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ON THE QUANTUM COMPLEXITY OF INTEGRATION OF A FUNCTION WITH UNKNOWN SINGULARITY

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In this paper we study the quantum complexity of the integration of a function with an unknown singularity. We assume that the function has r continuous derivatives, with the derivative of order r being Hölder continuous with the exponent ρ on the whole integration interval except the one singular point. We show that the ε-complexity of this problem is of order $\varepsilon^{-1/(r+\rho+1)}$. Since the classical deterministic complexity of this problem is $\varepsilon^{-1/(r+\rho)}$, quantum computers give a speed-up for this problem for all values of parameters r and ρ.

Keywords: integration, unknown singularities, quantum algorithms, optimality, complexity

1 Introduction

The problem of the integration of functions is well known and investigated for many different settings and assumptions on the regularity of functions. Many quadrature rules are known, such as Newton-Cotes rules or Gaussian quadrature. The investigation of the complexity of the integration on the classical computer in the deterministic and randomized setting started in 1959 with the work of Bakhvalov [1], where the Hölder class of function is considered. The Sobolev class of functions is investigated in [2]. The result on the complexity of integration on the classical computer may also be found in [3, 4, 5].

Besides classical computation, there is also progress in studying computation on a quantum computer. One of the first fundamental works treating quantum computation is that of Shor [6] presenting the quantum algorithm for discrete factoring. This algorithm has a polynomial cost in terms of the number of bits of the input and there is no classical algorithm known with this property. The second milestone work on quantum computation was the database search algorithm of Grover [7], which shows the quadratic speed-up of a quantum computer over the classical one for this problem. The advantage of quantum computations is also shown for other discrete problems, such as computing the mean, median, and quantiles, see e.g. [8, 9, 10, 11]. Moreover, a number of continuous problems were studied in the quantum setting. The first work considering the quantum complexity of the continuous problem was that of Novak [12] treating the integration of a function from the Hölder class. The integration in the Sobolev class was investigated by Heinrich [13]. Other problems such as maximization, approximation, path integration, solving ordinary differential equations, searching for roots of functions, solving eigenvalue problems were studied, see [14, 15, 16, 17, 18, 19, 20, 21, 22].

It is an interesting task to examine the continuous problem for a function that is losing regularity at some unknown points. The first rigorous analysis of the problem from that point of view was the work of Plaskota and Wasilkowski [23] where the integration of a function with unknown singularities in the deterministic model of computation was investigated. The approximation of a function in a similar class was considered in [24, 25]. The problem of solving initial-value problems with a right-hand side function which loses the regularity on some unknown hypersurface was considered in the series of papers by Kacewicz and Przybyłowicz, see [26, 27, 28, 29].

In this paper we study a similar problem as in [23]. Here we deal with the complexity of the integration in the quantum model of computation. We show that the quantum complexity of the integration of the singular functions under consideration is asymptotically equal to that of smooth functions. We provide the optimal algorithm which is obtained by a procedure localizing the singularity similar to that of [23] and [29] combined with ideas found in Novak's quantum integration algorithm for smooth functions. We show that for the class $F_1^{r,\rho}$ of functions which are r times continuously differentiable with the rth derivative satisfying the Hölder condition with exponent ρ on whole integration interval except one unknown singular point, the quantum ε-complexity of this problem is $\varepsilon^{-1/(r+\rho+1)}$. It is known that the classical deterministic complexity of this problem is of order $\varepsilon^{-1/(r+\rho)}$, thus quantum computers yield a speed-up for the whole range of class parameters.

The paper is organized as follows. In Section 2 the problem is formulated and basic definitions are presented. Section 3 contains the known results on the integration of both smooth and piecewise regular functions in the deterministic and quantum models of computation. In Section 4 the algorithm for localizing the singularity and computing the integral of a piecewise regular function on a quantum computer is presented. The main theorems about the error and the cost of algorithms and the complexity of the problem can be found in Section 5.

