Quantum Information and Computation, Vol. 23, No. 7&8 (2023) 0562–0576

ON UNITAL QUBIT CHANNELS

MAN-DUEN CHOI
Department of Mathematics, University of Toronto,
Toronto, Ontario, Canada M5S 2E4.
choi@math.toronto.edu

CHI-KWONG LI
Department of Mathematics, College of William & Mary,
Williamsburg, Virginia 23187-8795, USA.
ckli@math.wm.edu

In memory of Professor Chandler Davis.

Received January 3, 2023
Revised April 20, 2023

A canonical form for unital qubit channels under local unitary transforms is obtained. In particular, it is shown that the eigenvalues of the Choi matrix of a unital quantum channel form a complete set of invariants of the canonical form. It follows immediately that every unital qubit channel is the average of four unitary channels. More generally, a unital qubit channel can be expressed as the convex combination of unitary channels with convex coefficients $p_1, \ldots, p_m$ as long as $2(p_1, \ldots, p_m)$ is majorized by the vector of eigenvalues of the Choi matrix of the channel. A unital qubit channel in the canonical form will transform the Bloch sphere onto an ellipsoid. We look into the detailed structure of the natural linear maps sending the Bloch sphere onto a corresponding ellipsoid.

Keywords: Unital qubit channel, completely positive linear map, Choi matrix.

1 Introduction

The study of unital qubit channels is motivated by their importance in understanding the behavior of quantum systems subject to noise and errors. Unital qubit channels preserve the set of quantum states that form a basis for a qubit system, making them a natural class of channels to consider in the context of quantum error correction and fault tolerance. Furthermore, unital qubit channels form an important class of quantum channels in their own right, and studying their properties lead to better understanding of the fundamental properties of general quantum channels. Finally, unital qubit channels play a key role in quantum communication, where they can limit the capacity and reliability of quantum communication systems. In fact, there are interesting open research problems in this area. For example, determining the capacity of unital qubit channels is an important and challenging problem in quantum information theory. Another interesting problem is understanding the role of unital qubit channels in quantum state discrimination, which is a key task in many quantum information processing applications. One may see [1, 6, 8, 9, 14, 15] and their references for some general background.

In this paper, we obtain some basic results on unital qubit channels. To facilitate our discussion, let us introduce some technical background. In the mathematical framework of quantum mechanics, quantum states are density matrices, i.e., positive semidefinite matrices with trace one, and quantum channels (operations) are trace preserving completely positive linear maps $\Phi : M_n \to M_m$, where M_k denotes the set of $k \times k$ complex matrices, admitting the operator sum representation

$$\Phi(A) = \sum_{j=1}^{r} F_j A F_j^* \qquad \text{for all } A \in M_n \tag{1.1}$$

for some $m \times n$ matrices $F_1, \ldots, F_r$ satisfying $\sum_{j=1}^{r} F_j^* F_j = I_n$.

For a linear map $\Phi : M_n \to M_m$, its Choi matrix is defined as

$$C(\Phi) = (\Phi(E_{ij}))_{1 \le i,j \le n} = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij}) \in M_n(M_m) \equiv M_n \otimes M_m,$$

where $\{E_{ij} : 1 \le i, j \le n\}$ is the standard basis for M_n. By a result in [3], a linear map $\Phi : M_n \to M_m$ is a quantum channel if and only if $C(\Phi)$ is positive semidefinite and $\operatorname{tr}\Phi(E_{ij}) = \delta_{ij}$, the Kronecker delta.

We say that two quantum channels $\Phi, \Psi : M_n \to M_m$ are *unitarily equivalent* if there are unitary matrices $U \in M_n$ and $V \in M_m$ such that $\Phi(A) = V\Psi(UAU^*)V^*$ for all $A \in M_n$. It turns out that this happens if and only if $C(\Phi) = (U^t \otimes V)C(\Psi)(U^t \otimes V)^*$, where X^t denotes the transpose of the matrix X. This follows readily from the following lemma.

Lemma 1.1 *Let $\Phi : M_n \to M_k$ be a linear map. Then for any $R, S \in M_n$,*

$$\sum_{i,j} E_{ij} \otimes \Phi(RE_{ij}S) = \sum_{i,j} R^t E_{ij} S^t \otimes \Phi(E_{ij}).$$

Proof. Let $R = (r_{ij}) = \sum r_{ij}E_{ij}$ and $S = (s_{ij}) = \sum s_{ij}E_{ij}$. Then

$$\begin{aligned} \sum_{i,j} E_{i,j} \otimes \Phi(RE_{ij}S) &= \sum_{i,j,p,q} E_{ij} \otimes \Phi(r_{pi}s_{jq}E_{pq}) \\ = \quad \sum_{i,j,p,q} r_{pi}s_{jq}E_{ij} \otimes \Phi(E_{p,q}) &= \sum_{p,q} R^t E_{pq} S^t \otimes \Phi(E_{p,q}), \end{aligned}$$

which is the same as $\sum_{i,j} R^t E_{ij} S^t \otimes \Phi(E_{ij})$. ∎

By Lemma 1.1, if $\Phi(A) = V\Psi(UAU^*)V^*$ for all $A \in M_n$, then

$$\begin{aligned} C(\Phi) &= (I \otimes V)(\sum_{ij} E_{ij} \otimes \Psi(UE_{ij}U^*))(I \otimes V)^* \\ &= (I_n \otimes V)(U^t \otimes I_m)C(\Psi)(U^t \otimes I_m)^*(I_n \otimes V)^* \\ &= (U^t \otimes V)C(\Psi)(U^t \otimes V)^*. \end{aligned}$$

In Section 2, we make use of the above observation to show that if $\Phi : M_2 \to M_2$ is a unital qubit channel such that $C(\Phi)$ has eigenvalues $\lambda_1 \ge \cdots \ge \lambda_4 \ge 0$ summing up to two, then there are unitary matrices $U, V \in M_2$ such that the map $A \mapsto V\Phi(UAU^*)V^*$ has the form

$$A \mapsto \frac{1}{2}(\lambda_1 A + \lambda_2 XAX + \lambda_3 YAY + \lambda_4 ZAZ), \tag{1.2}$$

where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.3}$$

are the Pauli matrices. Consequently, two unital qubit channels are unitarily equivalent if and only if their Choi matrices have the same eigenvalues.

Using the results in Section 2, we show that every unital qubit channel Φ can be written as the average of four unitary channels, i.e., maps of the form $A \mapsto UAU^*$ for some unitary U in Section 3. More generally, suppose k positive eigenvalues $\lambda_1, \ldots, \lambda_k$, and $p_1, \ldots, p_m$ are positive numbers summing up to 1. Then there are unitary $V_1, \ldots, V_m$ such that Φ has the form $A \mapsto \sum_{j=1}^m p_j V_j A V_j^*$ $(p_1, \ldots, p_m)$ is majorized by $\frac{1}{2}(\lambda_1, \ldots, \lambda_4)$; see Section 3 for the definition of majorization relation between two nonnegative vectors of different sizes.

