacent edges form an angle of $\Omega(1)$ can be *represented* on a planar square lattice in the following sense: each vertex a of G is mapped to some lattice site $\phi(a)$ inside the square $[-O(|V|), O(|V|)]^2$, and each edge ab of G is mapped to a lattice path $\phi(ab)$ of length $O(1)$ from $\phi(a)$ to $\phi(b)$ that does not cross any other vertices or any other path. To see this, one can look at Fig. 9 or follow these steps: draw a fine square grid on the plane. If the spacing between points on the grid is small enough, moving each vertex a of G to a vertex in the lattice (and redrawing the edges) still leaves the graph planar, with $O(1)$-length edges and $\Omega(1)$ angles. Now for each edge, draw a lattice path that stays close to the edge. If the grid is fine enough, these paths can never cross outside an $O(1)$-size square (indicated in grey in Fig. 9) around the vertices of the graph, because of the angle condition. By further refining the grid if necessary, one can reroute each of the paths stemming out of a vertex a inside of a's square, so that no two different paths

collide. It is easy to see that we only need the grid to have spacing $\Omega(1)$, and that all the other conditions above are satisfied.

Clearly, this embedding can be found efficiently, given the adequate embedding of G. If H is a Hamiltonian that has G as (Pauli) interaction graph, one can use the subdivision gadget $O(1)$ times in parallel to map each edge ab to a path of the same length as $\phi(ab)$. The

Hamiltonian $\tilde{H}$ thus obtained has interaction graph $\phi(G)$ and $\lambda(\tilde{H})$ is $O(\epsilon)$-close to $\lambda(H)$. These arguments together with our previous results and Lemma 2 prove Theorem 5.

5 Universal Quantum Adiabatic Computation

In Ref. [1] the authors show that their perturbation-theoretic reduction of 3-LOCAL HAMILTONIAN to 2-LOCAL HAMILTONIAN also reduces 3-local adiabatic computations to 2-local ones. The goal of this Section is to show that an analogous result can be carried out in the present context, namely that 2-local Hamiltonians with nearest-neighbor interactions on qubits on a 2D lattice suffice for universal adiabatic quantum computation.

Let us describe in more detail what our goal is. We will construct a (classically) poly-time computable map Φ that takes as input a classical description $\langle Q\rangle$ of a quantum circuit Q and outputs a description of an adiabatic quantum computation on a 2D lattice. Suppose Q acts on n qubits and has T gates. Then

$$\Phi(\langle Q\rangle) = (\langle H_0\rangle, \ldots, \langle H_p\rangle).$$

Here

1. $p \in \mathbb{N}$ is a constant independent of Q;
2. N, the number of qubits on which H_i acts is $\mathrm{poly}(n, T)$;
3. for each $i \in \{0, \ldots, p\}$, $\langle H_i\rangle$ describes a 2-local nearest-neighbor Hamiltonian on qubits, acting on the same subset of $N = \mathrm{poly}(n, T)$ sites of the square lattice;
4. $\|H_i\| = \mathrm{poly}(n, T)$ for all $i \in \{0, \ldots, p\}$;
5. Let
$$H(s) = \sum_{i=0}^{p} s^i H_i \, (s \in [0, 1]).$$
The spectral gap between the ground-state and first excited state of $H(s)$ is $1/\mathrm{poly}(n, T)$ for all s.
6. The ground-state of $H(0)$ is $|0\rangle^{\otimes N}$ and the ground-state of $H(1)$ encodes the result of the computation of Q on input $|0\rangle^{\otimes n}$ (a more precise description is given in [1, Section 7] or [5]).

Of course, all occurrences of poly above correspond to fixed polynomials that do not depend on Q. Notice that for any Hamiltonian satisfying the above conditions one has that

$$\sup_{s\in[0,1]} \left\| \frac{d^j H(s)}{ds^j} \right\| \leq \mathrm{poly}(n, T), \text{ for all } j = 0, 1, \ldots$$

This is sufficient to ensure that adiabatic computation implemented by $H(s)$, starting from $|0\rangle^{\otimes N}$, appropriately simulates the quantum circuit Q, that is, in polynomial time [6]. Note that the usual adiabatic computation, e.g. the universal adiabatic computation in [5], has $p = 1$.