2 Problem formulation and basic definitions

We consider here the problem of Riemann integration of the real-valued function $f : [a, b] \to \mathbf{R}$ for $a < b$. Let $I(f) = \int_a^b f(x)dx$. Let us define first the class of smooth functions i.e. the Hölder class given by

$$F_{\mathrm{reg}}^{r,\rho}([\alpha,\beta])=\Big\{f : [\alpha,\beta] \to \mathbf{R} \mid f \in C^r([\alpha,\beta]), |f^{(i)}(x)| \le D,\ i = 0, 1, \dots, r, \\ |f^{(r)}(x) - f^{(r)}(y)| \le H|x-y|^\rho \text{ for } x, y \in [\alpha,\beta]\Big\},$$

where $a \le \alpha < \beta \le b$, $C^r([\alpha,\beta])$ is a class of r times continuously differentiable functions on $[\alpha,\beta]$, $r \ge 0$, $0 < \rho \le 1$, and D, H are positive constants. Let us now define the class of functions with at most p unknown singularities. Let

$$F_p^{r,\rho}([a,b]) = \{f : [a,b] \to \mathbf{R} \mid \text{there exist } \xi_1, \xi_2, \dots, \xi_p \in [a,b] : \\ f \in F_{\mathrm{reg}}^{r,\rho}([a,\xi_1)) \cap F_{\mathrm{reg}}^{r,\rho}([\xi_1,\xi_2)) \cap \dots \cap F_{\mathrm{reg}}^{r,\rho}([\xi_p,b])\Big\}.$$

In the singular points ξ_i $(i = 1, 2, \dots, p)$ the derivative is understood as the right-hand side derivative. It is easy to see that

$$F_{\mathrm{reg}}^{r,\rho}([a,b]) \subset F_1^{r,\rho}([a,b]) \subset F_2^{r,\rho}([a,b]) \subset \dots .$$

We are especially interested in the class $F_1^{r,\rho}([a,b])$ with at most one singular point. For that class of functions, we show how to localize the singular point to construct a quantum algorithm that preserves the order of convergence of the smooth case.

First, we present the model of computation. Let us start with the classical deterministic setting. We assume we can obtain information about a function by evaluating it at a number of points. These points may be selected non-adaptively or adaptively. For nonadaptive information, the points at which the function is evaluated are given a priori and do not depend on the function. Thus, the information is given by

$$N^{\mathrm{non}}(f) = [f(t_1), f(t_2), \ldots, f(t_n)].$$

For adaptive information $N^{ad}(f)$, we allow the number of evaluations and the successive evaluation points to depend on the previously computed values. Thus, $n = n(f)$, t_1 is given and $t_i = t_i(t_1, \ldots, t_{i-1})$ for $i = 2, 3, \ldots, n(f)$. For a given information vector $N(f)$ the approximation is obtained by some algorithm $U(f) = \varphi(N(f))$, where $\varphi : \mathbf{R}^n \to \mathbf{R}$ is some mapping. We are using the worst-case setting, thus the error of approximation U in the class of functions F is given by

$$\mathrm{e}^{\mathrm{det}}(U, F) = \sup_{f \in F} |I(f) - U(f)|.$$

The cost of the method U for a given function f denoted by $\mathrm{cost}^{\mathrm{det}}(U, f)$ is defined as the number of function evaluations n. Again, the global cost in class F is defined by the worst behavior

$$\mathrm{cost}^{\mathrm{det}}(U, F) = \sup_{f \in F} \mathrm{cost}^{\mathrm{det}}(U, f).$$

We are ready to define the complexity of the problem, that is the minimal cost needed to solve the problem within a given precision $\varepsilon > 0$.

$$\mathrm{comp}^{\mathrm{det}}(\varepsilon, F) = \inf\{\mathrm{cost}^{\mathrm{det}}(U, F) : U \text{ such that } \mathrm{e}^{\mathrm{det}}(U, F) \le \varepsilon\}.$$

A detailed definition of the deterministic model of computation may be found in [5].