One can identify the set of 2×2 density matrices:

$$\left\{ \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} : x, y, z \in \mathbf{R},\ x^2 + y^2 + z^2 \leq 1 \right\}$$

with the Bloch (unit) ball

$$\mathcal{B} = \{(x, y, z) : x, y, z \in \mathbf{R},\ x^2 + y^2 + z^2 \leq 1\} \subseteq \mathbf{R}^3.$$

In our discussion, both $\mathbf{R}^{1\times 3}$ *and* $\mathbf{R}^{3\times 1}$ *will be referred to as* $\mathbf{R}^3$. A unital qubit channel in the form (1.2) will transform the Bloch ball (linearly) to an ellipsoid (including interior) of the form

$$\mathcal{E} = \{(d_1 x, d_2 y, d_3 z) : x, y, z \in \mathbf{R},\ x^2 + y^2 + z^2 \leq 1\}$$

with $2(d_1, d_2, d_3) = (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)$; see Theorem 4.1. Note that $\mathcal{E}$ will degenerate to a point, a line segment, or an elliptical disk if 3, 2, or 1 of $d_1, \ldots, d_3$ are zero. Using the convention $0/0 = 0$, we have

$$\mathcal{E} = \{(x, y, z) \in \mathbf{R}^3 : (x/d_1)^2 + (y/d_2)^2 + (z/d_3)^2 \leq 1\}.$$

Section 4 is devoted to the complete classification of quantum channels sending the Bloch ball $\mathcal{B}$ onto an ellipsoid $\mathcal{E}$ for the unital qubit channel in the canonical form (1.2) in terms of (d_1, d_2, d_3).

2 A canonical form of unital qubit channels

We begin with the following simple lemma.

Lemma 2.1 *Let* $\Phi : M_2 \to M_2$ *be a unital linear map that preserves trace and Hermitian matrices. Then there are unitary* $U, V \in M_2$ *such that the map* $A \mapsto V\Phi(UAU^*)V^*$ *has Choi matrix of the form*

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & 1-a & c & 0 \\ 0 & c & 1-a & 0 \\ b & 0 & 0 & a \end{pmatrix} \quad \text{with} \quad b, c \geq 0 \quad \text{and} \quad 2a \geq 1 + b + c. \tag{2.1}$$

Proof. Let $P_0 \in M_2$ be a rank one orthogonal projection such that

$$\|\Phi(P_0)\| = \max\{\|\Phi(P)\| : P \text{ is a rank one orthogonal projection}\}.$$

Then there are unitary $U, V \in M_2$ such that $P_0 = UE_{11}U^*$ and $\Phi(P_0) = V^*\mathrm{diag}\,(1-a, a)V$ with $\|\Phi(P_0)\| = |a|$. So, $|a| \geq |1-a|$ implies that $a \geq 1/2$. Note that the map $\hat{\Phi} : A \mapsto V\Phi(UAU^*)V^*$ also preserves traces and Hermitian matrices; its Choi matrix $C(\hat{\Phi})$ has the form

$$\begin{pmatrix} a & 0 & d & b\mu_1 \\ 0 & 1-a & c\mu_2 & -d \\ \bar{d} & c\bar{\mu}_2 & 1-a & 0 \\ b\bar{\mu}_1 & -\bar{d} & 0 & a \end{pmatrix} \quad \text{with } d, \mu_1, \mu_2 \in \mathbf{C},\ |\mu_1| = |\mu_2| = 1, \text{ and } b, c \geq 0.$$

Moreover, $a = \max\{\|\hat{\Phi}(P)\| : P \text{ is a rank one orthogonal projection}\}$. We claim that $d = 0$. If not, we may let $w = (\cos\theta, \sin\theta|d|/d)^t$. Then the $(1,1)$ entry of $\hat{\Phi}(ww^*)$ equals

$$\begin{aligned} w^* \begin{pmatrix} a & d \\ \bar{d} & 1-a \end{pmatrix} w &= a\cos^2\theta + 2\sin\theta\cos\theta|d| + (1-a)\sin^2\theta \\ &= a + 2\sin\theta[|d|\cos\theta - (a-1/2)\sin\theta] > a \end{aligned}$$

if $\sin\theta > 0$ and $\cos\theta > (a-1/2)\sin\theta/|d|$, which is a contraction. Thus, $d = 0$ as asserted.

Let $D_1 = \mathrm{diag}\,(e^{i\theta}, 1)$ and $D_2 = \mathrm{diag}\,(e^{i\phi}, 1)$ such that $\mu_1 e^{i\phi}$ and $\mu_2 e^{-i\phi}$ have the same argument $-\theta$. We may further replace $\hat{\Phi}$ by the map $\tilde{\Phi} : T \mapsto D_2(\hat{\Phi}(D_1 T D_1^*))D_2^*$. Then the Choi matrix $C(\tilde{\Phi})$ has the asserted form (2.1) with $b, c \geq 0$.

To prove that $2a \geq 1 + b + c$, consider $P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\tilde{\Phi}(P) = \frac{1}{2}\begin{pmatrix} 1 & b+c \\ b+c & 1 \end{pmatrix}$ so that

$$a = \|\Phi(P_0)\| \geq \|\tilde{\Phi}(P)\| = \frac{1}{2}(1 + b + c). \qquad \blacksquare$$

Applying the above lemma to unital qubit channels, we have the following.

Theorem 2.2 *Let $\Phi : M_2 \to M_2$ be a unital linear map that preserve trace and Hermitian matrices. Then the Choi matrix of Φ has real eigenvalues summing up to two. If the eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_4$, then there are unitary $U, V \in M_2$ such that the map $A \mapsto V\Phi(UAU^*)V^*$ has the form*

$$A \mapsto \frac{1}{2}(\lambda_1 A + \lambda_2 ZAZ + \lambda_3 XAX + \lambda_4 YAY), \tag{2.2}$$

where X, Y, Z are the Pauli matrices described in (1.3)*, and its Choi matrix has the form*

$$\frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_2 & 0 & 0 & \lambda_1-\lambda_2 \\ 0 & \lambda_3+\lambda_4 & \lambda_3-\lambda_4 & 0 \\ 0 & \lambda_3-\lambda_4 & \lambda_3+\lambda_4 & 0 \\ \lambda_1-\lambda_2 & 0 & 0 & \lambda_1+\lambda_2 \end{pmatrix}. \tag{2.3}$$

In particular, Φ is a unital qubit channel if and only if $\lambda_4 \geq 0$.

Proof. By Lemma 2.1, there are unitary $U, V \in M_2$ such that the Choi matrix of the map $A \mapsto V\Phi(UAU^*)V^*$ has the form (2.1). Since $b, c \geq 0$ and $2a \geq 1 + b + c$, we see that

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (a+b, a-b, 1-a+c, 1-a-c).$$

Thus, the map $A \mapsto V\Phi(UAU^*)V^*$ has the form (2.2) with Choi matrix (2.3).

The last assertion follows from the fact that Φ is a unital qubit channel if and only if its Choi matrix is positive semidefinite. ■

By Theorem 2.2, we see that the four eigenvalues of the Choi matrix of a unital qubit channel form a complete set of invariants under unitary equivalence.

Corollary 2.3 *Let $\Phi, \Psi : M_2 \to M_2$ be unital linear maps that preserve trace and Hermitian matrices. Then Φ and Ψ are unitarily equivalent if and only if $C(\Phi)$ and $C(\Psi)$ have the same eigenvalues (counting multiplicity). In particular, two unital qubit channels are unitarily equivalent if and only if their Choi matrices have the same eigenvalues.*

Proof. The if part follows from Theorem 2.2. The only if part follows from Lemma 1.1. ■

Several remarks are in order. First, up to unitary equivalence the Choi matrix always has the form (2.3), and the operator sum representation has the form (2.2).