There are several ways to map a circuit Q to a corresponding $H(s)$. One could modify the 5-local construction in this paper in order to show that one can do universal quantum computation using a quantum adiabatic computation with a 5-local Hamiltonian on a spatially sparse graph. Then we could apply the perturbation gadgets to derive a 2-local Hamiltonian with similar properties. However, an easier route to the desired result is the following. In [5] it was shown how to map a circuit Q to a corresponding Hamiltonian $H^{(6)}(s)$ on a 2D lattice. That construction satisfies all but one of the above requirements, as it acts on 6-dimensional qudits rather than qubits. However, we can embed 6-dimensional qudits in states of 3 qubits. This implies that the 2-local interactions between these particles will be mapped onto 6-local interactions. Then we can apply the perturbation gadgets to 'massage' this Hamiltonian on a spatially sparse hypergraph to a 2-local Hamiltonian as we have done in our QMA construction.

Let us first review the 6-dim particle Hamiltonian, see Sec. 4.2 in [5]. The four phases of the particles, the unborn, the first, second and dead phase, can be described by two qubits in the states $|\text{unborn}\rangle = |00\rangle$,$|\text{first}\rangle = |01\rangle$, $|\text{second}\rangle = |10\rangle$ and $|\text{dead}\rangle = |11\rangle$. The third qubit holds the actual computational degree of freedom. In the 6-dimensional representation of the unborn and dead phase the computational degree of freedom is assumed to be fixed. If we represent those states as 3-qubit states, we fix the third qubit to be in the state $|0\rangle$. Hence we obtain 6 states: 2 'first' states $|010\rangle, |011\rangle$, two 'second' states $|100\rangle, |101\rangle$ and one unborn state $|000\rangle$ and one dead state $|110\rangle$. With this mapping the entire Hamiltonian in Sec. 4.2 in [5] can be rewritten in terms of 6-local interaction between qubits. Since we embed the 6-dim particle Hamiltonian in a higher dimensional space, we need to make sure that states outside the embedded space (i.e. $|001\rangle$ and $|111\rangle$) are penalized in the Hamiltonian, i.e. do not contribute to the ground-space. In Table 1 in [5] a list of forbidden configurations is given. In this list we can replace every unborn state by two unborn states $|\text{unborn}, 1\rangle$ and $|\text{unborn}, 0\rangle$ and similarly for the dead states. This implies a small modification of H''_{clock}. As a consequence we get that the space of legal shapes $\mathcal{S}$ is the same for this embedded Hamiltonian as for the original 6-dim particle Hamiltonian. It then follows that one can apply Lemma 4.6 and 4.7 bounding the spectral gap of the Hamiltonian in the space of legal states. We note in passing the Hamiltonian $H^{(6)}(s)$ has only linear terms in s (that correspond to $p = 1$ above) and terms independent of s (corresponding to $p = 0$).

Our second step is to analyze how this desired 6-local Hamiltonian can be implemented using a 2-local Hamiltonian on a 2D lattice. It is clear that one can apply the perturbation gadgets in Section 3 and 4 of this paper and map a 6-local Hamiltonian on a spatially sparse hypergraph onto a 2-local Hamiltonian on a subgraph of the 2D lattice. To go from a 3-local to a 2-local Hamiltonian we will use our alternative 3-to-2-local gadget described in Section 3.1. One needs to show the following properties of the perturbation method in order for these reductions to work:

1. The 2-local adiabatic path Hamiltonian $H^{(2)}(s)$ obtained through the perturbation gadgets simulates the 6-local adiabatic path Hamiltonian. This implies that the ground-state of the 2-local Hamiltonian should be approximately the ground-state of the desired 6-local Hamiltonian and the gap for the 2-local Hamiltonian is approximately the gap of the 6-local Hamiltonian. This requires showing that the perturbative method that

we employ does not only reproduce the lowest-eigenvalue but also the ground-state and the gap above the ground-state.