In the quantum setting the information is gathered by applying the so-called *quantum query* or *quantum oracle* call, which is some unitary operation that plays the role of computing the function values. The output of a quantum algorithm is a random variable U^ω on some probabilistic space (Ω, P). For a detailed explanation of the quantum query and the quantum model, the reader is referred to [30, 31]. The error in the quantum model of computation is defined by

$$\mathrm{e}^{\mathrm{quant}}(U, F) = \sup_{f \in F} \inf\{\varepsilon : \; P(|I(f) - U^\omega(f)| \ge \varepsilon) \le 3/4\}.$$

We consider hybrid quantum algorithms, which combine classical and quantum computations. The cost is, thus, defined as a sum of a number of classical function computations (adaptive or not) and the number of quantum queries. It is known that any hybrid quantum algorithm with many measurements may be rewritten as a pure quantum algorithm with one measurement, see [30, 13]. The ε-complexity is defined similarly to the deterministic setting

$$\mathrm{comp}^{\mathrm{quant}}(\varepsilon, F) = \inf\{\mathrm{cost}^{\mathrm{quant}}(U, F) : U \text{ such that } \mathrm{e}^{\mathrm{quant}}(U, F) \le \varepsilon\}.$$

3 Known results

We present in this section the useful results treating the integration of smooth functions and the known classical complexity bounds for integrating piecewise regular functions.

Consider first the integration in the classical deterministic setting. Let $Q_N(\alpha, \beta, f)$ be the N points midpoint rule approximating the integral $\int_\alpha^\beta f(x)dx$, that is

$$Q_N(\alpha, \beta, f) = \frac{\beta - \alpha}{N} \sum_{i=1}^{N} f\left(\alpha + \frac{2i-1}{2N}(\beta - \alpha)\right).$$

It is well known that for $f \in F_{\mathrm{reg}}^{0,\rho}([\alpha, \beta])$

$$\left| \int_\alpha^\beta f(x)dx - Q_N(\alpha, \beta, f) \right| \le C(\beta - \alpha)^{1+\rho} N^{-\rho}, \tag{1}$$

where C is some constant.

There are known complexity bounds for the integration problem both for smooth and piecewise smooth functions. For a smooth function form the Hölder class the complexity bounds are given int the following theorem (see [4]).

Theorem 1 *There exist constants c and C, such that for sufficiently small $\varepsilon > 0$*

$$c\ \varepsilon^{-\frac{1}{r+\rho}} \le \mathrm{comp}^{\mathrm{det}}(\varepsilon, F_{\mathrm{reg}}^{r,\rho}([a,b])) \le C\ \varepsilon^{-\frac{1}{r+\rho}}.$$

For a function with at most one irregularity point the rate of convergence from the smooth case is preserved when an adaptive algorithm is used (see [23, 32]). The complexity bounds are given in the following theorem.

Theorem 2 *There exist constants c and C, such that for sufficiently small $\varepsilon > 0$*

$$c\ \varepsilon^{-\frac{1}{r+\rho}} \le \mathrm{comp}^{\mathrm{det}}(\varepsilon, F_1^{r,\rho}([a,b])) \le C\ \varepsilon^{-\frac{1}{r+\rho}}.$$

It is known that the nonadaptive algorithms cannot preserve this rate of convergence. There are the following bounds (see [23]).

Theorem 3 *For every deterministic approximation U using m nonadaptive function evaluations*

$$\mathrm{e}^{\mathrm{det}}(U, F_1^{r,\rho}([a,b])) \ge \frac{D(b-a)}{2m}.$$

Let us now pass to the quantum setting. We need the following results on the summation problem. Let $g : \{0, 1, \ldots, N-1\} \to [0,1]$. Consider the problem of computing the discrete mean, that is the number $\displaystyle\frac{1}{N}\sum_{i=0}^{N-1} g(i)$ with precision $\varepsilon > 0$. Denote the quantum ε-complexity of this problem by $\mathrm{comp}^{\mathrm{quant}}(\varepsilon, N)$. There are known complexity bounds for this problem, see[33, 11, 3]

$$\mathrm{comp}^{\mathrm{quant}}(\varepsilon, N) = \Theta(\min\{N, \varepsilon^{-1}\}). \tag{2}$$

There are the following bounds on the quantum complexity of integration of Hölder regular functions, see [4].

