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NEW ENTANGLEMENT-ASSISTED QUANTUM MDS CODES DERIVED FROM CYCLIC CODES

SUJUAN HUANG
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
huangsujuan1019@163.com

SHIXIN ZHU
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
zhushixin@hfut.edu.cn

PAN WANG
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
panwang_hfut@163.com

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Entanglement-assisted quantum error-correcting codes, which can be seen as a generalization of quantum error-correcting codes, can be constructed from arbitrary classical linear codes by relaxing the self-orthogonality properties and using pre-shared entangled states between the sender and the receiver, and can also improve the performance of quantum error-correcting codes. In this paper, we construct some families of entanglement-assisted quantum maximum-distance-separable codes with parameters $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 2 + c, d; c]]_q$, where q is a prime power with the form $q = am \pm \ell$, $a = \frac{\ell^2-1}{3}$ is an odd integer, $\ell \equiv 2 \ (\bmod\ 6)$ or $\ell \equiv 4 \ (\bmod\ 6)$, and m is a positive integer. Most of these codes are new in the sense that their parameters are not covered by the codes available in the literature.

Keywords: Entanglement-assisted quantum error-correcting codes, Cyclic codes, Cyclotomic coset, Defining set

1 Introduction

Quantum error-correcting(QEC) codes can preserve coherent states against noise and other unwanted interactions in quantum communication and quantum computation . For any prime power q, an $[[n, k, d]]_q$ QEC code is a q^k-dimensional subspace of the Hilbert space $\mathbb{C}^{q^n}$ with minimum distance d, which can correct up to $\lfloor \frac{d-1}{2} \rfloor$ quantum errors. It is well-known that QEC codes can be constructed from classical linear codes with certain self-orthogonality (or dual-containing) properties. However, such properties of some famous codes are hard to determine. In 2006, a more general framework called entanglement-assisted stabilizer formalism was introduced [2, 20], the associated codes are the so-called entanglement-assisted quantum error-correcting (EAQEC) codes, which not only can be contructed from any classical linear codes without self-orthogonality properties by utilizing pre-shared entangled states between the sender and the receiver, but also can increase the communication capacity. After that, many EAQEC codes with good parameters have been constructed. (Please see, for example,[10, 11, 12, 19, 20, 25, 32, 47, 48] and the relevant references therein).

Assume that q is a prime power. A q-ary EAQEC code with minimum distance d, denoted by $[[n, k, d; c]]_q$, encodes k information qudits into n channel qudits with the aid of c pairs of maximally entangled states and can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors. If $c = 0$, it is indeed the standard $[[n, k, d]]_q$ QEC code. Hence, EAQEC codes can be seen as the generalization of QEC codes. In this paper, QEC codes are also regarded as EAQEC codes. Similar to QEC codes, the parameters of EAQEC codes satisfy the following well-known entanglement-assisted (EA) quantum Singleton bound.

Theorem 1.1: [1, 2, 16, 25](EA-quantum Singleton bound) For any $[[n, k, d; c]]_q$ EAQEC code with $d \leq \frac{n+2}{2}$, its parameters satisfy

$$2d \leq n - k + c + 2,$$

where $0 \leq c \leq n - 1$.

An EAQEC code achieving such bound is called an entanglement-assisted quantum maximum-distance-separable (EAQMDS) code. If $c = 0$, it is indeed the quantum Singleton bound and a quantum code achieving such bound is called a quantum maximum-distance-separable (MDS) code. For the case $d > \frac{n+2}{2}$, Grassl [15] gave some examples of EAQEC codes beating such bound. As we said before, EAQEC codes can be constructed from any classical codes. However, it is still hard to construct such codes due to the difficulty in determining the number of maximally pre-shared entangled states. Thanks to [18], a relationship between the number of maximally pre-shared entangled states required to construct an EAQEC code from a classical code and the hull of classical code was obtained, and some EAQEC codes with flexible parameters were also constructed. Since then, many families of EAQMDS codes were obtained via the computation of the hull dimension of generalized Reed-Solomon codes, Goppa codes, *et al.* [3, 11, 14, 17, 27, 35, 41, 42, 46]. Very recently, due to the excellent work of Chen [4, 5, 6], many families of EAQMDS codes with flexible parameters were obtained via the Euclidean and Hermitian hulls of linear codes. Allahmadi *et al.* [1] presented two new interesting constructions of EAQEC codes, which indicate that EAQEC codes can be constructed through LCD codes and the related concatenation constructions.

