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THE SIGNALING DIMENSION IN GENERALIZED PROBABILISTIC THEORIES

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The signaling dimension of a given physical system quantifies the minimum dimension of a classical system required to reproduce all input/output correlations of the given system. Thus, unlike other dimension measures - such as the dimension of the linear space or the maximum number of (jointly or pairwise) perfectly discriminable states - which examine the correlation space only along a single direction, the signaling dimension does not depend on the arbitrary choice of a specific operational task. In this sense, the signaling dimension summarizes the structure of the entire set of input/output correlations consistent with a given system in a single scalar quantity. For quantum theory, it was recently proved by Frenkel and Weiner in a seminal result that the signaling dimension coincides with the Hilbert space dimension.

Here, we derive analytical and algorithmic techniques to compute the signaling dimension for any given system of any given generalized probabilistic theory. We prove that it suffices to consider extremal measurements with ray-extremal effects, and we bound the number of elements of any such measurement in terms of the linear dimension. For systems with a finite number of extremal effects, we recast the problem of characterizing the extremal measurements with ray-extremal effects as the problem of deriving the vertex description of a polytope given its face description, which can be conveniently solved by standard techniques such as the double description algorithm. For each such measurement, we recast the computation of the signaling dimension as a linear program, and we propose a combinatorial branch and bound algorithm to reduce its size. We apply our results to derive the extremal measurements with ray-extremal effects of a composition of two square bits (or squits) and prove that their signaling dimension is five, even though each squit has a signaling dimension equal to two.

Keywords: signaling dimension, generalized probabilistic theory, GPT, square bit, squit, extremal measurements

1 Introduction

Generalized probabilistic theories [1, 2, 3], of which classical and quantum theories represent particular instances, represent the most general mathematical description of a physical theory. A generalized probabilistic theory is specified in terms of the set of states and the set of effects of its systems. The former correspond to the admissible preparation procedures for the system, while the latter represent the admissible building blocks of measurements. A rule to generate composite systems and their allowed dynamics can be specified, further enriching the structure of the theory.

Within any given generalized probabilistic theory, an operationally relevant problem is to quantify the dimension of any given system. Several different quantifiers [4, 5] of dimension had been introduced, including the dimension of the linear space of the states and effects of the system, or the maximum number of (jointly or pairwise) perfectly distinguishable states of the system. On the one hand, the former quantifier lacks a direct operational interpretation in terms of the input/output correlations achievable by the system; on the other hand, the latter quantifier is dependent on the arbitrary choice of an operational task, and hence investigates the space of input/output correlations along a single direction only.

In stark contrast with that, the signaling dimension [6] does not depend on the arbitrary choice of a specific operational task, and hence summarizes the structure of the entire set of input/output correlations that is consistent with a given system in a single scalar quantity. Formally, the signaling dimension of the system quantifies the minimum dimension of any simulating classical system, that is, any classical system that can reproduce all the input-output correlations of the given system.

For quantum theory, it was recently proved [7] by Frenkel and Weiner in a groundbreaking result that the signaling dimension coincides with the Hilbert space dimension. Subsequently, the problem of computing the signaling dimension in different contexts has drawn considerable attention, for instance in the case [8, 9, 10, 11, 12] of arbitrary classical and quantum channels, as well as in the case [13, 14, 15, 16] of channels in generalized probabilistic theories.

In this work, we derive analytical and algorithmic techniques to compute the signaling dimension for any given system of any given generalized probabilistic theory. We split the problem of computing the signaling dimension in two steps: i) the characterization of the extremal measurements of any given system, that is relevant in its own right for optimization problems other than the signaling dimension, and ii) the actual computation of the signaling dimension given the characterization of extremal measurements.

Concerning the first step, we prove that, when computing the signaling dimension, it suffices to consider extremal measurements with ray-extremal effects, and we show that the number of elements of any such a measurement is upper bounded by the dimension of the linear space. For systems whose set of admissible effects is a polytope, we recast the problem of characterizing the extremal measurements with ray-extremal effects as the problem of deriving the vertices description of a polytope given its faces description. Such a problem can be conveniently solved by standard techniques such as the double description algorithm.

Regarding the second step, that is, the actual computation of the signaling dimension given the characterization of extremal measurements, we recast it as a series of linear programs, one for each extremal measurement. We propose a combinatorial, branch and bound algorithm to reduce such a size and make it practically tractable. We provide an implementation [17] of the such an algorithm, as well as of the other algorithms discussed in this work, released under a free software license.