Theorem 4 *There exist constants c and C, such that for sufficiently small $\varepsilon > 0$ it holds*

$$c\,\varepsilon^{-\frac{1}{r+\rho+1}} \leq \mathrm{comp}^{\mathrm{quant}}(\varepsilon, F^{r,\rho}_{\mathrm{reg}}([a,b])) \leq C\,\varepsilon^{-\frac{1}{r+\rho+1}}.$$

The integration algorithm of Novak first approximate f by a piecewise polynomial $P_n(f)$ using n classical deterministic evaluations of the function. Then

$$\int_a^b f(x)dx = \int_a^b P_n(f)(x)dx + \int_a^b [f - P_n(f)](x)dx.$$

Since the integral of the polynomial can be computed exactly it suffices to approximate the last integral. This is done using a quantum algorithm. Hence, the function integration error is equal to that of the quantum algorithm approximating the third integral. Since the nonadaptive approximation algorithms cannot preserve the rate of convergence for the approximation problem when the function is piecewise regular (see [24, 25]) this algorithm cannot be used directly in the class $F^{r,\rho}_1([a,b])$. Our aim is to adapt the algorithm of Novak for functions with at most one singularity.

4 Algorithm

We will present in this section the quantum algorithm for integrating a piecewise smooth function with one unknown singularity. First, we will define a test for localizing the singular point. For a function $f \in F^{r,\rho}_1([a,b])$ let $w^s_f(\alpha,\beta)$ be Lagrange polynomial of order s defined on $s+1$ equidistant points in $[\alpha,\beta]$, i.e.

$$w^s_f(\alpha,\beta)(x) = \sum_{i=0}^{s} f(t_i)\phi_i(x),$$

where $t_i = \alpha + i\,(\beta-\alpha)/s$, and $\phi_i(x) = \prod_{j=0,j\neq i}^{s} \frac{x-x_j}{x_i-x_j}$, $i = 0,1,\ldots,s$. Then we define a test as

$$A_f(\alpha,\hat{\alpha},\hat{\beta},\beta) = \max_{0\leq i\leq r} \frac{\left|w^r_f(\hat{\beta},\beta)(z_i) - w^r_f(\alpha,\hat{\alpha})(z_i)\right|}{\hat{h}^{r+\rho}},$$

where $\alpha < \hat{\alpha} < \hat{\beta} < \beta$, $z_i = \hat{\alpha} + i\,(\hat{\beta}-\hat{\alpha})/r$ and $\hat{h} = \beta - \alpha$. This quantity has the following properties (see [29, 32]).

Lemma 1 *There exists constant $C > 0$, such that for $a \leq \alpha < \hat{\alpha} < \hat{\beta} < \beta \leq b$, $\beta - \alpha \leq 2(\hat{\beta} - \hat{\alpha})$, $f \in F^{r,\rho}_1([a,b])$, if $\xi_1 \in (\hat{\alpha},\hat{\beta}]$ then we have*

$$\sup_{x\in[\hat{\alpha},\hat{\beta}]} \left|f(x) - w^r_f(\hat{\alpha},\hat{\beta})\right| \leq C(1 + A_f(\alpha,\hat{\alpha},\hat{\beta},\beta))\;\hat{h}^{r+\rho}.$$

Lemma 2 *There exists constant $C^* > 0$, such that for $a \leq \alpha < \hat{\alpha} < \hat{\beta} < \beta \leq b$ and $f \in F^{r,\rho}_{\mathrm{reg}}([\alpha,\beta])$*

$$A_f(\alpha,\hat{\alpha},\hat{\beta},\beta) \leq C^*.$$

We are now ready to define the algorithm localizing the singularity and computing the quantum approximation of the integral. Let $\varepsilon_1 > 0$ be the algorithm parameter. The algorithm for computing the integral $I(f) = \int_a^b f(x)dx$ works in the following steps.