Second, the roles of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in (2.2) are symmetric because a unital qubit channel with Choi matrix

$$\frac{1}{2}\begin{pmatrix} \lambda_{j_1}+\lambda_{j_2} & 0 & 0 & \lambda_{j_1}-\lambda_{j_2} \\ 0 & \lambda_{j_3}+\lambda_{j_4} & \lambda_{j_3}-\lambda_{j_4} & 0 \\ 0 & \lambda_{j_3}-\lambda_{j_4} & \lambda_{j_3}+\lambda_{j_4} & 0 \\ \lambda_{j_1}-\lambda_{j_2} & 0 & 0 & \lambda_{j_1}+\lambda_{j_2} \end{pmatrix}$$

for any permutation (j_1, j_2, j_3, j_4), the Choi matrix has eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. By Theorem 2.2, it is equivalent to the channel with Choi matrix (2.3). In the following, we will often consider linear map on M_2 of the form $A \mapsto \mu_1 A + \mu_2 XAX + \mu_3 YAY + \mu_4 ZAZ$; its Choi matrix will have eigenvalues $2\mu_1, \ldots, 2\mu_4$.

Third, under unitary similarity, the eigenvalues form a complete set of invariants of Hermitian matrices. In general, two Hermitian matrices in $M_n(M_m)$ with the same eigenvalues may not be similar under a unitary matrix of the form $U \otimes V$. It is interesting that the Choi matrices of two unital qubit channels are unitarily similar if and only if they are unitarily similar via a matrix of the form $U \otimes V$.

Example 2.4 *Define $\Lambda : M_2 \to M_2$ by $\Lambda(A) = \frac{1}{2}(\mathrm{tr}\, A)I$ for all $A \in M_2$. Then Λ is the only unital qubit channel satisfying $V\Lambda(UAU^*)V^* = \Lambda(A)$ for all unitary $U, V \in M_2$. Moreover, $\Lambda(A) = \frac{1}{4}(A + XAX + YAY + ZAZ)$, and the Choi matrix $C(\Lambda) = \frac{1}{2}I_4$ with four equal eigenvalues.*

Using this example and the previous remark, we can prove the following.

Corollary 2.5 *Let $\Phi, \Psi : M_2 \to M_2$ be linear maps of the form*

$$A \mapsto \mu_1 A + \mu_2 XAX + \mu_3 YAY + \mu_4 ZAZ \quad \text{and} \quad A \mapsto \nu_1 A + \nu_2 XAX + \nu_3 YAY + \nu_4 ZAZ,$$

respectively, where $\mu_j, \nu_j \in \mathbf{R}$ for $j = 1, \ldots, 4$, and X, Y, Z are the Pauli matrices defined as in (1.3). Then Φ and Ψ are unitarily equivalent if and only if $(\mu_1, \ldots, \mu_4)$ is a permutation of $(\nu_1, \ldots, \nu_4)$.

Proof. Suppose Φ and Ψ are unitarily equivalent. By Lemma 1.1 and the remark after it, there are unitary $U, V \in M_2$ such that $C(\Phi) = (U^t \otimes V)C(\Psi)(U^t \otimes V)^*$. So, $C(\Phi)$ and $C(\Psi)$ have the same eigenvalues, which are $2\mu_1, \ldots, 2\mu_4$, and $2\nu_1, \ldots, 2\nu_4$, respectively. Thus, $(\mu_1, \ldots, \mu_4)$ is a permutation of $(\nu_1, \ldots, \nu_4)$.

Conversely, if $(\mu_1, \ldots, \mu_4)$ is a permutation of $(\nu_1, \ldots, \nu_4)$, then $\sum_{j=1}^4 \mu_j = \sum_{j=1}^4 \nu_j = a$. Suppose Λ is defined as in Example 2.4. Then $\hat{\Phi} = \Phi + (1-a)\Lambda$ and $\hat{\Psi} = \Psi + (1-a)\Lambda$ are unital linear maps preserving trace and Hermitian matrices. Let $\xi = (1-a)/2$. Then

$$C(\hat{\Phi}) = C(\Phi) + \xi I \quad \text{and} \quad C(\hat{\Psi}) = C(\Psi) + \xi I$$

have the same eigenvalues $2\mu_1 + \xi, \ldots, 2\mu_4 + \xi$. By Corollary 2.3, $\hat{\Phi}$ and $\hat{\Psi}$ are unitarily equivalent. So, there are unitary $U, V \in M_2$ such that $C(\hat{\Phi}) = (U^t \otimes V)C(\hat{\Psi})(U^t \otimes V)^*$. Clearly, we also have $C(\Phi) = (U^t \otimes V)C(\Psi)(U^t \otimes V)^*$. Hence, $\Phi(A) = V\Psi(UAU^*)V^*$ for all A. So, Φ and Ψ are unitarily equivalent. ■

Instead of using the unitary equivalence via the Choi matrices, we can show that a map on M_2 of the form $A \mapsto \mu_1 A + \mu_2 XAX + \mu_3 YAY + \mu_4 ZAZ$ is unitarily equivalent to a map on M_2 of the form $A \mapsto \mu_{j_1} A + \mu_{j_2} XAX + \mu_{j_3} YAY + \mu_{j_4} ZAZ$ for any permutation (j_1, j_2, j_3, j_4) of $(1, 2, 3, 4)$ as follows. Let $H = (X+Z)/\sqrt{2}$ and $H_1 = (X+Y)/\sqrt{2}$. Using the anti-commuting relations $XY = -YX = iZ, YZ = -ZY = iX, ZX = -XZ = iY$, we have

$$\begin{aligned} A &\mapsto \Psi_1(A) = H\Psi(HAH)H = \mu_1 A + \mu_3 ZAZ + \mu_2 XAX + \mu_4 YAY, \\ A &\mapsto \Psi_2(A) = H_1\Psi(H_1AH_1)H_1 = \mu_1 A + \mu_2 ZAZ + \mu_4 XAX + \mu_3 YAY, \\ A &\mapsto \Psi_3(A) = Z\Psi(A)Z = \mu_2 A + \mu_1 ZAZ + \mu_4 XAX + \mu_3 YAY. \end{aligned}$$

The composition of Ψ_2 and Ψ_3 yields

$$A \mapsto \Psi_4(A) = ZH_1\Psi(H_1AH_1)H_1Z = \mu_2 A + \mu_1 ZAZ + \mu_3 XAX + \mu_4 YAY.$$

It is known that the group of permutations of $\{1, 2, 3, 4\}$ is generated by the transpositions (2-cycles) $(1, 2), (2, 3), (3, 4)$. Thus, we may get any permutation $(\mu_{i_1}, \ldots, \mu_{i_4})$ of $(\mu_1, \ldots, \mu_4)$ by a composition of Ψ_1, Ψ_2, Ψ_4.

Remark 2.6 We note that one can derive Lemma 2.1 via the correspondence between linear maps on $\mathbf{R}^3$ and unital trace-preserving maps $\Phi : M_2 \to M_2$ that preserve Hermitian matrices; see [9, 10, 12, 14]. We gave a direct proof of Lemma 2.1 and used it to deduce Theorem 2.2. Moreover, we showed that the matrix in (2.3) is a canonical form for the Choi matrix of a unital trace-preserving map $\Phi : M_2 \to M_2$ that preserves Hermitian matrices, where $\lambda_1 \geq \cdots \geq \lambda_4$ are the eigenvalues of $C(\Phi)$. This led to Corollary 2.3 and Corollary 2.5, which provide simple tests for two unital qubit channels to be unitarily equivalent.

3 A unital qubit channel as combination of unitary channels

It is known that unitary channels are the extreme points of the set of unital qubit channels; e.g., see [7]. We have the following.