Recently, some research showed that EAQEC codes can be directly derived from QEC codes or EAQEC codes. Lai and Brun [26] first showed that any (nondegenerate) standard $[[n, k, d]]$ stabilizer code can be transformed into an $[[n - c, k, d; c]]$ EAQEC code that can correct errors on the qudits of both the sender and the receiver, where $0 \leq c \leq n - k$. Particularly, the obtained EAQEC codes are equivalent to standard stabilizer codes. Galindo *et al.* [12] generalized [26] to arbitrary finite fields, and they got some EAQEC codes from QEC codes by considering Euclidean, Hermitian and symplectic duality, respectively. Very recently, a surprising and interesting result was given by Grassl *et al.* [16]. They showed that any EAQEC code can be derived from a pure QEC code, i.e., if there is a pure QEC code with parameters $[[n,k,d]]_q$, an EAQEC code with parameters $[[n-c,k,d;c]]_q$ exists for all $c<d$. Luo *et al.* [36] presented three new propagation rules for constructing EAQEC codes from EAQEC codes and discussed how each of them affects the error handling.

Due to their rich algebraic structure, constacyclic codes including cyclic codes and negacyclic codes are preferred objects on the construction of EAQMDS codes. Lu *et al.* [33] and Chen *et al.* [7] respectively utilized the decomposition of the defining set of constacyclic codes to determine the number of maximally pre-shared entangled states c, which transmitted the determination of c into determining a subset of the defining set of the underlying codes, and they also constructed some EAQMDS codes with large minimum distance. After that, many families of EAQMDS codes with lengths divide $q^2 \pm 1$ have been constructed via such technique.(Please see, [7, 8, 9, 13, 21, 22, 24, 28, 29, 30, 31, 33, 34, 38, 39, 40, 43, 44] and the relevant references therein).

As listed above, EAQMDS codes with lengths divide q^2-1, i.e., $\frac{q^2-1}{a}$ have been constructed. However, almost all the a either divides $q+1$ or divides $q-1$. Very recently, EAQMDS codes of length $\frac{q^2-1}{a}$ have been constructed in [45], where a either divides $q+\ell$ or divides $q-\ell$ and $\ell>1$ is an odd integer. Going on the line of such study, in this paper, based on the decomposition of the defining set of cyclic codes, we construct some families of EAQMDS codes with parameters $[[\frac{q^2-1}{a}, \frac{q^2-1}{a}-2d+2+c, d; c]]_q$ by exploiting less pre-shared maximally entangled states c, where q is a prime power with the form $q=am\pm\ell$, $a=\frac{\ell^2-1}{3}$ is an odd integer, $\ell \equiv 2 \pmod 6$ or $\ell \equiv 4 \pmod 6$, and m is a positive integer.

The paper is organized as follows. In Section 2, some notations and basic results of cyclic codes and EAQEC codes are presented. In Section 3, some new families of EAQMDS codes with small pre-shared entangled states are derived from cyclic codes. The conclusion is given in Section 4.