1. Set $m = \left\lceil \varepsilon_1^{-1/(r+\rho+1)} \right\rceil$, $h = (b-a)/m$, $\delta = h^{r+\rho+2}$, $t_i = a + i\ h$ for $i = 0, \dots m$, and $M = \{t_0, t_1, \dots, t_m\}$.

2. Compute $i_0 = \arg\max_{0 \le i \le m-1} A_f(t_i, t_i+\delta, t_{i+1}-\delta, t_{i+1})$. If the maximum is achieved for two distinct indicators then go to Step 6.

3. Set $B = \emptyset$ and $[\alpha, \beta] = [t_{i_0}, t_{i_0+1}]$.

4. If $\beta - \alpha \le 4\delta$ go to Step 6. Otherwise, compute $v = (\alpha+\beta)/2$ and put $B \coloneqq B \cup \{v\}$.

5. Compute $A_f(\alpha, \alpha+\delta, v-\delta, v)$ and $A_f(v, v+\delta, \beta-\delta, \beta)$. If these quantities are equal then go to Step 6. Else choose as a new working interval $[\alpha, \beta]$ the interval $[\alpha, v]$ or $[v, \beta]$ for which the value of the test is greater. Go to Step 4.

6. Let $M \coloneqq M \cup B \eqqcolon \{\hat{t}_0, \hat{t}_1, \dots, \hat{t}_{m'}\}$.

7. Let $B' = \{\hat{t}_i + \delta\ :\ \hat{t}_{i+1} - \hat{t}_i > 4\delta,\ i = 0, 1, \dots, m'-1\}$ and $B'' = \{\hat{t}_{i+1} - \delta\ :\ \hat{t}_{i+1} - \hat{t}_i > 4\delta,\ i = 0, 1, \dots, m'-1\}$. Set $M \coloneqq M \cup B' \cup B'' \eqqcolon \{\bar{t}_0, \bar{t}_1, \dots, \bar{t}_{m''}\}$.

8. Define the approximation
$$q(x) = \begin{cases} w_f^r(\bar{t}_i, \bar{t}_{i+1})(x), & \text{if } x \in [\bar{t}_i, \bar{t}_{i+1}) \text{ and } \bar{t}_i \in B'; \\ 0, & \text{if } x \in [\bar{t}_i, \bar{t}_{i+1}) \text{ and } \bar{t}_i \notin B'. \end{cases}$$
with $q(b)$ given by continuity.

9. Let $Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f-q)$ for $\bar{t}_i \in B'$ be the midpoint rule with $N_i = \lceil (\bar{t}_{i+1} - \bar{t}_i)\varepsilon^{-\rho} \rceil$ approximating integral $\int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx$. Let $\tilde{Q}$ be the approximation of $\sum_{\bar{t}_i \in B'} Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q)$ computed by the optimal quantum algorithm for computing the mean within the precision ε_1 with a probability not less than $3/4$. We will describe in more detail how to compute $\tilde{Q}$ in the proof of Theorem 5 in the following section.

10. The final approximation U is given by
$$U(f) = I(q) + \tilde{Q}.$$

The algorithm uses the bisection method to localize the singularity. The number of bisection steps is $O(log(m))$. Thus, the total number of subintervals generated by the algorithm is $O(m + log(m)) = O(m)$. The aim of the bisection part of the algorithm is to catch the singularity point within a small interval of length $O(\delta)$. Due, to Lemma 1, we may still miss the dangerous singularity in the marginal part of the interval. That is why the additional points are added in Step 7. If the algorithm still misses the singularity, then we are sure that the singularity is not dangerous in the sense that function f is properly approximated by q. This will be shown in Lemma 3.