As a running example thorough our work, we consider a composition of two systems. For each of such systems, the set of admissible states is geometrically represented by a square, hence such systems are also known as square bits, or squits. Square systems have been originally introduced as an implementation of the Popescu-Rohrlich [18, 19, 2, 20] correlations. In that case, the composition includes all the entangled states consistent with the squit local structure, leaving no room for entangled effects, a trade-off first noticed in Ref [21]. However, alternative composition rules can be considered with a richer set of entangled effects, thus allowing for a richer characterization of extremal measurement and a non-trivial computation of the signaling dimension.

Incidentally, we show that alternative compositions, such as the one considered in Ref. [22], are inconsistent, that is, they contain well-formed experiments that nonetheless give rise to negative probabilities. We classify all the consistent composition rules of two squits and focus on the instance considered in Ref. [13], a "dual version" of the Popescu-Rohrlich boxes that includes all possible entangled effects but only local preparations. We apply the present algorithmic techniques to derive the extremal measurements with ray-extremal effects of such a model and prove that its signaling dimension five. Incidentally, this shows the tightness of the lower bound in Ref. [13] for the signaling dimension of such a model.

The paper is structured as follows. In Section 2.1 we formalize the problem of computing the signaling dimension as an optimization problem. In Section 2.2 we introduce our running example by discussing the completely positive compositions of two square bits. In Section 3.1 we derive analytical and algorithmic results on the characterization of the extremal measurements with ray-extremal effects of any given system. In Section 3.2 we provide a combinatorial, branch and bound algorithm for the exact, closed-form computation of the signal dimension. We summarize our results and discuss possible future developments in Section 4.

2 Formalization

In this section we introduce the object of study of this work, that is, the signaling dimension, and the running example given by the composition of two squit systems.

2.1 Signaling dimension

A physical system S of linear dimension $\ell(S) \in \mathbb{N}$ can be represented by a pair $(\mathcal{S}, \mathcal{E})$, where $\mathcal{S} \subseteq \mathbb{R}^\ell$ and $\mathcal{E} \subseteq \mathbb{R}^\ell$ are the sets of admissible states and effects, respectively. The probability of measuring the effect $e \in \mathcal{E}$ given the state $\omega \in \mathcal{S}$ is given by $e \cdot \omega$, and the effect $\overline{e}$ that gives unit probability for any deterministic preparation is called unit effect. This effect is unique in causal theories [3], and it corresponds to the identity operator in the quantum case. For any given system S, let $\mathcal{P}_S^{m \to n}$ denote the set of m-input/n-output conditional probability distributions that can be generated by system S with shared randomness. That is, p is an element of $\mathcal{P}_S^{m \to n}$ if and only if there exists states $\{\omega_{x|\lambda}\}_{x,\lambda}$ and measurements $\{E_{y|\lambda}\}_{y,\lambda}$ such that

$$p_{y|x} = \sum_\lambda q_\lambda \omega_{x|\lambda} \cdot E_{y|\lambda},$$

for some probability distribution $\{q_\lambda\}_\lambda$. The *signaling dimension* [13] κ of a system S is the minimum dimension d of a classical system C_d that can reproduce the input/output correlations attainable by S, that is

$$\kappa(S) := \min_{d \in \mathbb{N}} d \qquad \text{s.t.} \qquad \mathcal{P}_S^{m \to n} \subseteq \mathcal{P}_{C_d}^{m \to n},$$

for any $m, n \in \mathbb{N}$. In particular, for a classical system C_d of dimension d, the sets $\mathcal{S}$ and $\mathcal{E}$ of admissible states and effects, respectively, are known to be represented by regular $d-1$ simplices in a linear space of dimension $\ell(C_d) = d$. For instance, for the (classical) bit, trit, and quart, the states are represented by a segment, a triangle, and a tetrahedron, respectively, and so are the effects.

The signaling dimension was originally introduced [13] in connection with the no-hyper-signaling principle, which is a scaling rule stipulating that the signaling dimension of the composition of any given systems cannot be larger than the product of the signaling dimension of each system. In other words, the no-hypersignaling principle constraints the time-like correlations that any given system can exhibit, and allows to rule out as unphysical even systems whose space-like correlations are instead compatible with quantum or even classical theory. In particular, the fact that quantum theory satisfies such a scaling rule follows only as a consequence of the aforementioned recent result [7] by Frenkel and Weiner.