Theorem 3.1 *Let Φ be a unital qubit channel. Then Φ is the average of four unitary channels.*

Proof. Note that the Choi matrix of the unitary channel $A \mapsto VAV^*$ equals vv^* with $v = (\alpha, -\beta, \bar{\beta}, \bar{\alpha})^t$ if $V = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}$. As a result, if $\alpha, \beta \in \mathbf{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$, then for

$$V_1 = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}, \quad V_2 = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad V_3 = \begin{pmatrix} \bar{\alpha} & \beta \\ -\bar{\beta} & \alpha \end{pmatrix}, \quad V_4 = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix},$$

we can let

$$v_1 = (\alpha, -\beta, \bar{\beta}, \bar{\alpha})^t, \quad v_2 = (\alpha, \beta, -\bar{\beta}, \bar{\alpha})^t, \quad v_3 = (\bar{\alpha}, -\bar{\beta}, \beta, \alpha)^t, \quad v_4 = (\bar{\alpha}, \bar{\beta}, -\beta, \alpha)^t,$$

so that the Choi matrix of the unital channel $A \mapsto \frac{1}{4}\sum_{j=1}^4 V_j A V_j^*$ equals

$$\frac{1}{4}\sum_{j=1}^4 v_j v_j^* = \begin{pmatrix} |\alpha|^2 & 0 & 0 & \frac{1}{2}(\alpha^2+\bar{\alpha}^2) \\ 0 & |\beta|^2 & \frac{-1}{2}(\beta^2+\bar{\beta}^2) & 0 \\ 0 & \frac{-1}{2}(\beta^2+\bar{\beta}^2) & |\beta|^2 & 0 \\ \frac{1}{2}(\alpha^2+\bar{\alpha}^2) & 0 & 0 & |\alpha|^2 \end{pmatrix}.$$

Now, suppose Φ is a unital channel. By Theorem 2.2, up to a local unitary similarity transform, a unital qubit channel has Choi matrix

$$\frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_2 & 0 & 0 & \lambda_1-\lambda_2 \\ 0 & \lambda_3+\lambda_4 & \lambda_3-\lambda_4 & 0 \\ 0 & \lambda_3-\lambda_4 & \lambda_3+\lambda_4 & 0 \\ \lambda_1-\lambda_2 & 0 & 0 & \lambda_1+\lambda_2 \end{pmatrix}$$

with $\lambda_1, \dots, \lambda_4 \geq 0$ summing up to two. Let $\alpha = \frac{1}{\sqrt{2}}(\sqrt{\lambda_1} + i\sqrt{\lambda_2}), \beta = \frac{1}{\sqrt{2}}(\sqrt{\lambda_4} + i\sqrt{\lambda_3})$. Then

$$|\alpha|^2 = \frac{1}{2}(\lambda_1+\lambda_2), \quad \alpha^2+\bar{\alpha}^2 = \lambda_1-\lambda_2, \quad |\beta|^2 = \frac{1}{2}(\lambda_3+\lambda_4), \quad \beta^2+\bar{\beta}^2 = \lambda_4-\lambda_3.$$

As a result, up to a unitary similarity transform, Φ has the form $A \mapsto \frac{1}{4}\sum_{j=1}^4 V_j A V_j^*$ as defined above. ■

The set $\mathcal{S}$ of unital qubit channels is linearly isomorphic to the compact convex set of Choi matrices of unital qubit channels, that are 4×4 positive semidefinite matrices in block form $(C_{ij})_{1 \leq i,j \leq 2}$ with $C_{11}, C_{12}, C_{21}, C_{22} \in M_2$ satisfying $C_{11} + C_{22} = I_2$, $\operatorname{tr} C_{11} = \operatorname{tr} C_{22} = 1$ and $\operatorname{tr} C_{12} = \operatorname{tr} C_{21} = 0$. Thus, the set of Choi matrices has real affine dimension 9, and it is known that the Choi matrices of unitary channels are extreme points of the convex set; see [7]. By the Caratheodory theorem, every element in $\mathcal{S}$ is a convex combination of no more than 10 extreme points. It is remarkable that Theorem 3.1 ensures that every element $\Phi \in \mathcal{S}$ can be written as the average of four extreme points.

In the following, we will determine all the possible convex coefficients $p_1, \dots, p_m$ so that $\Phi = \sum_{j=1}^m p_j \Psi_j$ for some unitary channels $\Psi_1, \dots, \Psi_m$. Theorem 3.2 asserts that the set of such $(p_1, \dots, p_m)$ can be completely determined by the eigenvalues of the Choi matrix of Φ. To achieve this, we need the concept of majorization.

Recall that a vector $u \in \mathbf{R}^m$ is *majorized* by another vector $v \in \mathbf{R}^m$, denoted by $u \prec v$, if the sum of the k largest entries of u is not larger than that of v for $k = 1, \dots, m$, with equality holding for $k = m$. We extend the notion of majorization to nonnegative vectors of different sizes as follows. Suppose the sum of entries of the nonnegative vectors $u \in \mathbf{R}^m$ and $v \in \mathbf{R}^n$ are the same. We can change u, v to $\tilde{u}, \tilde{v}$ by adding zero entries to the vector with lower dimension. Then we say that $u \prec v$ if $\tilde{u} \prec \tilde{v}$.

Our main theorem is the following.

Theorem 3.2 *Let $\Phi : M_2 \to M_2$ be a unital qubit quantum channel such that the Choi matrix of Φ has k positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$. Suppose $p_1, \ldots, p_m$ are positive real numbers summing up to one. Then there are unitary matrices $V_1, \ldots, V_m \in M_2$ such that*

$$\Phi(A) = \mu_1 V_1 A V_1^* + \cdots + \mu_m V_m A V_m^* \qquad \text{for all } A \in M_2 \tag{3.1}$$

if and only if $(\mu_1, \ldots, \mu_m) \prec \frac{1}{2}(\lambda_1, \ldots, \lambda_k)$.

From Theorem 3.2, one can deduce the following corollary from which Theorem 3.1 will follow.

Corollary 3.3 *Suppose Φ is a unital qubit channel such that $C(\Phi)$ has k positive eigenvalues. Then Φ can be written as the average of m unital channels for any positive integer m satisfying $k \leq m$.*

Proof. If $C(\Phi)$ has eigenvalues $\lambda_1, \ldots, \lambda_k$, then $(1/m, \ldots, 1/m) \prec \frac{1}{2}(\lambda_1, \ldots, \lambda_k)$. So, $C(\Phi)$ is the average of m unitary channels by Theorem 3.2. ■

To prove Theorem 3.2, we first obtain some auxiliary results, which are of independent interest.