Note that the value of q on the marginal subintervals of length δ is set to 0 only to simplify the proof and may be changed to any bounded value. The values of q at points $\bar{t}_i \in M$ are also irrelevant.

We will show that such constructed q is a good approximation of f on all intervals such that q is not identically equal to 0.

Lemma 3 *There exists constant $C' > 0$ such that for every $f \in F_1^{r,\rho}([a,b])$ and every $\bar{t}_i \in B'$ we have*

$$\sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - q(x)| \leq C' h^{r+\rho}.$$

Proof. We proceed in a similar way as in [29, 32]. Suppose first that $\xi_1 \notin (\bar{t}_i, \bar{t}_{i+1}]$. So, $f \in F_{\text{reg}}^{r,\rho}([\bar{t}_i, \bar{t}_{i+1}])$. From the well-known bounds on the error of polynomial interpolation of a smooth function we have

$$\sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - q(x)| = \sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - w_f^r(\bar{t}_i, \bar{t}_{i+1})| \leq \frac{D}{r!} h^{r+\rho}$$

Suppose know that $\xi_1 \in (\bar{t}_i, \bar{t}_{i+1}]$. Using Lemma 1 with $\alpha = \bar{t}_{i-1}$, $\hat{\alpha} = \bar{t}_i$, $\hat{\beta} = \bar{t}_{i+1}$, $\beta = \bar{t}_{i+2}$ we have

$$\begin{aligned}
\sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - q(x)| &= \sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - w_f^r(\bar{t}_i, \bar{t}_{i+1})| \\
&\leq C \left(1 + A_f(\bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \bar{t}_{i+2})\right) (t_{i+2} - t_{i-1})^{r+\rho} \\
&\leq C \left(1 + A_f(\bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \bar{t}_{i+2})\right) h^{r+\rho}.
\end{aligned}$$

The test $A_f(\bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \bar{t}_{i+2})$ was computed in Step 2 or Step 5 of the algorithm and was not selected as the largest. So, in this step, some other interval was selected, with a larger value of the test, but then due to Lemma 2 it is less than C^* (f is smooth on the selected interval since the singularity is unique). Thus, we have $A_f(\bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \bar{t}_{i+2}) \leq C^*$ and

$$\sup_{x \in [\bar{t}_i, \bar{t}_{i+1}]} |f(x) - q(x)| \leq C \left(1 + C^*\right) h^{r+\rho}.$$

□

5 Main result

The algorithm presented in the previous section has the following properties.

Theorem 5 *There exist constants C_1 and C_2, such that for sufficiently small $\varepsilon_1 > 0$ we have*

$$e^{\text{quant}}(U, F_1^{r,\rho}([a,b])) \leq C_1 \varepsilon_1$$

and

$$\text{cost}^{\text{quant}}(U, F_1^{r,\rho}([a,b]) \leq C_2 \varepsilon_1^{-\frac{1}{r+\rho+1}}.$$

Proof. Let us find the bounds on the error of the algorithm presented in the previous section. The difference between the output of the algorithm and the true value of the integral is bounded by

$$\begin{aligned}
&|U(f) - I(f)| = |I(q) + \tilde{Q} - I(f)| = |I(q) + \tilde{Q} - I(q) - I(f-q)| \\
&\leq \left| \tilde{Q} - \sum_{\bar{t}_i \in B'} Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) \right| + \left| \sum_{\bar{t}_i \in B'} Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_a^b (f(x) - q(x)) dx \right|
\end{aligned}$$

$$\leq \left|\tilde{Q} - \sum_{\bar{t}_i \in B'} Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q)\right| + \sum_{\bar{t}_i \in B'} \left|Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right|$$
$$+ \sum_{\bar{t}_i \in M \setminus B'} \int_{\bar{t}_i}^{\bar{t}_{i+1}} |f(x) - q(x)|dx =: I_1 + I_2 + I_3. \tag{3}$$