2. One needs to verify that $H^{(2)}(s)$ is of the form $\sum_{i=0}^{p} s^i H_i$, with p constant and $\max_i \|H_i\| \leq \text{poly}(n, T)$.

Our 2-local simulator Hamiltonian $H^{(2)}(s)$ is determined by applying the perturbative gadgets in Sections 3.1 and 3.2, on the 6-local Hamiltonian $H^{(6)}(s)$. In [1] it was shown how to generate, not only the lowest eigenvalues, but also the ground-state with the perturbative technique. This implies that both the ground-state of the target Hamiltonian as well as the gap above this ground-state can be generated perturbatively. Since the total number of applications of the perturbation theory is constant, one can apply this argument for each step and thus show that the 6-local target Hamiltonian can be effectively generated by a simulator Hamiltonian $H^{(2)}$.

We now fulfill our second task, i.e. we show that $H^{(2)}(s) = \sum_{i=0}^{p} s^i H_i$, with p constant and $\|H_i\|$ polynomially bounded. In [1] such arguments were developed for the 3-to-2 local perturbation gadget and basically identical arguments can be given here. The original Hamiltonian $H^{(6)}$ is at most linear in s. If a gadget is applied on a term which is linear in s, for example a 6-local term such $sA \otimes B = A(s) \otimes B$, we obtain a new Hamiltonian of which the terms are at most quadratic in s. Similarly each application of the perturbation gadgets takes a Hamiltonian $H'(s) = \sum_{j=0}^{p'} s^j H'_j$ to another Hamiltonian $H''(s) = \sum_{i=0}^{p''} s^i H''_i$ where $p'' \leq 2p'$. Assuming that the norm of each H'_i is polynomial in n and T, then the norms of each H''_j are also $\text{poly}(n, T)$. Thus the final Hamiltonian $H^{(2)}$, obtained after a constant number of gadget applications, is indeed of the desired form.

6 Discussion and Acknowledgements

The drawback of the reductions performed by our perturbation theory method is that the 2-local Hamiltonian that we construct has large variability in the norms of the 2-local terms. In other words, 2-local terms have constant norm whereas others can be fairly high degree polynomials in n. Such dependence on n may be undesirable from a practical point of view, e.g. if one wants to perform universal adiabatic quantum computation.

It is possible that a less stringent but still rigorous perturbation theory could be developed in which only the expectation values of *local* observables with respect to the ground-space are perturbatively generated. If such expectation values are reproduced with constant accuracy (not scaling as $1/\text{poly}(n)$), then the perturbation theory need not be accurately reproduce the entire ground-space as in Lemma 3. For adiabatic quantum computation this method would suffice since one can measure a single output qubit to extract the answer of the computation.

One of the reasons why finding QMA-complete problems is of interest is that it may give us a hint at what problems can be solved in BQP. One example is the unresolved status of the 2-local Hamiltonian problem on qubits in one dimension. Another example is to find a quantum extension of classical 2-local Hamiltonian problems which can be solved efficiently. We thank David DiVincenzo for an inspiring discussion about superexchange. We would like to thank Sergey Bravyi for pointing out an improvement in the proof of Lemma 2. We acknowledge support by the NSA and the ARDA through ARO contract number W911NF-04-C-0098.

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Appendix A General Perturbation Theorem

In order to give a more complete background in the perturbation method we will prove in Theorem A.1 that under the right conditions the entire operator $\tilde{H}|_{<\lambda_*}$ is approximated by H_{eff}, not only its eigenvalues. In Ref. [1] a similar result was proven, namely that the ground-state of $\tilde{H}$ is approximately the ground-state of H_{eff}. We extend their result to the case when the ground-space is degenerate in Lemma A.1 of this Appendix. To a certain extent our proof-technique is similar to the one used in Ref. [1], however we will use complex z and contour integration in parts of the proofs.

Some of our notation has been given in Section 3 for the specific cases considered in this paper. Here we consider the more general setting as defined in Ref. [1].