Lemma 3.4 *Let $\theta, \eta_1, \eta_2, \nu_1, \nu_2$ be nonnegative numbers such that $\eta_1 \geq \nu_1 \geq \nu_2 \geq \eta_2$ and $\eta_1 + \eta_2 = \nu_1 + \nu_2 = d$. Then there are $\theta_1, \theta_2 \in [0, 2\pi)$ with $\nu_1 e^{i\theta_1} + \nu_2 e^{i\theta_2} = \eta_1 + \eta_2 e^{i\theta}$.*

Proof. By the given conditions,

$$\nu_1 - \nu_2 \leq \eta_1 - \eta_2 \leq |\eta_1 + \eta_2 e^{i\theta}| \leq \eta_1 + \eta_2 = \nu_1 + \nu_2.$$

Thus, there is $\phi \in [0, 2\pi)$ such that $|\nu_1 + e^{i\phi}\nu_2| = |\eta_1 + \eta_2 e^{i\theta}|$. Hence, there is $\theta_1 \in \mathbf{R}$ such that $e^{i\theta_1}(\nu_1 + e^{i\phi}\nu_2) = \eta_1 + \eta_2 e^{i\theta}$. Let $\theta_2 \equiv \phi + \theta_1 \pmod{2\pi}$. The result follows. ■

Corollary 3.5 *Suppose $\Psi(A) = \eta_1 V_1 A V_1^* + \eta_2 V_2 A V_2^*$ for all $A \in M_2$, where $\eta_1 \geq \eta_2 \geq 0$, $V_1, V_2 \in M_2$ are unitary matrices. Then for any ν_1, ν_2 such that $\eta_1 \geq \nu_1 \geq \nu_2 \geq \eta_2$ with $\eta_1 + \eta_2 = \nu_1 + \nu_2$, there are unitary matrices $U_1, U_2 \in M_2$ such that $\Psi(A) = \nu_1 U_1 A U_1^* + \nu_2 U_2 A U_2^*$ for all $A \in M_2$.*

Proof. Consider the special case when $V_1 = I_2$ and $V_2 = D = \operatorname{diag}(1, e^{i\theta})$. Define $\Psi(A) = \eta_1 A + \eta_2 DAD^*$. It follows that

$$C(\Psi) = \eta_1 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \eta_2 \begin{pmatrix} 1 & 0 & 0 & e^{-i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta} & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 3.4, there are $\theta_1, \theta_2 \in [0, 2\pi)$ such that $\eta_1 + \eta_2 e^{i\theta} = \nu_1 e^{i\theta_1} + \nu_2 e^{i\theta_2}$. Let $\tilde{U}_1 = \operatorname{diag}(1, e^{i\theta_1}), \tilde{U}_2 = \operatorname{diag}(1, e^{i\theta_2})$. Then $\Psi(A) = \nu_1 \tilde{U}_1 A \tilde{U}_1^* + \nu_2 \tilde{U}_2 A \tilde{U}_2^*$.

Next, consider general unitary $V_1, V_2 \in M_2$. Then $V_1^* V_2 = \alpha W D W^*$ with a complex unit α, a unitary $W \in M_2$ and $D = \operatorname{diag}(1, e^{i\theta})$. Then

$$\begin{aligned} \Psi(A) &= \eta_1 V_1 A V_1^* + \eta_2 V_2 A V_2^* = \eta_1 V_1 A V_1^* + \eta_2 V_1 (V_1^* V_2 A_2 V_2^* V_1) V_1^* \\ &= \eta_1 V_1 A V_1^* + \eta_2 V_1 (\alpha W D W^* A \bar{\alpha} W D^* W^*) V_1^* = V_1 W \Phi(W^* A W) W^* V_1^* \end{aligned}$$

with $\Phi(X) = \eta_1 X + \eta_2 DXD^*$ for $X \in M_2$. By the special case, there are diagonal unitary matrices $\tilde{U}_1, \tilde{U}_2 \in M_2$ such that

$$\Phi(W^*AW) = \eta_1(W^*AW) + \eta_2 D(W^*AW)D^* = \nu_1\tilde{U}_1(W^*AW)\tilde{U}_1^* + \nu_2\tilde{U}_2(W^*AW)\tilde{U}_2^*.$$

Let $U_j = V_1 W\tilde{U}_j W^*$ for $j = 1, 2$. Then $\Psi(A) = \nu_1 U_1 AU_1^* + \nu_2 U_2 AU_2^*$ for all $A \in M_2$. ■

Lemma 3.6 *Let $m \geq 2$ and $\mathbf{u} = (u_1, \ldots, u_m), \mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{R}^m$ with entries arranged in descending order, and $\mathbf{u} \prec \mathbf{v}$. Then one of the following holds.*

(1) *There is $i \in \{1, \ldots, m\}$ such that $u_i = v_i$.*

(2) *There is $j \in \{1, \ldots, m-1\}$ with $v_j > u_j \geq u_{j+1} > v_{j+1}$. If $\delta = \min\{v_j - u_j, u_{j+1} - v_{j+1}\}$, then the vector $\tilde{\mathbf{v}}$ obtained from $\mathbf{v}$ by replacing the pair of entries (v_j, v_{j+1}) to $(v_j - \delta, v_{j+1} + \delta)$ will satisfy $\mathbf{u} \prec \tilde{\mathbf{v}} \prec \mathbf{v}$, and $\mathbf{u}$ and $\tilde{\mathbf{v}}$ will have the same value at the jth or $(j+1)$st position.*

Proof. Direct verification. ■

In general, suppose $\mathbf{y} = (y_1, \ldots, y_m)$ with $y_1 \geq \cdots \geq y_m$, and $\mathbf{x} = (x_1, \ldots, x_m)$ is obtained from $\mathbf{y}$ by changing a pair of entries (y_p, y_q) to $(x_p, x_q) = (y_p - \delta, y_q + \delta)$ with $\delta \in (0, (y_p - y_q)/2)$ for some $p < q$ and $y_q - y_p > 0$. We say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by a pinching. By Lemma 3.6, if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^m$ satisfy $\mathbf{x} \prec \mathbf{y}$, we may remove the entries x_ℓ, y_ℓ whenever $x_\ell = y_\ell$, and apply (2) to get a vectors $\tilde{\mathbf{y}}$ such that the pair of vectors $\mathbf{x}$ and $\tilde{\mathbf{y}}$ have one more common entry than the pair of vectors $\mathbf{x}$ and $\mathbf{y}$. Thus, one can change $\mathbf{y}$ to $\mathbf{x}$ by at most $m-1$ pinchings.

Corollary 3.7 *Suppose Φ is a qubit channel having the form*

$$A \mapsto \sum_{j=1}^{k} \nu_j U_j AU_j^* \tag{3.2}$$

for some nonnegative real numbers $\nu_1, \ldots, \nu_k$ summing up to 1, and unitary $U_1, \ldots, U_k \in M_2$. If $(\mu_1, \ldots, \mu_m) \prec (\nu_1, \ldots, \nu_k)$, then Φ admits a representation of the form $A \mapsto \sum_{j=1}^m \mu_j V_j AV_j^*$ for some unitary $V_1, \ldots, V_m \in M_2$.

Proof. We may assume that $m = k$ by adding zeros to the vector with lower dimension. Assume $(\mu_1, \ldots, \mu_m) \neq (\nu_1, \ldots, \nu_m)$ to avoid trivial consideration. Furthermore, we may assume that the entries of the two vectors are arranged in descending order. Suppose $(\mu_1, \ldots, \mu_m)$ is obtained from $(\nu_1, \ldots, \nu_m)$ by a pinching, say, $(\mu_p, \mu_q) = (\nu_p - \delta, \nu_q + \delta)$ with $\delta \in (0, (\nu_q - \nu_p)/2)$ for some $p < q$. By Lemma 3.5, we can replace the expression $\nu_p U_p AU_p^* + \nu_q U_q AU_q^*$ in (3.2) by $\mu_p V_p AV_p^* + \mu_q V_q AV_q^*$ for some unitary matrices $V_p, V_q \in M_2$. By the remark after Lemma 3.6, we can reduce $(\nu_1, \ldots, \nu_m)$ by at most $m-1$ pinchings. Thus, we can repeat the above argument for at most $m-1$ times to get the conclusion. ■

Now we are ready to present the following.

Proof of Theorem 3.2. Suppose $\Psi(A) = \sum_{j=1}^\ell V_j AV_j^*$ for all $A \in M_n$. Then $C(\Psi) = \sum_{j=1}^\ell v_j v_j^*$, where for $j = 1, \ldots, \ell$, $v_j \in \mathbf{C}^{mn}$ has the first m entries equal to the first column of V_j, the next m entries equal to the second column of V_j, etc. Suppose $R = [v_1|\cdots|v_\ell]$.