We start with the bounds on I_3. For $x \in [\bar{t}_i, \bar{t}_{i+1})$, where $\bar{t}_i \in M \setminus B'$ we have $q(x) = 0$. The length of such an interval is equal to δ when the interval was created by adding the points from B' or B'' and is less than 4δ if the interval is the result of the bisection. Thus, taking into account the fact that f is bounded by D, we have for $\bar{t}_i \in M \setminus B'$

$$\int_{\bar{t}_i}^{\bar{t}_{i+1}} |f(x) - q(x)|dx = \int_{\bar{t}_i}^{\bar{t}_{i+1}} |f(x)|dx \leq D(\bar{t}_{i+1} - \bar{t}_i) \leq 4\delta D = 4Dh^{r+\rho+2}.$$

Since the number of intervals is $O(m)$, we get

$$I_3 = O\left(m4Dh^{r+\rho+2}\right) = O\left(m^{-(r+\rho+1)}\right) = O(\varepsilon_1). \tag{4}$$

We will now find the bounds on I_2 – the error of the midpoint rule for the function $f - q$. Consider interval $[\bar{t}_i, \bar{t}_{i+1})$ for $\bar{t}_i \in B'$. Suppose now that $[\bar{t}_i, \bar{t}_{i+1}]$ does not contain the singular point ξ_1. In this case $f \in F^{r,\rho}_{\text{reg}}([\bar{t}_i, \bar{t}_{i+1}])$ and q is a polynomial, so $f - q \in F^{0,\rho}_{\text{reg}}([\bar{t}_i, \bar{t}_{i+1}))$. From (1) we get

$$\left|Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right| \leq C(\bar{t}_{i+1} - \bar{t}_i)^{1+\rho} N_i^{-\rho}$$

for some constant C. Since $N_i = \lceil \varepsilon_1^{-1/\rho} (\bar{t}_{i+1} - \bar{t}_i)\rceil$, we have

$$\begin{aligned}\left|Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right| &\leq C(\bar{t}_{i+1} - \bar{t}_i)^{1+\rho}(\bar{t}_{i+1} - \bar{t}_i)^{-\rho}\varepsilon_1 \\ &= C\varepsilon_1(\bar{t}_{i+1} - \bar{t}_i).\end{aligned} \tag{5}$$

On the interval $[\bar{t}_i, \bar{t}_{i+1}]$ containing the singular point ξ_1, we use Lemma 3. We have that for $x \in [\bar{t}_i, \bar{t}_{i+1}]$ we have $|f(x) - q(x)| \leq C'h^{r+\rho}$. Since the midpoint rule is the mean of the function values multiplied by the length of the interval, we get

$$\begin{aligned}&\left|Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right| \\ &\quad \leq |Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q)| + \left|\int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right| \leq 2C'h^{r+\rho}(\bar{t}_{i+1} - \bar{t}_i) \\ &\quad = O(h^{r+\rho+1}) = O(\varepsilon_1).\end{aligned} \tag{6}$$

Summing up the bounds (5) and (6) for $\bar{t}_i \in B'$ we get

$$I_2 = \sum_{\bar{t}_i \in B'} \left|Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f - q) - \int_{\bar{t}_i}^{\bar{t}_{i+1}} (f(x) - q(x))dx\right| \leq (b - a)C\varepsilon_1 + O(\varepsilon_1) = O(\varepsilon_1). \tag{7}$$

Let us now bound I_1. First, we will show how to compute $\tilde{Q}$. Note that from the definition of the midpoint rule we have

$$Q \coloneqq \sum_{\bar{t}_i \in B'} Q_{N_i}(\bar{t}_i, \bar{t}_{i+1}, f-q) = \sum_{\bar{t}_i \in B'} \frac{\bar{t}_{i+1} - \bar{t}_i}{N_i} \sum_{j=1}^{N_i} (f-q)(z_i^j),$$

where $z_i^j = \bar{t}_i + \frac{2j-1}{2N_i}(\bar{t}_{i+1} - \bar{t}_i)$ and $N_i = \lceil \varepsilon_1^{-1/\rho}(\bar{t}_{i+1} - \bar{t}_i) \rceil$. Let $N = \sum_{i:\bar{t}_i \in B'} N_i$. Then

$$Q = \frac{1}{N} \sum_{\bar{t}_i \in B'} \sum_{j=1}^{N_i} \frac{(\bar{t}_{i+1} - \bar{t}_i)N}{N_i}(f-q)(z_i^j).$$