Assume that H and V are operators acting on the Hilbert space $\mathcal{L}$ and $\tilde{H} = H + V$. H has a spectral gap Δ such that no eigenvalues lie in the interval $[\lambda_* - \Delta/2, \lambda_* + \Delta/2]$ for some

cutoff λ_*. Let $\mathcal{L}_-$ (resp. $\mathcal{L}_+$) be the span of all eigenvectors of H whose eigenvalues are less than λ_* (respectively larger than λ_*). We will use the *resolvent* $G(z) \equiv (zI - H)^{-1}$ of H with *complex* $z \in \mathbb{C}$ and let $\tilde{G}(z) = (zI - \tilde{H})^{-1}$ be the resolvent of $\tilde{H}$. The definition of the self-energy $\Sigma_-(z)$ is given by

$$\Sigma_-(z) = zI_- - \tilde{G}_{--}^{-1}(z). \tag{A.1}$$

see also Eqs. (9)-(10).

The perturbation theory result of Kempe *et al.* states that under suitable technical conditions, –namely if $\Sigma_-(z)$ is close to a fixed operator H_{eff} for all z in some range–, all eigenvalues of $\tilde{H} = H + V$ that lie below the cutoff λ_* are close to those of H_{eff}. Our result shows that the *entire operator* $\tilde{H}$ restricted to its low-lying energy levels is close to H_{eff} under a slightly stronger assumption.

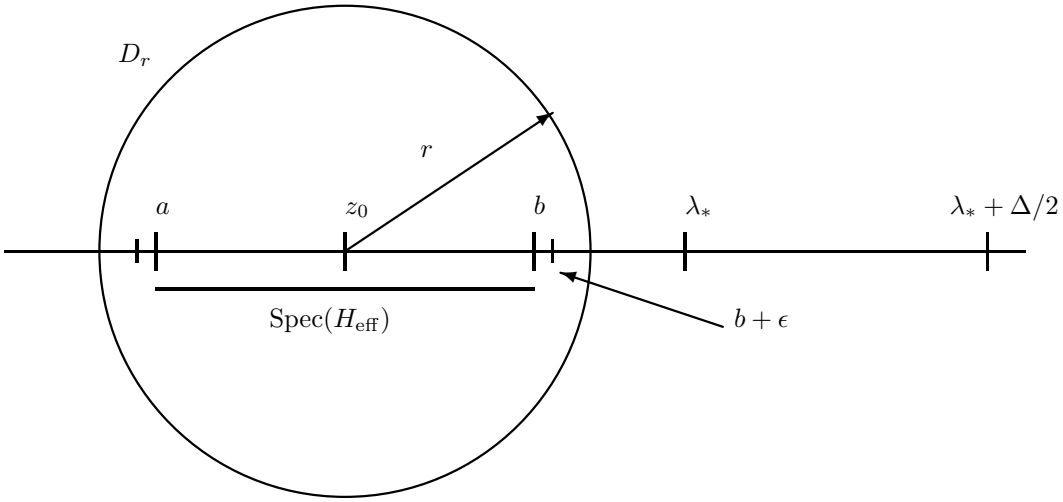

Fig. A.1. The disk D_r in the complex plane, the spectrum of H_{eff} and the other parameters in Theorem A.2.

Theorem A.1 *Given is a Hamiltonian H such that no eigenvalues of H lie between $\lambda_- = \lambda_* - \Delta/2$ and $\lambda_+ = \lambda_* + \Delta/2$. Let $\tilde{H} = H + V$ where $||V|| \leq \Delta/2$. Let there be an effective Hamiltonian H_{eff} with $\text{Spec}(H_{\text{eff}}) \subseteq [a, b]$, $a < b$. We assume that $H_{\text{eff}} = \Pi_- H_{\text{eff}} \Pi_-$. Let D_r be a disk of radius r in the complex plane centered around $z_0 = \frac{b+a}{2}$. Let r be such that $b + \epsilon < z_0 + r < \lambda_*$ (see Figure A.1). Let $w_{\text{eff}} = \frac{b-a}{2}$. Assume that for all $z \in D_r$ we have $\|\Sigma_-(z) - H_{\text{eff}}\| \leq \epsilon$. Then*

$$\|\tilde{H}_{<\lambda_*} - H_{\text{eff}}\| \leq \frac{3(||H_{\text{eff}}|| + \epsilon)\|V\|}{\lambda_+ - ||H_{\text{eff}}|| - \epsilon} + \frac{r(r + z_0)\epsilon}{(r - w_{\text{eff}})(r - w_{\text{eff}} - \epsilon)}. \tag{A.2}$$