2 Preliminaries

Assume that q is a prime power and $\mathbb{F}_{q^2}$ is the Galois field with q^2 elements. A q^2-ary linear code $\mathcal{C}$ of length n with dimension k, denoted by $[n,k]_{q^2}$, is a k-dimensional linear subspace of $\mathbb{F}_{q^2}^n$. The number of nonzero components of $\mathbf{c}\in\mathcal{C}$, denoted by $\mathrm{wt}(\mathbf{c})$, is called the weight of the codeword $\mathbf{c}$. The minimum nonzero weight of all codewords in $\mathcal{C}$, denoted by $d(\mathcal{C})$, is called the minimum distance of $\mathcal{C}$. $[n,k,d]_{q^2}$ is used to denote an $[n,k]_{q^2}$ linear code with minimum distance d. It is well-known that the parameters of $\mathcal{C}$ satisfy the Singleton bound: $d\le n-k+1$, and if the minimum distance d of the code $\mathcal{C}$ achieves such bound, it is the so-called MDS code.

Given any two vectors $\mathbf{x}=(x_0,x_1,\dots,x_{n-1})$, and $\mathbf{y}=(y_0,y_1,\dots,y_{n-1})\in\mathbb{F}_{q^2}^n$, their Hermitian inner product is defined as

$$\langle \mathbf{x},\mathbf{y}\rangle_h := x_0y_0^q + x_1y_1^q + \cdots + x_{n-1}y_{n-1}^q.$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal if $\langle \mathbf{x},\mathbf{y}\rangle_h=0$. For a q^2-ary linear code $\mathcal{C}$ of length n, its Hermitian dual code, denoted by $\mathcal{C}^{\perp_h}$, is defined as

$$\mathcal{C}^{\perp_h} := \{\mathbf{x}\in\mathbb{F}_{q^2}^n : \langle \mathbf{x},\mathbf{y}\rangle_h = 0 \text{ for all } \mathbf{y}\in\mathcal{C}\}.$$

Actually, $\mathcal{C}^{\perp_h}$ is a q^2-ary linear code with dimension $n-\dim(\mathcal{C})$. If $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$, then $\mathcal{C}$ is called an Hermitian dual-containing code, and $\mathcal{C}$ is called an Hermitian self-dual code if $\mathcal{C}^{\perp_h}=\mathcal{C}$.

Assume that τ: $\tau(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$ is the cyclic shift on $\mathbb{F}_{q^2}^n$. A q^2-ary linear code $\mathcal{C}$ of length n is said to be cyclic if $\tau(\mathcal{C}) = \mathcal{C}$. Defining a map

$$\sigma : \mathbb{F}_{q^2}^n \longrightarrow \mathcal{R} = \frac{\mathbb{F}_{q^2}[x]}{\langle x^n - 1\rangle},$$

$$(c_0, c_1, \ldots, c_{n-1}) \longmapsto c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}.$$

Then a q^2-ary linear code $\mathcal{C}$ of length n is a cyclic code if and only if $\sigma(\mathcal{C}) = \{\sigma(\mathbf{c}) : \mathbf{c} \in \mathcal{C}\}$ is an ideal of the quotient ring $\mathcal{R}$. As we know, each ideal of $\mathcal{R}$ is principal, so each cyclic code $\mathcal{C}$ is generated by a monic divisor $f(x)$ of $x^n - 1$, which has the minimal degree in $\mathcal{C}$.

Assume that $\gcd(n, q) = 1$, and m is an integer, where $0 \leq m \leq n - 1$. The q^2-cyclotomic coset of m modulo n, denoted by C_m, is defined as

$$C_m := \{mq^{2s} \pmod{n} : 0 \leq s \leq s_m - 1\},$$

where s_m is the smallest positive integer such that $mq^{2s_m} \equiv m \pmod{n}$, and it is also called the size of C_m, i.e., $|C_m| = s_m$, where $|C_m|$ denotes the cardinality of the set C_m.

Let $\mathcal{C}$ be a q^2-ary cyclic code of length n with generator polynomial $f(x)$, then the set $Z = \{0 \leq i \leq n-1 : f(\eta^i) = 0\}$ is called the defining set of $\mathcal{C}$, where η is a primitive n-th root of unity in some extension field of $\mathbb{F}_{q^2}$. It is obvious that the defining set Z of $\mathcal{C}$ is a union of some q^2-cyclotomic cosets and $\dim(\mathcal{C}) = n - |Z|$. The minimum distance of $\mathcal{C}$ satisfy the following well-known bound.