2.2 Compositions of square bits

Here we introduce a toy model theory, i.e. the square bit (or squit) and its compositions, that will serve as a running example across all this work. The squit system S has linear dimension $\ell(S) = 3$, namely its states ω and effects e are described by vectors in $\mathbb{R}^3$. We now specify the convex sets $\mathcal{S}$ and $\mathcal{E}$. The extremal states of the squit (that is, the four vertices of the square) are given by the four vectors $\{\omega_k = U_k^s \omega_0\}_k$, with $\omega_0 = (1, 0, 1)^T$. Here, $\{U_k^s\}_{k,s}$ are the reversible transformations of the system (that is, the symmetries of the square or, more generally, the transformation that leave the set of admissible states invariant, as opposed to those transformations that shrink it) given by the following rotations and reflections

$$U_k^s = \begin{pmatrix} \cos \frac{\pi k}{2} & -s \sin \frac{\pi k}{2} & 0 \\ \sin \frac{\pi k}{2} & s \cos \frac{\pi k}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $k \in \{0, 1, 2, 3\}$ and $s = \pm 1$.

By explicit computation, the effect space dual to such a state space (that is, the set of effects e such that $e \cdot \omega \geq 0$) is given (up to a positive scaling factor) by the convex hull of $\{e_k = U_k^s e_0\}_k$, with $e_0 = (1, 1, 1)^T$.

Any composition of two squits necessarily includes the factorized extremal states and effects, that is

$$\Omega_{4i+j} := \omega_i \otimes \omega_j, \tag{1}$$

and

$$E_{4i+j} := e_i \otimes e_j, \tag{2}$$

respectively, where $i, j \in \{0, 1, 2, 3\}$, so that the model has to include at least the bipartites states $\Omega_0, \ldots \Omega_{15}$ and the bipartite effects $E_0, \ldots E_{15}$. Additionally, by explicit computation the set of effects dual to Eq. (1) includes [20, 21] the eight entangled effects E_{16} up to E_{23}

given by the columns of

$$\begin{pmatrix}
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix},$$

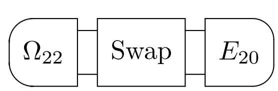
respectively. Analogously, the state space dual to Eq. (2) includes [20, 21] the eight entangled states Ω_{16} up to Ω_{23} given (up to a positive scaling factor) by the columns of

$$\begin{pmatrix}
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{pmatrix},$$

respectively. Notice that the bipartite system $S \otimes S$ has linear dimension $\ell(S \otimes S) = \ell(S)^2 = 9$.

However, it is well known that not all of the eight entangled states and the eight entangled effects can be included in the same composite system. Indeed, composite systems must be completely positive, that is, they must generate non-negative probabilities when connected in any possible way allowed by the (reversible) dynamics of the system. In general, to find the reversible dynamics of the composite system (that is, the symmetries of their sets of admissible states and effects), one can use the algorithm introduced in the appendix of Ref. [23]. In the particular case of two squits, it is known from Ref. [24] that the only non-trivial bipartite reversible dynamics is the swap operator.

In Ref. [22], Janotta considered a theory including the following four entangled states: Ω_{16}, Ω_{18}, Ω_{22}, and Ω_{23} [see Eqs. (10), (11), (12), and (9) therein], as well as the following four entangled effects (up to a positive scaling factor): E_{17}, E_{19}, E_{20}, and E_{21} [see Eqs. (18), (19), (17), and (20) therein]. Such a theory is not even positive, and therefore not completely positive, that is, it includes events whose probability of occurrence is strictly negative. As an example, take the following event, obtained by wiring (the Swap gate represents the swap operator) the entangled state Ω_{23} with the entangled effect E_{20} (both included in Janotta's model), and whose probability is strictly negative:

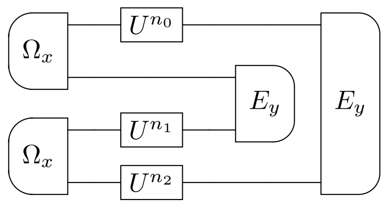


$< 0.$

To find the completely positive compositions of two squits, we first observe that the following relations hold

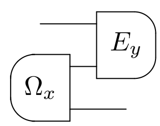
$$\mathrm{Swap}\,\Omega_{20} = \Omega_{23}, \tag{3}$$
$$\mathrm{Swap}\,\Omega_{21} = \Omega_{22}, \tag{4}$$
$$\mathrm{Swap}\,E_{20} = E_{23}, \tag{5}$$
$$\mathrm{Swap}\,E_{21} = E_{22}. \tag{6}$$

Moreover, any such a composition must satisfy



$\geq 0,$ (7)

where U is given by

$U \propto$ 

,

and $n_i = 0, 1$ with $i \in \{0, 1, 2\}$.