Before we prove the theorem, let us make a few comments about how it can be applied. We have assumed that H_{eff} has no support in $\mathcal{L}_+$; this will be the case in typical applications since H_{eff} approximates $\Sigma_-(z)$ which has support only on $\mathcal{L}_-$. It is not hard to modify the theorem if H_{eff} has (necessarily small) support outside $\mathcal{L}_-$.

The r.h.s in Eq. (A.2) contains the energy scale $||H_{\rm eff}||$ which is not invariant under shifts by αI. In applying the theorem to a Hamiltonian $\tilde{H}$ one can always shift this Hamiltonian $\tilde{H}$ by αI, without changing its eigenvalues or eigenvectors, such that $H_{\rm eff}$ has a spectrum centered around 0. In that case $||H_{\rm eff}|| = \min_\alpha ||H_{\rm eff} + \alpha I|| = w_{\rm eff}$, the effective width. Thus one may replace $||H_{\rm eff}||$ by $w_{\rm eff}$ in the application of the Theorem.

In the construction using mediator qubits, we will choose $\lambda_- = 0$ and thus $\lambda_* = \Delta/2$. $\mathcal{L}_-$ is the space in which the mediator qubits are in the state $|00\ldots0\rangle$ and $H_{\rm eff}$ is of the form $H_{\rm target} \otimes |00\ldots0\rangle\langle 00\ldots0|$. In order for the right-hand-side of Eq. (A.2) to be small, we need to take the spectral gap Δ to be sufficiently large (some poly(n)). This will directly bound the first term on the right hand side. Now consider the second term and the choice for r. In our applications $H_{\rm eff}$ is derived from the perturbative expansion of $\Sigma_-(z)$. Since $\tilde{H}$ and H on n qubits have norm poly(n), the Hamiltonian $H_{\rm eff}$ (related to the target Hamiltonian) will also have norm poly(n). Hence a, b and thus z_0 are at most poly(n). Note that we need to take $z_0 + r > b + \epsilon$ which implies that $\Sigma_-(z)$ has to be approximately equal to $H_{\rm eff}$ in a range of z which is larger than what is needed in Theorem 7. Secondly, it is necessary that the eigenvalues of $\tilde{H}$ are bounded away from λ_*, the difference between the largest eigenvalue below λ_* and the smallest eigenvalue above λ_* needs to be at least $1/{\rm poly}(n)$. For our mediator qubit gadgets, one could take r (for example) to scale as $\Delta^{1/k}$ for some constant $k > 1$ in order for these conditions to be fulfilled.

Proof. (of Theorem A.1) We start from Theorem 3 in Ref. [1] (stated as Theorem A.1 in this paper) which shows that under the assumptions in the Theorem one has $|\lambda_j(\tilde{H}_{<\lambda_*}) - \lambda_j(H_{\rm eff})| \leq \epsilon$ for each $1 \leq j \leq \dim(\mathcal{L}_-)$. We can draw a contour C in the complex plane, the disk D_r in Figure A.1, that encloses all the eigenvalues of $\tilde{H}_{<\lambda_*}$ and none of the higher eigenvalues of $\tilde{H}$. The radius r needs to be chosen such that $b + \epsilon < z_0 + r$ to include all the eigenvalues of $H_{<\lambda_*}$. At the same time $z_0 + r < \lambda_*$ such that none of the higher eigenvalues of $\tilde{H}$ are included in the contour integral. Using Cauchy's contour integral formula we can write

$$\tilde{H}_{<\lambda_*} = \frac{1}{2\pi i} \oint_C z\, \tilde{G}(z)\, dz. \tag{A.3}$$

The remainder of our proof proceeds in two parts. In the first part we show that $\tilde{H}_{<\lambda_*}$ is close to $\Pi_- \tilde{H}_{<\lambda_*} \Pi_-$; this is expressed in Eq. (A.8). In the second part we show that $\Pi_- \tilde{H}_{<\lambda_*} \Pi_-$ is close to $H_{\rm eff}$, expressed in Eq. (A.12).