Theorem 2.1: [37](BCH bound) Let δ be an integer in the range $2 \leq \delta \leq n$. Suppose that $\mathcal{C}$ is a cyclic code of length n with defining set Z. If Z consists of $\delta - 1$ consecutive elements, then $d(\mathcal{C}) \geq \delta$.

The following lemma gives a criterion for verifying that $\mathcal{C}$ contains its Hermitian dual code $\mathcal{C}^{\perp_h}$.

Lemma 2.1: [23] Let $\mathcal{C}$ be a cyclic code of length n over $\mathbb{F}_{q^2}$ with defining set Z. Then $\mathcal{C}$ contains its Hermitian dual code $\mathcal{C}^{\perp_h}$ if and only if $Z \bigcap Z^{-q} = \emptyset$, where $Z^{-q} = \{-qz (\bmod\ n) : z \in Z\}$.

As we said before, scholars had proposed several methods to construct EAQMDS codes. Among these methods, the most frequently used one is to decompose the defining set of the associated codes, please see [7, 33] *et al.* Similar to such method, we can get the following result.

Theorem 2.2: Let $\mathcal{C}$ be a q^2-ary cyclic code of length n with defining set Z. Suppose that $\mathcal{Z} = Z \cap Z^{-q}$, where $Z^{-q} = \{-qz \ (\bmod\ n) : z \in Z\}$. If the parameters of $\mathcal{C}$ are $[n, n-|Z|, d]_{q^2}$, then there is an EAQEC code with parameters $[[n, n - 2|Z| + |\mathcal{Z}|, d; |\mathcal{Z}|]]_q$.

3 New EAQMDS codes of length $n = \frac{q^2-1}{a}$ with $a = \frac{\ell^2-1}{3}$

In this section, we will construct some new families of EAQMDS codes of length $n = \frac{q^2-1}{a}$, where $q = am \pm \ell$, and $a = \frac{\ell^2-1}{3}$ is an odd positive integer. Since $q^2 \equiv 1(\bmod\ n)$, then the q^2-cyclotomic coset C_x modulo n is $C_x = \{x\}$ for each x in the range $1 \le x \le n$.

3.1 The Case $q = am + \ell$

In this subsection, we consider that q is a prime power of the form $q = am+\ell$, where $a = \frac{\ell^2-1}{3}$ is an odd positive integer. As $a = \frac{\ell^2-1}{3}$ should be an odd integer, one can get $\ell \equiv 2(\bmod\ 6)$ or $\ell \equiv 4(\bmod\ 6)$. We first consider the case $\ell \equiv 2(\bmod\ 6)$ and a useful lemma is given in the following.

Lemma 3.1: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set

$$Z = \bigcup_{j=0}^{\delta} C_{1+j}, \quad 0 \le \delta \le (2t+1)m - 1,$$

then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof: According to Lemma 2.1, we only need to consider that $Z \cap Z^{-q} = \emptyset$. Suppose that $Z \cap Z^{-q} \neq \emptyset$, then there exist two integers i and j, where $0 \le i, j \le (2t+1)m - 1$, such that

$$1 + i \equiv -q(1 + j)(\bmod\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0(\bmod\ n).$$

Since $\ell = 6t + 2$, $\quad a = \frac{\ell^2-1}{3}$, then

$$q = am + \ell = (12t^2 + 8t + 1)m + 6t + 2,$$

$$n = \frac{q^2 - 1}{a} = (12t^2 + 8t + 1)m^2 + (12t + 4)m + 3.$$

If $km \le j \le (k+1)m - 1$, where $0 \le k \le 2t$. Then we have

$$kn+[12t^2+8t+1-k(6t+2)]m+6t+3-3k \le 1+q+i+qj \le (1+k)n-k(6t+2)m-3(1+k)-(4t+1)m,$$

Hence,

$$kn < 1 + q + i + qj < (1 + k)n,$$

which is a contradiction.