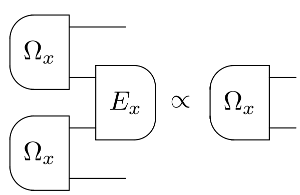
By explicit computation, Eqs. (3), (4), (5), (6), and 7 above imply that the only possible compositions are:

PR model all eight entangled states, no entangled effect, free local dynamics, so called since it produces Popescu-Rohrlich correlations;

HS model no entangled state, all eight entangled effects, free local dynamics, so called since it violates the no-hypersignaling principle [13];

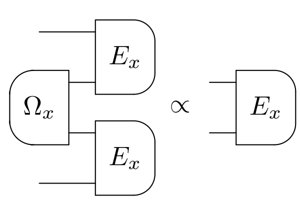
four frozen models only one entangled state Ω_x and one entangled effect E_x, with $x \in \{16, 17, 18, 19\}$, no non-trivial local dynamics (hence the name of the models).

Any other composition of two squits is necessarily not completely positive and possibly not even positive, as Janotta's model. Finally, it is very easy to see that such models are completely positive, by observing that for each of them one has



,

as well as



,

and finally

$$\begin{aligned} \mathrm{Swap}\,\Omega_x &= \Omega_x, \\ \mathrm{Swap}\,E_x &= E_x, \end{aligned}$$

for any $x \in \{16, 17, 18, 19\}$.

3 Main results

In this section we introduce our main results, that is, analytical and algorithmic techniques for the carachterization of all the extremal measurements with ray-extremal effects of any given system, as well as for the computation of its signaling dimension.

3.1 Extremal measurements with ray-extremal effects

It is well-known [25, 26] that extremal quantum measurements can comprise effects that are not ray-extremal (that is, they are not rank-one projectors). However, it is also known [27] that, whenever optimizing an objective function that is convex in the measurement, one can restrict to extremal measurements with ray-extremal effects. In the following we extend this result to generalized probabilistic theories.

Before proceeding, let us give a convenient definition of measurement. To do so, we first stipulate to normalize effects so that any measurement of the system can be expressed as a probability distribution p over the effects such that $\sum_y p_y e_y = \overline{e}$, the unit effect with unit probability over any state. As a comparison, in quantum theory such a choice would be equivalent to defining effects as operators with trace equal to the Hilbert space dimension (the same normalization as the identity operator), so that finite positive operator-valued measures would actually be uniquely identified by the corresponding probability distributions. Hence, for instance, in quantum theory rank-one projectors are the only ray-extremal effects. Any other projector, while an extremal effect, is not ray-extremal.

The relevance of extremal measurements with ray-extremal effects (or, equivalently, extremal normalized effects) is made clear by the following two trivial observations: i) the maximum of any convex objective function of the measurement is attained by an extremal measurement, and ii) the maximum of any given objective function of the measurement that is non-decreasing under fine graining is attained by a measurement with extremal normalized effects. It is thus important to characterize those measurements with ray-extremal effects that are extremal: this is achieved by the following lemma, which was first proved in the supplemental material of Ref. [13].