First part. We have

$$\begin{aligned} \|\tilde{H}_{<\lambda_*} - \Pi_- \tilde{H}_{<\lambda_*} \Pi_-\| &= \|\Pi_+ \tilde{H}_{<\lambda_*} \Pi_+ + \Pi_+ \tilde{H}_{<\lambda_*} \Pi_- + \Pi_- \tilde{H}_{<\lambda_*} \Pi_+\| \\ &\leq 2\,\|\Pi_+ \tilde{H}_{<\lambda_*}\| + \|\tilde{H}_{<\lambda_*} \Pi_+\|, \end{aligned} \tag{A.4}$$

using standard properties of the operator norm $||.||$. Let $\tilde{\Pi}_{<\lambda_*}$ be the projector onto the space spanned by the eigenvectors of $\tilde{H}_{<\lambda_*}$ with eigenvalues below λ_*. We can insert $\tilde{\Pi}_{<\lambda_*}$ before or after $\tilde{H}_{<\lambda_*}$ and use that for projectors P_1, P_2, $||P_1 P_2|| = ||P_2 P_1||$, so that

$$||\tilde{H}_{<\lambda_*} - \Pi_- \tilde{H}_{<\lambda_*} \Pi_-\| \leq 3\,\|\tilde{H}_{<\lambda_*}\|\,\|\Pi_+ \tilde{\Pi}_{<\lambda_*}\| \leq 3\,(||H_{\rm eff}|| + \epsilon)\,\|\Pi_+ \tilde{\Pi}_{<\lambda_*}\|. \tag{A.5}$$

In order to bound this, we first derive

$$||\Pi_+ H \tilde{\Pi}_{<\lambda_*}|| = ||\Pi_+ H \Pi_+ \tilde{\Pi}_{<\lambda_*}|| \geq \lambda_+ ||\Pi_+ \tilde{\Pi}_{<\lambda_*}||. \tag{A.6}$$

On the other hand, we have

$$||\Pi_+ H\tilde{\Pi}_{<\lambda_*}|| \leq ||\Pi_+\tilde{H}\tilde{\Pi}_{<\lambda_*}|| + ||V|| \leq (||H_{\rm eff}|| + \epsilon)||\Pi_+\tilde{\Pi}_{<\lambda_*}|| + ||V||. \quad (A.7)$$

Putting the last three equations together gives the final bound

$$\|\tilde{H}_{<\lambda_*} - \Pi_-\tilde{H}_{<\lambda_*}\Pi_-\| \leq \frac{3(||H_{\rm eff}|| + \epsilon)\|V\|}{\lambda_+ - (||H_{\rm eff}|| + \epsilon)}. \quad (A.8)$$

Second part. We consider

$$\Pi_-\tilde{H}_{<\lambda_*}\Pi_- = \frac{1}{2\pi i}\oint_C z\,\Pi_-\tilde{G}(z)\Pi_-\,dz, \quad (A.9)$$

and recall that $\Pi_-\tilde{G}(z)\Pi_- = \tilde{G}_{--}(z) = (zI_- - \Sigma_-(z))^{-1}$, Eq. (A.1). By showing that this operator is close to $\Pi_-(zI - H_{\rm eff})^{-1}\Pi_- = (zI_- - H_{\rm eff})^{-1}$, we will be able to deduce that $\Pi_-\tilde{H}_{<\lambda_*}\Pi_-$ is close to $H_{\rm eff}$.