Then $C(\Psi) = RR^*$ has the same nonzero eigenvalues as $R^*R \in M_\ell$, which is a Hermitian matrix with diagonal entries

$$(v_1^*v_1, \ldots, v_\ell^* v_\ell) = (\mathrm{tr}\,(V_1^*V_1), \ldots, \mathrm{tr}\,(V_\ell^* V_\ell)).$$

Then $\ell \geq k$ and the majorization relation holds by the result in [13].

For the sufficiency, we see that Φ can be written in the form $A \mapsto \frac{1}{2}\sum_{j=1}^k \lambda_j U_j A U_j^*$ by Theorem 2.2. By Corollary 3.7, If $(\mu_1, \ldots, \mu_m) \prec \frac{1}{2}(\lambda_1, \ldots, \lambda_k)$, there are unitary $V_1, \ldots, V_m \in M_n$ such that Φ can be written in the form (3.1). ∎

We note that Corollary 3.5, Corollary 3.7, and Corollary 3.3 were obtained as Lemma 1.1, Theorem 1.2, and Corollary 1.4 in [8], where the proofs were done using the idea in [4] as indicated by the authors.

We conclude this section with some comments on the unusual convexity features of the set $\mathcal{S}$ of unital qubit channels, whose extreme points are unitary channels.

By Theorem 3.2, if $\Phi \in \mathcal{S}$ such that $C(\Phi)$ has rank k with positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$, then Φ can be written as a convex combination of unitary channels $A \mapsto \sum_{j=1}^m \mu_j U_j A U_j^*$ as long as $(\mu_1, \ldots, \mu_m) \prec \frac{1}{2}(\lambda_1, \ldots, \lambda_k)$. Consequently, the four eigenvalues of $C(\Phi)$ can be characterized as the supremum of set of four numbers $p_1 \geq \cdots \geq p_4$ such that Φ admits a representation of the form $A \mapsto \frac{1}{2}(p_1 V_1 A V_1^* + \cdots + p_4 V_4 A V_4^*)$ for some unitary matrices $V_1, \ldots, V_4 \in M_2$. Of course, $(\lambda_1, \cdots, \lambda_4)$ can also be characterized as $(\mathrm{tr}\, V_1^*V_1, \ldots, \mathrm{tr}\, V_4^* V_4)$ such that Φ admits a representation of the form $A \mapsto V_1 A V_1^* + \cdots + V_4 A V_4^*$ for some matrices $V_1, \ldots, V_4 \in M_2$ satisfying $\mathrm{tr}\, V_i^* V_j = 0$ for all $1 \leq i, j \leq 4$. If Φ is a unitary channel so that Φ is a unital qubit channel so that $C(\Phi)$ has eigenvalues $1, 0, 0, 0$, then Φ can be written as $\sum_{j=1}^m \mu_j U_j A U_j^*$ for any choice of convex coefficients $\mu_1, \ldots, \mu_m$. In some sense, the eigenvalues of $C(\Phi)$ can be viewed as a measure on how close Φ is to a unitary channel, an extreme point of $\mathcal{S}$.

4 Unital qubit channels and the Bloch ball

A unital qubit channel $\Phi : M_2 \to M_2$ in the canonical form (1.2) is determined by the conditions

$$\Phi(I_2) = I_2, \quad \Phi(X) = d_1 X, \quad \Phi(Y) = d_2 Y, \quad \Phi(Z) = d_3 Z, \tag{4.1}$$

where $d_1 = (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)/2$, $d_2 = (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)/2$, and $d_3 = (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)/2$. The map Φ will transform the Bloch (unit) ball to the ellipsoid $\mathcal{E}$ specified by (d_1, d_2, d_3):

$$\{(d_1 x, d_2 y, d_3 z) : x, y, z \in \mathbf{R}, x^2 + y^2 + z^2 \leq 1\}$$

$$= \{(x, y, z) \in \mathbf{R}^3 : (x/d_1)^2 + (y/d_2)^2 + (z/d_3)^2 \leq 1\}.$$

If Φ is defined as in (4.1), then

$$\Phi(E_{11}) = \Phi(I) + \Phi(Z) = \frac{1}{2}\begin{pmatrix} 1 + d_3 & 0 \\ 0 & 1 - d_3 \end{pmatrix}, \ \Phi(E_{22}) = I - \Phi(E_{11}) = \frac{1}{2}\begin{pmatrix} 1 - d_3 & 0 \\ 0 & 1 + d_3 \end{pmatrix},$$

and

$$\Phi(E_{21})^t = \Phi(E_{12}) = \frac{1}{2}(\Phi(X) + \Phi(iY)) = \frac{1}{2}\begin{pmatrix} 0 & d_1 + d_2 \\ d_1 - d_2 & 0 \end{pmatrix}.$$

By these observations and Theorem 2.2, we have the following.

Theorem 4.1 *Let $\Psi : M_2 \to M_2$ be a unital trace-preserving linear map that preserves Hermitian matrices. Then Ψ is unitarily equivalent to Φ of the form (4.1) sending the Bloch sphere onto the ellipsoid specified by (d_1, d_2, d_3). The Choi matrix $C(\Phi)$ has the form*

$$\frac{1}{2}\begin{pmatrix} 1+d_3 & 0 & 0 & d_1+d_2 \\ 0 & 1-d_3 & d_1-d_2 & 0 \\ 0 & d_1-d_2 & 1-d_3 & 0 \\ d_1+d_2 & 0 & 0 & 1+d_3 \end{pmatrix} \tag{4.2}$$

with eigenvalues $\frac{1}{2}(1+d_1+d_2+d_3), \frac{1}{2}(1+d_3-d_1-d_2), \frac{1}{2}(1-d_3+d_1-d_2), \frac{1}{2}(1-d_3-d_1+d_2)$.

One can now connect the results in Sections 2 and 3 to the map $(x, y, z) \mapsto (d_1x, d_2y, d_3z)$. For example, by Corollary 2.5, we have the following.

Theorem 4.2 *Let $\Phi, \Psi : M_2 \to M_2$ be linear map such that Φ is defined by (4.1) and Ψ is defined by*

$$\Psi(I_2) = I_2, \quad \Psi(X) = \tilde{d}_1 X, \quad \Psi(Y) = \tilde{d}_2 Y, \quad \Psi(Z) = \tilde{d}_3 Z.$$

Then the following conditions are equivalent.

(a) *Φ and Ψ are unitarily equivalent.*

(b) $(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3, \tilde{d}_1 - \tilde{d}_2 - \tilde{d}_2, -\tilde{d}_1 + \tilde{d}_2 - \tilde{d}_3, -\tilde{d}_1 - \tilde{d}_2 + \tilde{d}_3)$ *is a permutation of* $(d_1 + d_2 + d_3, d_1 - d_2 - d_3, -d_1 + d_2 - d_3, -d_1 - d_2 + d_3)$.

(c) $(|\tilde{d}_1|, |\tilde{d}_2|, |\tilde{d}_3|)$ *is a permutation of* $(|d_1|, |d_2|, |d_3|)$, *and* $\tilde{d}_1\tilde{d}_2\tilde{d}_3 = d_1d_2d_3$.

Proof. By Corollary 2.3, (a) holds if and only if $C(\Phi)$ and $C(\Psi)$ have the same eigenvalues, equivalently, condition (b) holds by (4.2).