Lemma 1 (Characterization of extremal measurements with ray-extremal effects)**.** *For any measurement $M = \{p_y > 0, e_y\}$ with extremal normalized effects $\{e_y\}$, the following conditions are equivalent:*

1. *M is extremal,*
2. *$\{e_y\}$ are linearly independent.*

Proof. Let us first prove by contradiction that 1 implies 2. We start by assuming that there exists an extremal measurement $\{p_y, e_y\}$ with $p > 0$ such that $\{e_y\}$ are not linearly independent, i.e. $|\{e_y\}| > \dim \mathrm{span}(\{e_y\})$. Since $\{e_y\}$ are normalized, they belong to an affine subspace of dimension $\dim \operatorname{span}(\{e_y\})-1$. Thus, applying Caratheodory's theorem, the unit effect $\overline{e}$, that belongs to $\operatorname{span}(\{e_y\})$ by hypothesis, can be decomposed in terms of a subset of $\{e_y\}$ with cardinality $\dim \operatorname{span}(\{e_y\})$. In other words, there exists a probability distribution p'_y, whose support has cardinality not greater than $\dim \operatorname{span}(\{e_y\})$, such that $\sum_y p'_y e_y = \overline{e}$. By taking $\lambda > 0$ such that $p - \lambda p' \geq 0$ (such a λ always exists since $p > 0$) and $p''_y := (1-\lambda)^{-1}(p_y - \lambda p'_y)$, it immediately follows that also $\{p''_y, e_y\}$ is a measurement. Then measurement $M = \{p_y, e_y\}$ can be decomposed as $\lambda\{p'_y, e_y\} + (1-\lambda)\{p''_y, e_y\}$, i.e. it is not extremal, thus leading to a contradiction.

Let us now prove that 2 implies 1. Since $\{e_y\}$ are extremal, they cannot be further decomposed, so any convex decomposition of M would necessarily involve a subset of the effects $\{e_y\}$. Since such effects $\{e_y\}$ are linearly independent, the decomposition of $\overline{e}$ is unique and, since $p > 0$, no subset of $\{e_y\}$ can be a measurement. Therefore, the statement follows. $\square$

As an immediate consequence of Lemma 1, for any extremal measurement with ray-extremal effects one has that the number n of such effects is upper bounded by the linear dimension of the system, a result that generalizes a theorem by Davies [27] to generalized probabilistic theories.

If the extremal normalized effects are finite in number and given by the columns of matrix E, a probability distribution p is a measurement on the extremal normalized effects if and only if it satisfies the following linear equalities

$$Ep = \overline{e}, \tag{8}$$

as well as following the linear inequalities

$$p \geq 0. \tag{9}$$

Therefore, probability distribution p is an extremal measurement on the extremal normalized effects if and only if it is a vertex of such a polytope. Notice that Eqs. (8) and (9) characterize such a polytope by giving its faces description. The extremal measurements with extremal normalized effects can therefore be found by passing from the faces description to the vertices description, a standard problem that can be solved e.g. with the double description method [28]. This is summarized by the following proposition.

Proposition 1. *For any system with a finite number of extremal normalized effects given by the columns of matrix E, the extremal measurements with extremal normalized effects are the vertices of the polytope given (in faces description) by Eqs.* (8) *and* (9)*, and can be found by the double description method.*

As an application, let us go back to our running example given by the HS model, consisting of the composition of two squits that includes eight entangled effects. The aforementioned procedure produces in this case a set of 408 extremal measurements with extremal normalized effects. Such a set can be partitioned according to the equivalence class induced by the reversible transformations of the system, so that two such measurements belong to the same class if and only if they are equivalent up to a reversible transformation. By taking a single representative for each equivalence class, the reduced set of extremal measurements with extremal normalized effects is left with 15 elements only, as reported in Table 1.