For all $z \in D_r$, $\|\Sigma_-(z) - H_{\rm eff}\| \leq \epsilon$ by assumption. In order to bound $\|(zI_- - \Sigma_-(z))^{-1} - (zI_- - H_{\rm eff})^{-1}\|$, we will use the following

$$||(A-B)^{-1} - A^{-1}|| = ||(I - A^{-1}B)^{-1}A^{-1} - A^{-1}|| \leq \left((1 - ||A^{-1}||\,||B||)^{-1} - 1\right)||A^{-1}||, \quad (A.10)$$

when $||A^{-1}||\,||B|| < 1$. We choose $A = zI_- - H_{\rm eff}$ and $B = \Sigma_-(z) - H_{\rm eff}$. For $z \in C$ (i.e. on the contour) $||A^{-1}|| \leq (r - w_{\rm eff})^{-1}$ and thus $||A^{-1}||\,||B|| \leq \frac{\epsilon}{r - w_{\rm eff}} \leq 1$. It follows that

$$\sup_{z\in C}\|(zI_- - \Sigma_-(z))^{-1} - (zI_- - H_{\rm eff})^{-1}\| \leq \frac{\epsilon}{(r - w_{\rm eff} - \epsilon)(r - w_{\rm eff})}.$$

Now we will use the following for an operator-valued function $F(z)$ and a contour C with radius r around a real-valued z_0:

$$\left\|\frac{1}{2\pi i}\oint_C z\,F(z)\,dz\right\| \leq r(r + z_0)\sup_{z\in C}||F(z)||. \quad (A.11)$$

Using this bound and the resolvent for $H_{\rm eff}$, we find that

$$\begin{aligned} ||\Pi_-\tilde{H}_{<\lambda_*}\Pi_- - H_{\rm eff}|| &= \left\|\frac{1}{2\pi i}\oint_C z\left((zI_- - \Sigma_-(z))^{-1} - (zI_- - H_{\rm eff})^{-1}\right)\right\| \\ &\leq \frac{r(r + z_0)\epsilon}{(r - w_{\rm eff})(r - w_{\rm eff} - \epsilon)}. \end{aligned} \quad (A.12)$$

where we have used that $H_{\rm eff} = \Pi_- H_{\rm eff}\Pi_-$. Putting Eqs. (A.8) and (A.12) together gives the desired result, Eq. (A.2). □.

The proof technique used in this theorem can be easily adapted to prove properties of the low-lying eigenspace of $\tilde{H}$; this is the content of the following Lemma. For the resulting bound of Eq. (A.13) to be useful one needs (1) to take Δ large enough compared to $||V||$ (which bounds the first term in Eq. (A.13)) and (2) the gap $\Delta_{\rm eff}$ of the effective Hamiltonian $H_{\rm eff}$, defined as $\Delta_{\rm eff} \equiv \lambda_{1,\rm eff} - \lambda_{0,\rm eff}$, needs to be bounded away from zero. In particular in the Lemma we can take $r = \Delta_{\rm eff} - 2\epsilon$ and then the second term in Eq. (A.13) can be (upper)-bounded by $\frac{\epsilon\lambda_{1,\rm eff}}{(\Delta_{\rm eff} - 2\epsilon)(\Delta_{\rm eff} - 3\epsilon)}$.

If $\Delta_{\rm eff} \geq \frac{1}{\rm poly}(n)$ we can thus take a polynomially small ϵ to bound the second term in Eq. (A.13) by some other inverse polynomial.

Lemma A.1 *Given is a Hamiltonian H such that no eigenvalues of H lie between $\lambda_- = \lambda_* - \Delta/2$ and $\lambda_+ = \lambda_* + \Delta/2$. Let the perturbed Hamiltonian $\tilde{H} = H + V$ where V is a small perturbation with $||V|| \leq \Delta/2$. We assume that $H_{\text{eff}} = \Pi_- H_{\text{eff}} \Pi_-$ and $\text{Spec}(H_{\text{eff}}) \subseteq [a, b]$. Let $0 < \epsilon < \Delta$ and assume that for all $z \in D_r$, a disk of radius r centered around $z_0 = \lambda_{0,\text{eff}}$ with $\epsilon < r < \Delta_{\text{eff}} - \epsilon$, we have $\|\Sigma_-(z) - H_{\text{eff}}\| \leq \epsilon$. Let $\Pi_{0,\text{eff}}$ be the projector onto the ground-space of H_{eff} with degeneracy d. Let $\tilde{\Pi}_{\text{low}}$ be the projector onto the d lowest-lying eigenvectors of $\tilde{H}$. Then we can bound*