To prove the equivalence of (b) and (c), let $T = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$. Then $TT^* = I_3$ and $T^*T = I_4 - J_4/4$, where $J_4 \in M_4$ has all entries equal to 1. Moreover, for any permutation matrix $P \in M_4$ and $PT^*T = T^*TP$. Consequently, the map $X \mapsto TXT^*$ defines a group homomorphism from the group of permutation matrices in M_4 to the group of the 24 matrices $Q_1, \ldots, Q_{24} \in M_3$, where each Q_j is the product of a permutation matrix in M_3 and a diagonal orthogonal matrix with determinant 1.

Note that $(d_1+d_2+d_3, d_1-d_2-d_3, -d_1+d_2-d_3, -d_1-d_2+d_3) = 2(d_1, d_2, d_3)T$. Thus, if (b) holds, then there is a permutation matrix $P \in M_4$ such that $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)T = (d_1, d_2, d_3)TP$. Hence, $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) = (d_1, d_2, d_3)TPT^* = (d_1, d_2, d_3)Q_j$ for some $j = 1, \ldots, 24$, i.e., condition (c) holds.

If (c) holds, then $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) = (d_1, d_2, d_3)Q$ for some $Q \in \{Q_j : 1 \leq j \leq 24\}$ so that $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)T = (d_1, d_2, d_3)(TPT^*)T = (d_1, d_2, d_3)T(T^*T)P = (d_1, d_2, d_3)TP$ for some permutation $P \in M_4$. Thus, condition (b) holds. ∎

Note that if $\Phi : M_2 \to M_2$ satisfying (4.1) is a unital qubit channel, then it will sends the set of density matrices back to itself. Hence, $|d_j| \leq 1$ for $i = 1, \ldots, 3$. However, the converse is not true in general. For example, if we let $(d_1, d_2, d_3) = (1, 1, 0)$, then

$$\Phi : \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 1 & x-iy \\ x+iy & 1 \end{pmatrix} \quad \text{with} \quad C(\Phi) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \tag{4.3}$$

is not a quantum channel because $C(\Phi)$ is not positive semidefinite.

We have shown that up to unitarily equivalence, every unital qubit channel can be written as a convex combination of the following four maps

$$A \mapsto A, \qquad A \mapsto XAX, \qquad A \mapsto YAY, \qquad A \mapsto ZAZ.$$

The corresponding linear transformations of the Bloch balls are

$$(x, y, z) \mapsto (x, y, z),\ (x, y, z) \mapsto (x, -y, -z),\ (x, y, z) \mapsto (-x, y, -z),\ (x, y, z) \mapsto (-x-y, z),$$

which are the identity map, the rotation of π of the (y, z)-plane, (x, z)-plane, and (x, y)-plane, respectively. Equivalently, (d_1, d_2, d_3) is a convex combination of vectors

$$(1, 1, 1),\ (1, -1, -1),\ (-1, 1, -1),\ (-1, -1, 1).$$

As a result, $\Phi : M_2 \to M_2$ in the canonical form (1.2) transforming the Bloch ball by the map $(x, y, z) \mapsto (d_1 x, d_2 y, d_3 z)$ is a unital qubit channel if and only if the eigenvalues of $C(\Phi)$ are nonnegative, i.e.,

$$1 + d_1 + d_2 + d_3, 1 + d_3 - d_1 - d_2, 1 - d_3 + d_1 - d_2, 1 - d_3 - d_1 + d_2 \in [0, \infty). \tag{4.4}$$

One can determine the set of extreme points of the compact convex set $(d_1, d_2, d_3) \in \mathbf{R}^3$ satisfying (4.4). Every extreme point of the set must attain equality for at least three of the linear inequality constraints. A direct computation shows that there are four extreme points, namely,

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1),$$

and the set of real vectors (d_1, d_2, d_3) satisfying (4.4) is the convex hull of these four points, which is a regular tetrahedron. These observations can be used to determine a linear map $\Phi : M_2 \to M_2$ satisfying (4.1) is a unital qubit channel as shown in the following.

Theorem 4.3 *Suppose* $\Phi : M_2 \to M_2$ *is a linear map as defined by* (4.1). *Then* Φ *is a unital qubit channel if and only if any one of the following holds.*

(a) (d_1, d_2, d_3) *belongs to the regular tetrahedron with vertices*

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).$$

(b) *The vector* $(1 + d_1 + d_2 + d_3, 1 + d_1 - d_2 - d_3, 1 - d_1 + d_2 - d_3, 1 - d_1 - d_2 + d_3)$ *is nonnegative.*

In Theorem 4.3, (a) and (b) establish a correspondence between the regular tetrahedron in $\mathbf{R}^3$ with vertices $(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)$, and the convex set

$$\left\{(\lambda_1,\ldots,\lambda_4) : \lambda_1,\ldots,\lambda_4 \geq 0, \sum_{j=1}^{4} \lambda_j = 2\right\}.$$

Mathematically, it is simple to verify the correspondence. Nevertheless, it is physically astonishing that the unital qubit channels serve as the natural link between these two convex sets. Namely, each quantum channel Φ determines four eigenvalues $\lambda_1,\ldots,\lambda_4$ of $C(\Phi)$ as well as the shrinking effects (d_1,d_2,d_3) of the Block sphere in $\mathbf{R}^3$. For example, if $(d_1,d_2,d_3) = (1,1,0)$, i.e., the Bloch sphere is transformed to the unit disk on the (x,y)-plane, then it does not correspond to a unital qubit channel. Also, if $(d_1,d_2,d_3) = (1,1,-1)$, i.e., the Bloch sphere is transformed to itself by a reflection about the (x,y)-plane, then it does not correspond to a unital qubit channel.

In fact, it not hard to show that for a nonngative number γ the map $(x,y,z) \mapsto \gamma(x,y,-z)$ corresponds to a unital qubit channel if and only if $\gamma \in [0,1/3]$. It turns out that this and other limitations on admissible scalings of the Bloch sphere corresponding to unital qubit maps can be deduced from the next theorem, where the assumption $d_1 \geq d_2 \geq |d_3|$ is imposed without loss of generality by virtue of Theorem 4.2 (c).

Theorem 4.4 *Suppose* $\Phi : M_2 \to M_2$ *is a linear map as defined by* (4.1) *with* $d_1 \geq d_2 \geq |d_3|$. *Then* Φ *is a unital qubit channel if and only if* $1 + d_3 \geq d_1 + d_2$*, equivalently,* $(d_1,d_2,d_3) \in \mathrm{conv}\{(0,0,0),(1,1,1),(1,0,0),(1,1,-1)/3\}$.

Proof. By Theorem 4.3, under the assumption that $d_1 \geq d_2 \geq |d_3|$, condition (b.2) reduces to $1 + d_3 \geq d_1 + d_2$. The convex set

$$S = \{(d_1,d_2,d_3) \in \mathbf{R}^3 : d_1 \geq d_2 \geq |d_3|, 1 + d_3 \geq d_1 + d_2\}$$

is defined by the four inequalities $d_1 \geq d_2, d_2 \geq d_3, d_2 \geq -d_3$, and $1+d_3 \geq d_1+d_2$; an extreme point of the set S must attain at least three of these inequalities. Thus the extreme points of S are $(0,0,0),(1,1,1),(1,0,0),(1,1,-1)/3$, and

$$S = \mathrm{conv}\{(0,0,0),(1,1,1),(1,0,0),(1,1,-1)/3\}.$$

■

By Theorem 4.4, Φ is a unital qubit channel if and only if the Bloch ball is transformed by a convex combination of the maps $\psi_1 : (x,y,z) \mapsto (0,0,0)$, $\psi_2 : (x,y,z) \mapsto (x,y,z)$, $\psi_3 : (x,y,z) \mapsto (x,0,0)$, $\psi_4 : (x,y,z) \mapsto (0,0,-z)$. These maps on the Bloch sphere (the boundary of the Bloch ball) have the following effects, respectively:

- shrinking the sphere to the origin,
- leaving the sphere invariant,
- shrinking the Bloch sphere to the line segment joining $(-1,0,0)$ to $(1,0,0)$,
- shrinking the Bloch sphere by a factor of $1/3$ followed by a refection over the (x,y)-plane.