M	#	$\mathbf{E_0}$	$\mathbf{E_1}$	$\mathbf{E_2}$	$\mathbf{E_3}$	$\mathbf{E_4}$	$\mathbf{E_5}$	$\mathbf{E_6}$	$\mathbf{E_7}$	$\mathbf{E_8}$	$\mathbf{E_9}$	$\mathbf{E_{10}}$	$\mathbf{E_{11}}$	$\mathbf{E_{12}}$	$\mathbf{E_{13}}$	$\mathbf{E_{14}}$	$\mathbf{E_{15}}$	$\mathbf{E_{16}}$	$\mathbf{E_{17}}$	$\mathbf{E_{18}}$	$\mathbf{E_{19}}$	$\mathbf{E_{20}}$	$\mathbf{E_{21}}$	$\mathbf{E_{22}}$	$\mathbf{E_{23}}$
0	2																	120		120					
1	4	60		60						60		60													
2	4	60		60							60		60												
3	6	30	30									30	30							60					60
4	6	30					30					30					30					60			60
5	6	40										40							40	40		40			40
6	7	30	30					30		30		30					30								60
7	8	20	20			20						40					20			40		40			40
8	8	20	20					40		20	20						40	40							40
9	8	40	20					20			20		40			20				40					40
10	8	30					30						30			30				30	30	30			30
11	9	20	20			20		20			20	20					40					40			40
12	9	15	15			15		30			30						45	30				30			30
13	9	20	20			20			20			20	20		20	20				80					
14	9	24		24			24						48		24					24	24			24	24
M	#	$\mathbf{E_0}$	$\mathbf{E_1}$	$\mathbf{E_2}$	$\mathbf{E_3}$	$\mathbf{E_4}$	$\mathbf{E_5}$	$\mathbf{E_6}$	$\mathbf{E_7}$	$\mathbf{E_8}$	$\mathbf{E_9}$	$\mathbf{E_{10}}$	$\mathbf{E_{11}}$	$\mathbf{E_{12}}$	$\mathbf{E_{13}}$	$\mathbf{E_{14}}$	$\mathbf{E_{15}}$	$\mathbf{E_{16}}$	$\mathbf{E_{17}}$	$\mathbf{E_{18}}$	$\mathbf{E_{19}}$	$\mathbf{E_{20}}$	$\mathbf{E_{21}}$	$\mathbf{E_{22}}$	$\mathbf{E_{23}}$

Table 1. Set of all extremal measurements (up to reversible transformations) for the composition of two squits named HS model with eight entangled effects. Measurements are labelled by M and represented by a probability distribution p over the set of extremal normalized effects $\{E_j\}$ (rescaled by a factor of 240 so that each entry is an integer). The number of non-null values of p is also reported for convenience in the column indicated by the symbol #. We recall that factorized effects are those from E_0 to E_{15} (included), while the other effects are entangled.

3.2 The signaling dimension

We have already shown that, in order to compute the signaling dimension of any given system, it suffices to consider the m-input/n-output conditional probability distributions generated by the extremal measurements with ray-extremal effects upon the input of all the states of the system. Here and in the following we take any such a conditional probability distribution p arranged as an $m \times n$ stochastic matrix, where x and y label the row and the column, respectively. For any such a p, it suffices to compute the minimal dimension d of the classical polytope $\mathcal{P}_d^{m \to n}$ that contains p. The maximum over such p's of the minimal dimension d constitutes the signaling dimension of the given system.

The straightforward way of achieving this would be to check p against all the inequalities that characterize the facets of $\mathcal{P}_d^{m \to n}$. However, known algorithms for the characterization of such facets require as input the vertices of $\mathcal{P}_d^{m \to n}$, which are too many to make this approach feasible for the instances of the problem we are interested in. This includes standard algorithms such as the double description method [28], as well as fine-tuned algorithms such as the adjacency decomposition algorithm by Doolittle and Chitambar of Ref. [10], that can exploit the symmetries of $\mathcal{P}_d^{m \to n}$ in order to reduce the number of facets they output by producing one representative facet for each equivalence class under symmetry. For instance, for $m = 16$ (the number of extremal states in the HS composition of the squit), $n = 9$ (the maximum number of elements of extremal measurements, attained by four such measurements as per Table 2), and $d = 5$ (the minimum value we are interested in), the number of such vertices is $\sim 2.4 \cdot 10^{13}$ (see Lemma 2 below). From this issue, the need arises to devise techniques that do not requrire the explicit enumeration of all the vertices of $\mathcal{P}_d^{m \to n}$, as done in the following.

For any given conditional probability distribution p and any classical dimension d, in order to prove that d is the minimum dimension such that p belongs to the classical polytope $\mathcal{P}_d^{m \to n}$, one needs to provide:

- an explicit convex decomposition $\mathbf{x}$ of p in terms of the vertices of $\mathcal{P}_d^{m \to n}$, and
- an explicit linear witness (or game, or separating hyperplane) g such that $p \cdot g > q \cdot g$

for any $q \in \mathcal{P}_{d-1}^{m \to n}$.