$$||\tilde{\Pi}_{\text{low}} - \Pi_{0,\text{eff}}|| \leq \frac{3\|V\|}{\lambda_+ - (\lambda_{0,\text{eff}} + \epsilon)} + \frac{\epsilon(\lambda_{0,\text{eff}} + r)}{r(r - \epsilon)}. \tag{A.13}$$

Proof. As in the proof of Theorem A.1, we first prove that $\Pi_- \tilde{\Pi}_{\text{low}} \Pi_-$ is close to $\tilde{\Pi}_{\text{low}}$. Then we show that $\Pi_- \tilde{\Pi}_{\text{low}} \Pi_-$ is close to the projector onto the ground state of H_{eff}, $\Pi_{0,\text{eff}}$. For both parts we will use that due to the assumptions in the Lemma, Theorem 7 implies that for all $i = 0, \ldots, d-1$, $|\lambda_i(H_{\text{eff}}) - \lambda_i(\tilde{H})| \leq \epsilon$. Let $\lambda_{0,\text{eff}}$ be the lowest (degenerate) eigenvalue of H_{eff}. We can first bound

$$||\tilde{\Pi}_{\text{low}} - \Pi_- \tilde{\Pi}_{\text{low}} \Pi_-|| \leq 3||\Pi_+ \tilde{\Pi}_{\text{low}}||. \tag{A.14}$$

As before we bound $||\Pi_+ H \tilde{\Pi}_{\text{low}}||$ in two different directions:

$$(\epsilon + \lambda_{0,\text{eff}})||\Pi_+ \tilde{\Pi}_{\text{low}}|| + ||V|| \geq ||\Pi_+ H \tilde{\Pi}_{\text{low}}|| \geq \lambda_+ ||\Pi_+ \tilde{\Pi}_{\text{low}}||. \tag{A.15}$$

These inequalities together with the previous equation give us the first bound

$$||\tilde{\Pi}_{\text{low}} - \Pi_- \tilde{\Pi}_{\text{low}} \Pi_-|| \leq \frac{3\|V\|}{\lambda_+ - (\lambda_{0,\text{eff}} + \epsilon)} \tag{A.16}$$

In order to prove the other part we draw a circular contour C of radius r centered around $z_0 = \lambda_{0,\text{eff}}$ with $\epsilon < r < \Delta_{\text{eff}} - \epsilon$ such that it encloses only the lowest d eigenvalues of $\tilde{H}$. We will choose r such that $\Delta_{\text{eff}} - r > r$ or $z_0 + r$ is closer to $\lambda_{0,\text{eff}}$ than to $\lambda_{1,\text{eff}}$. We have

$$\Pi_- \tilde{\Pi}_{\text{low}} \Pi_- = \frac{1}{2\pi i} \oint_C \Pi_- \tilde{G}(z) \Pi_-. \tag{A.17}$$

We use that $||\Sigma_-(z) - H_{\text{eff}}|| \leq \epsilon$ for $z \in C$ and bound

$$\sup_{z \in C} \|(zI_- - \Sigma_-(z))^{-1} - (zI_- - H_{\text{eff}})^{-1}\| \leq \frac{\epsilon}{r(r - \epsilon)}, \tag{A.18}$$

using $||(zI_- - H_{\text{eff}})^{-1}|| \leq r^{-1}$ for $z \in C$. It follows that

$$||\Pi_- \tilde{\Pi}_{\text{low}} \Pi_- - \Pi_{0,\text{eff}}|| \quad \leq \quad \frac{\epsilon(\lambda_{0,\text{eff}} + r)}{r(r - \epsilon)}. \tag{A.19}$$

□.