Therefore, we conclude that $Z \cap Z^{-q} = \emptyset$ as desired. Then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$. □

Lemma 3.2: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t + 2$, and t, m are positive integers. Then

(1) $-qC_{(2t+2)m+1} = C_{(8t+3)m+4}$;

(2) For $1 \leq i \leq 3$, we have $-qC_{i[(2t+1)m+1]} = C_{i[(2t+1)m+1]}$.

Proof: (1) As $q = (12t^2 + 8t + 1)m + (6t + 2)$ and $n = (12t^2 + 8t + 1)m^2 + (12t + 4)m + 3$, then we have

$$\begin{aligned} -q[(2t+2)m+1] &= -[(12t^2+8t+1)m + 6t+2][(2t+2)m+1] \\ &= -(2t+2)n + (8t+3)m + 4 \\ &\equiv (8t+3)m + 4 \pmod{n}, \end{aligned}$$

which implies that $-qC_{(2t+2)m+1} = C_{(8t+3)m+4}$.
(2) For $1 \leq i \leq 3$, $-qC_{i[(2t+1)m+1]} = C_{i[(2t+1)m+1]}$ holds for the following reason

$$\begin{aligned} -qi[(2t+1)m+1] &= -i[(12t^2+8t+1)m+6t+2][(2t+1)m+1] \\ &= -i[(2t+1)n - (2t+1)m - 1] \\ &\equiv i[(2t+1)m+1] \pmod{n}. \end{aligned}$$

□

Lemma 3.3: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set

$$Z = \bigcup_{j=0}^{\delta} C_{1+j}, \quad or \quad Z = \bigcup_{j=m}^{\delta'} C_{1+j},$$

then

$$|Z \cap Z^{-q}| = \begin{cases} 0, & 0 \leq \delta \leq (2t+1)m-1; \\ 1, & (2t+1)m \leq \delta \leq (4t+2)m; \\ 2, & (4t+2)m+1 \leq \delta' \leq (6t+3)m+1; \\ 3, & (6t+3)m+2 \leq \delta' \leq (8t+3)m+2. \end{cases}$$

Proof: (1) Let $Z = \bigcup_{j=0}^{\delta} C_{1+j}$, where $0 \leq \delta \leq (2t+1)m - 1$. Then $|Z \cap Z^{-q}| = 0$ follows from Lemma 3.1.
(2) Let

$$Z = \bigcup_{j=0}^{\delta} C_{1+j} = Z_1 \cup Z_2 \cup C_{(2t+1)m+1},$$

where $Z_1 = \bigcup_{j=0}^{(2t+1)m-1} C_{1+j}$, $Z_2 = \bigcup_{j=(2t+1)m+1}^{\delta} C_{1+j}$ and $(2t+1)m + 1 \leq \delta \leq (4t+2)m$. Then

$$\begin{aligned} Z^{-q} \cap Z &= ({Z_1}^{-q} \cup {Z_2}^{-q} \cup -qC_{(2t+1)m+1}) \cap (Z_1 \cup Z_2 \cup C_{(2t+1)m+1}) \\ &= ({Z_1}^{-q} \cap Z_1) \cup ({Z_1}^{-q} \cap Z_2) \cup ({Z_1}^{-q} \cap C_{(2t+1)m+1}) \cup \\ &\quad ({Z_2}^{-q} \cap Z_1) \cup ({Z_2}^{-q} \cap Z_2) \cup ({Z_2}^{-q} \cap C_{(2t+1)m+1}) \cup \\ &\quad (-qC_{(2t+1)m+1} \cap Z_1) \cup (-qC_{(2t+1)m+1} \cap Z_2) \cup \\ &\quad (-qC_{(2t+1)m+1} \cap C_{(2t+1)m+1}) \end{aligned}$$