To obtain a convex decomposition $\mathbf{x}$ of p in terms of the vertices of $\mathcal{P}_d^{m \to n}$, whenever it exists, one can proceed as follows. The problem can be framed as the feasibility of the following linear program

$$\min_{\substack{A\mathbf{x}=\mathbf{b} \\ \mathbf{x}\geq 0}} \mathbf{c} \cdot \mathbf{x}, \tag{10}$$

where A is the $(mn) \times V$ matrix whose columns are the V vertices of $\mathcal{P}_d^{m \to n}$ rearranged as vectors with $m \times n$ entries (the particular rearranging is irrelevant as long as used consistently across the protocol); $\mathbf{b}$ is the vector with mn entries obtaining rearranging the conditional probability distribution p; $\mathbf{c}$ is the vector with V entries all equal to zero since, as said earlier, we are interested in the feasibility of the problem only (an alternative option is to take $\mathbf{c}$ to be the vectors with all entries equal to one, which also gives a constant objective function since $\mathbf{x}$ is constrained to be a probability distribution).

In turn, the number V of vertices of $\mathcal{P}_d^{m \to n}$ (equivalently, the number of columns of matrix A) can itself be expressed in terms of m, n, and d. To see this, let us denote with

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

the binomial coefficient and with

$$\left\{ {m \atop k} \right\} := \sum_{j=0}^{k} \frac{1}{k!} (-1)^{k-j} \binom{k}{j} j^m$$

the Stirling number of the second kind, i.e. the number of partitions of a set of m elements in k non-empty classes. Then the following result, first proved in the supplemental material of Ref. [13], holds.

Lemma 2. *The number V of vertices of $\mathcal{P}_d^{m \to n}$ is given by*

$$V = \sum_{k=1}^{d} k! \binom{n}{k} \left\{ {m \atop k} \right\}.$$

Proof. The statement follows by a a simple counting argument. First, one chooses $k \leq d$ non-null columns (as stated earlier, different columns correspond to different effects); since the matrix has a n columns, there are $\binom{n}{k}$ such possible choices. Then, for each such a choice one has to assign a single one to each row within the k columns that were chosen; in other words, one has to assign each row (playing here the role of the element of a set) to a column (playing here the role of an element of a partition of such a set). There are exactly $k! \left\{ {m \atop k} \right\}$ possible assignments, where the factorial $k!$ comes from the fact that here the elements of the partition are labeled, while the definition of the Stirling number of the second kind does not take this into account. $\square$

In typical instances of the problem (see our running example on the composition of two squits, discussed later on), V is too large for matrix A to be practically tractable. In this case, the following steps can reduce the complexity of the problem. First, observe that any row of p (as stated earlier, different rows correspond to different states) that is the convex combination of other rows can be eliminated without altering the result, thus reducing the effective value of m (and thus V) without loss of generality. Also, in typical applications the conditional probability distributions p's are rather sparse, a fact that can be exploited as follows. Any vertex of $\mathcal{P}_d^{m\to n}$ that contains an entry equal to one where p contains a zero will not contribute to the convex decomposition of p; hence can be discarded without loss of generality, thus helping to further reduce the number V of vertices. In the following, we denote with v such a reduced number of vertices, and we refer to it as the number of effective vertices. We have then the following proposition.

Proposition 2. *Any given $m \times n$ conditional probability distribution p belongs to the polytope $\mathcal{P}_d^{m\to n}$ if and only if q belongs to the polytope $\mathcal{Q}_d^{m'\to n}$, where q is $m' \times n$ and is the same as p where any row that is the convex combination of rows is removed, and $\mathcal{Q}_d^{m'\to n}$ is the same as $\mathcal{P}_d^{m'\to n}$ where any vertex that has an entry equal to one where q has an entry equal to zero has been removed.*

The process to reduce the number of vertices from V to v can be efficiently implemented as a combinatorial, branch and bound algorithm. The algorithm starts on the first row of matrix p, cycles over any choice of non-null entries of that row, and for each choice recursively calls itself on the second row. At each call, the bounding procedure consists of verifying if the number of columns from which an entry has been chosen so far is larger than d; if so, the branch is pruned. Any pre-determined branching strategy can be adopted.

If the above steps do not suffice to make the linear program in Eq. 10 tractable, the technique known as delayed column generation [29, 30, 31, 32] can be adopted, by observing that it is possible to efficiently generate the vertex of the classical polytope $\mathcal{P}_d^{m\to n}$ that minimizes the inner product with any given vector; this immediately gives the reduced cost to be used in delayed column generation.