In terms of Φ, it means that every unital qubit channel can be written as a convex combination of $\Psi_1, \dots, \Psi_4$ defined, respectively, by

$$A \mapsto \frac{1}{2}(\operatorname{tr} A)I_2, \quad A \mapsto A, \quad A \mapsto \frac{1}{2}(A + XAX), \quad \text{and} \quad A \mapsto \frac{1}{3}(A + XAX + YAY).$$

Using Theorem 4.2 (c), one may impose other assumptions on the scaling factors d_1, d_2, d_3 of the Bloch sphere, and deduce additional results on unital qubit channels. For example, we may assume that $d_1 \ge d_2 \ge d_3$ and $d_2 \ge 0$. In such a case, $(x, y, z) \mapsto (d_1x, d_2y, d_3z)$ will correspond to a unital qubit channel if and only if $d_1 \ge d_2 \ge d_3$, $d_2 \ge 0$, and $1 + d_3 \ge d_1 + d_2$, i.e., $(d_1, d_2, d_3) \in \operatorname{conv}\{(0,0,0), (1,1,1), (1,0,0), (0,0,-1)\}$. Thus, a unital qubit channel can be written as a convex combination of the maps

$$A \mapsto \frac{1}{2}(\operatorname{tr} A)I_2, \quad A \mapsto A, \quad A \mapsto \frac{1}{2}(A + XAX), \quad \text{and} \quad A \mapsto \frac{1}{2}(XAX + YAY).$$

5 Additional remarks and further research

One referee pointed out that the Choi matrix of a unital quantum channel corresponds to a maximally entangled 2-qubit state after normalization, as noted in our paper. Specifically, if $\rho = (\rho_{ij}) \in M_2(M_2)$ is a density matrix with $\rho_{11} + \rho_{22} = I_2/2$, then ρ represents a maximally entangled 2-qubit state. Corollary 2.3 further states that two maximally entangled 2-qubit states ρ_1 and ρ_2 in M_4 are unitarily similar if and only if there exist unitary matrices $U, V \in M_2$ such that $(U \otimes V)\rho_1(U \otimes V)^* = \rho_2$. It was also suggested that one may study unital qutrit or qudit channels. However, qutrit and qudit channels may not be mixed unitary so that our techniques do not apply. New techniques will be required to study these problems.

Another referee pointed out that the matrix in (2.1) can be interpreted as an X-state in M_4 after normalization, if it is positive semidefinite. In [2], the authors identified 2-qubit states that can be reduced to such states under a certain equivalence relation; see Theorem 1 and Section 2.3 in [2]. Our study may have other connections to the study of X-states, e.g., see [11] and related references. Also, the fact that every unital qubit channnel can be written as the average of k unitary channels, where k is the rank of the Choi matrix, can be restated as: The mixed-unitary rank of unital qubit channel is always the same as the Choi rank; see [5]. It would be interesting to identify other quantum channels with this property.

There is an one-one correspondence between affine maps on $\mathbf{R}^3$ and trace preserving linear maps on M_2 that preserve Hermitian matrices. Specifically, an affine map $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$\phi(a, b, c) = (a, b, c)A + (a_0, b_0, c_0)$$

for a real matrix $A \in M_3$ and a fixed vector $(a_0, b_0, c_0) \in \mathbf{R}^3$ corresponds to the linear map $\Phi : M_2 \to M_2$ satisfying $\Phi(I_2) = I_2 + a_0X + b_0Y + c_0Z$ and

$$\Phi(aX + bY + cZ) = \hat{a}X + \hat{b}Y + \hat{c}Z \quad \text{with} \quad (\hat{a}, \hat{b}, \hat{c}) = (a, b, c)A.$$

Clearly, the map Φ is unital if and only if $(a_0, b_0, c_0) = (0, 0, 0)$. In [12], the authors investigated the effect of the affine map $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ induced by a general qubit channel $\Phi : M_2 \to M_2$ on the Bloch sphere. It would be interesting to identify simple conditions on a real matrix $A \in M_3$ and $(a_0, b_0, c_0) \in \mathbf{R}^3$ that guarantee the affine map

$(a, b, c) \mapsto (a, b, c)A + (a_0, b_0, c_0)$ corresponds to a qubit channel. Some analysis of this problem has been done in [12].

Acknowledgements

Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo. His research was supported by the Simons Foundation Grant 851334. The authors gratefully acknowledge the valuable comments provided by the referees. They also like to express their gratitude to Professor Alexander Müller-Hermes, Professor Frederik vom Ende, and Professor Seok-Hyung Lie for their helpful suggestions.

References

1. J. Bae1 and L.-C. Kwek (2015), Quantum state discrimination and its applications, Journal of Physics A: Mathematical and Theoretical 48, 083001.
2. P. Caban, J. Rembieliński, K.A. Smoliński, Z. Walczak (2015), Classification of two-qubit states, Quantum Inf Process 14, pp. 46654690.
3. M.D. Choi (1975), Completely positive linear maps on complex matrices, Linear Algebra Appl. 10, pp. 285-290.
4. M.D. Choi and P.Y. Wu (1990), Convex combinations of projections, Linear Algebra Appl. 136, pp. 25-42.
5. M. Girard, D. Leung, J. Levick, C.K. Li, V. Paulsen, Y.T. Poon, and J. Watrous(2022), On the mixed-unitary rank of quantum channels, Comm. Math. Phys. 394 no. 2, pp. 919-951.
6. L. Gyongyosi, S. Imre and H. V. Nguyen (2018), A Survey on Quantum Channel Capacities, IEEE Communications Surveys & Tutorials 20, pp. 1149-1205.
7. L. Landau and R. Streater (1993), On Birkhoff's theorem for doubly stochastic completely positive maps of matrix algebras, Linear algebra Appl. 193, pp. 107127.
8. A. Müller-Hermes and C. Perry (2019), All unital qubit channels are 4-noisy operations, Letters in Mathematical Physics 109, pp. 1-9.
9. M.A. Nielsen and I.L. Chuang (2004), Quantum Computation and Quantum Information. Cambridge University Press.
10. A. Pasieka, D.W. Kribs, R. Laflamme, and R. Pereira (2009), On the geometric interpretation of single qubit quantum operations on the Bloch sphere, Acta Applicandae Mathematicae 108, Article number: 697.
11. N. Quesada, A. Al-Qasimi and D.F.V. James (2012), Quantum properties and dynamics of X states, Journal of Modern Optics 59, pp. 1322-1329.
12. M. B. Ruskai, S. Szarek (2002), E. Werner, An analysis of completely positive trace-preserving maps on M_2, Linear Algebra Appl. 347, pp. 159-187.
13. I. Schur (1923), Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie, Sitzungsber. Berl. Math. Ges. 22, pp. 920.
14. F. Verstraete and H. Verschelde (2022), On quantum channels, arXiv:quant-ph/0202124
15. M.M. Wilde (2017), Quantum Information Theory (2nd ed.), Cambridge University Press.