Since $\frac{(\bar{t}_{i+1} - \bar{t}_i)N}{N_i} \simeq b-a$ and due to Lemma 3 $(f-q)(z_i^j) \leq C^* h^{r+\rho}$, Q is a mean of N numbers of order $O(h^{r+\rho})$. Let $\bar{Q}$ be the quantum approximation of

$$h^{-(r+\rho)}Q = \frac{1}{N} \sum_{\bar{t}_i \in B'} \sum_{j=1}^{N_i} h^{-(r+\rho)} \frac{(\bar{t}_{i+1} - \bar{t}_i)N}{N_i}(f-q)(z_i^j),$$

the mean on N numbers of order $O(1)$, computed by the optimal quantum algorithm for computing the mean within the precision $\varepsilon_1 h^{-(r+\rho)}$ with a probability not less than $3/4$. Then, $\tilde{Q} = h^{r+\rho}\bar{Q}$. So, we have

$$I_1 = \left|\tilde{Q} - Q\right| = h^{r+\rho}|\bar{Q} - h^{-(r+\rho)}Q| \leq h^{r+\rho}\varepsilon_1 \; h^{-(r+\rho)} = \varepsilon_1 \tag{8}$$

with a probability not less than $3/4$. Returning to (3) with bounds (4), (7), and (8) we get $|U(f) - I(f)| = O(\varepsilon_1)$ with a probability not less than $3/4$. Thus,

$$e^{\text{quant}}(U, F_1^{r,\rho}([a,b])) \leq C_1 \varepsilon_1$$

for some constant C_1.

We will now find the bounds on the cost of the algorithm. Let us start with the cost of classical computations. The test is computed on each of m initial subintervals, and each of $\log(m)$ bisection steps. Each test needs $2r+2$ function values. So, the total cost of computing tests is $O(m)$. To construct q, the Lagrange polynomial of order r is computed on every interval $\bar{t}_i, \bar{t}_{i+1}$ for $\bar{t}_i \in B'$. Thus, the total cost of computing q is $O(m)$. So, the total classical cost is $O(m) = O(\varepsilon_1^{-1/(r+\rho+1)})$.
On a quantum computer the approximation $\bar{Q}$ of the mean of N numbers within the precision $\varepsilon_1 h^{-(r+\rho}$ with probability at least $3/4$ is computed. Due to (2) the total quantum cost is $O(\varepsilon_1^{-1}h^{r+\rho}) = O(\varepsilon_1^{-1}\varepsilon_1^{(r+\rho)/(r+\rho+1)}) = O(\varepsilon_1^{-1/(r+\rho+1)})$. The total cost is a sum of the classical and the quantum cost, so

$$\text{cost}^{\text{quant}}(U, F_1^{r,\rho}([a,b]) \leq C_2 \varepsilon_1^{-\frac{1}{r+\rho+1}}$$

for some constant C_2, which finishes the proof. □

From the theorem above and the lower complexity bound for the smooth case it follows.

Theorem 6 *For sufficiently small $\varepsilon > 0$*

$$\mathrm{comp}^{\mathrm{quant}}(\varepsilon, F_1^{r,\rho}([a,b])) = \Theta\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{r+\rho+1}}\right).$$

Proof. The upper bound is a direct result of Theorem 5. To get error bound ε_1 one needs to take $\varepsilon_1 = \varepsilon/C_1$. Then, the cost is bounded by $C_2 C_1^{1/(r+\rho+1)} \varepsilon^{-1/(r+\rho+1)}$.

Since $F_1^{r,\rho} \subset F_{\mathrm{reg}}^{r,\rho}$, the lower complexity bounds follow from Theorem 4. □

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