To obtain a linear witness g that separates the classical polytope $\mathcal{P}_d^{m\to n}$ from p, whenever it exists, one can proceed as follows. The problem can be framed as the following linear program, dual of the one in Eq. 10 except for the inclusion of a bounding box around variable $\mathbf{y}$:

$$\max_{\substack{A^T\mathbf{y}=\mathbf{c}\\ -\mathbf{u}\leq\mathbf{y}\leq\mathbf{u}}} \mathbf{b}\cdot\mathbf{y}, \tag{11}$$

where matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ are defined as above, and vector $\mathbf{u}$ is the vector with mn entries all equal to one. Notice that, if $\mathbf{c}$ has been taken to be the vector with all entries equal to one, then one does not need to include the bounding box condition. The witness (or game, or separating hyperplane) g is thus recovered by rearranging the entries of vector $\mathbf{y}$ as a matrix.

As noticed above for the primal problem in Eq. (10), the dual problem in Eq. (11) too is typically intractable, that is, there are too many constraints. In addition to the techniques discussed above to reduce the size of matrix A, one can in this case adopt the ellipsoid method [29, 30, 31, 32] or the cutting plane method [29, 30, 31, 32], by using as a separation oracle the aforementioned function that efficiently returns the vertex of the classical polytope $\mathcal{P}_{d-1}^{m\to n}$ that minimizes the inner product with any given vector.

As an application of the discussion of this section, let us return to our running example by considering the extremal measurements with ray-extremal effects for any composition of two squits given in Table 1. Considering the composition named HS-model that includes all eight entangled effects (and therefore no entangled state), upon the input of the 16 (factorized) extremal states each such measurement gives rise to a conditional probability distribution with 16 rows and a number of columns equal to the number of effects. Notice that, when computing the signaling dimension, it suffices to consider extremal measurements with at least four ray-extremal effects, that is, measurements number 3 up to 14 in Table 1.

The two steps discussed above (that is, the derivation of the convex decomposition of p or of the separating witness g) have been implemented in function `decompose.m`, and the results are summarized in Table 2. Hence, the following corollary follows.

M	**#**	**d**	$\mathbf{g}\cdot\mathbf{b}$	**v**	**V**
3	6	4		128	$\sim 9\cdot 10^{10}$
4	6	4		64	$\sim 9\cdot 10^{10}$
5	6	4		465	$\sim 9\cdot 10^{10}$
6	7	5	2	672	$\sim 4\cdot 10^{12}$
7	8	5	$1/3$	60752	$\sim 10^{13}$
8	8	5	$8/3$	7616	$\sim 10^{13}$
9	8	5	2	10040	$\sim 10^{13}$
10	8	4		576	$\sim 3\cdot 10^{11}$
11	9	5	$4/3$	37136	$\sim 2\cdot 10^{13}$
12	9	5	2	107504	$\sim 2\cdot 10^{13}$
13	9	5	$2/3$	8704	$\sim 2\cdot 10^{13}$
14	9	5	$8/5$	488092	$\sim 2\cdot 10^{13}$
M	**#**	**d**	$\mathbf{g}\cdot\mathbf{b}$	**v**	**V**

Table 2. Set of all extremal measurements as in Table 1. The two columns labeled with d and $\mathbf{g}\cdot\mathbf{b}$ denote the minimal dimension d of the classical polytope $\mathcal{P}_d^{m\to n}$ such that the conditional probability distribution p obtained by the measurement upon the input of the 16 extremal states belongs to $\mathcal{P}_d^{m\to n}$, and (whenever such a d is larger than 4) the maximum value attained by any witness g separating p from the classical polytope $\mathcal{P}_{d-1}^{m\to n}$, respectively. The two columns labeled with v and V represent the number of effective vertices for such a measurement and the total number of vertices of the classical polytope $\mathcal{P}_d^{m\to n}$. Even in the worst case, corresponding to the last row of the table, the protocol described here reduces the size of the problem by a factor $V/v \sim 4\cdot 10^7$.

Corollary 1. *The signaling dimension of the composition of two squits named HS model, including all eight possible entangled effects, is five.*

Incidentally, our results proves the tightness of the lower bound on the signaling dimension of such a model given in Ref. [13].

4 Conclusion

In this work we derived analytical and algorithmic techniques to characterize the extremal measurements and compute the signaling dimension of any given system of any given generalized probabilistic theory. As an example, we applied our results to the composition of two square bits. The algorithmic techniques we derived here, and whose implementation is made available online [17], can be directly applied to other system whose states and effects form polytopes, such as polygonal and polyhedral theories.

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