According to Lemma 3.1, $Z_1 \cap Z_1^{-q} = \emptyset$. Due to Lemma 3.2, one can get

$$\begin{gathered}
-qC_{(2t+1)m+1} \cap Z_1 = \emptyset, \\
-qC_{(2t+1)m+1} \cap Z_2 = \emptyset, \\
Z_1{}^{-q} \cap C_{(2t+1)m+1} = \emptyset, \\
Z_2{}^{-q} \cap C_{(2t+1)m+1} = \emptyset, \\
-qC_{(2t+1)m+1} \cap C_{(2t+1)m+1} = C_{(2t+1)m+1}.
\end{gathered}$$

Now we only have to proof that $Z_1{}^{-q} \cap Z_2 = Z_2{}^{-q} \cap Z_1 = \emptyset$, $Z_2{}^{-q} \cap Z_2 = \emptyset$.

Suppose that $Z_2{}^{-q} \cap Z_1 \neq \emptyset$, then there exist two integers i and j, where $0 \leq i \leq (2t+1)m-1$ and $(2t+1)m+1 \leq j \leq (4t+2)m$ such that

$$1 + i \equiv -q(1+j)(\bmod\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0(\bmod\ n).$$

As $0 \leq i \leq (2t+1)m-1$. If $km+1 \leq j \leq (k+1)m$, where $2t+1 \leq k \leq 4t+1$. Then we have

$$\begin{aligned}
&kn + [24t^2 + 16t + 2 - k(6t+2)]m + 12t + 5 - 3k \\
&\leq 1 + q + i + qj \leq \\
&(1+k)n - [(1+k)(6t+2) - 12t^2 - 10t - 2]m - (3k - 6t + 1),
\end{aligned}$$

Hence,

$$kn < 1 + q + i + qj < (1+k)n,$$

which is a contradiction. It shows that $Z_2{}^{-q} \cap Z_1 = \emptyset$, then $Z_1{}^{-q} \cap Z_2 = (Z_2{}^{-q} \cap Z_1)^{-q} = \emptyset$.

Finally, suppose that $Z_2{}^{-q} \cap Z_2 \neq \emptyset$, then there exist two integers i and j, where $(2t+1)m+1 \leq i, j \leq (4t+2)m$, such that

$$1 + i \equiv -q(1+j)(\bmod\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0(\bmod\ n).$$

If $km+1 \leq j \leq (k+1)m$, where $2t+1 \leq k \leq 4t+1$. Then we have

$$\begin{aligned}
&kn + [24t^2 + 18t + 3 - k(6t+2)]m + 12t + 6 - 3k \\
&\leq 1 + q + i + qj \leq \\
&(1+k)n - [(1+k)(6t+2) - 12t^2 - 12t - 3]m - 3(k - 2t),
\end{aligned}$$

which implies that $kn < 1 + q + i + qj < (1+k)n$. It is also a contradiction. Hence, $Z_2{}^{-q} \cap Z_2 = \emptyset$. Therefore,

$$Z^{-q} \cap Z = C_{(2t+1)m+1} = \{(2t+1)m+1\},$$

which means that $|Z \cap Z^{-q}| = 1$.

(3) Let $Z=\bigcup_{j=m}^{\delta'}C_{1+j}=Z_1\cup Z_2\cup Z_3\cup C_{(2t+1)m+1}\cup C_{(4t+2)m+2}$, where $Z_1=\bigcup_{j=m}^{(2t+1)m-1}C_{1+j}$, $Z_2=\bigcup_{j=(2t+1)m+1}^{(4t+2)m}C_{1+j}$, $Z_3=\bigcup_{j=(4t+2)m+2}^{\delta'}C_{1+j}$ and $(4t+2)m+2\leq\delta'\leq(6t+3)m+1$. Going on the line of the proofs similar to the above cases, one can get $|Z\cap Z^{-q}|=2$.

(4) The remaining case can be proved by using the same method, here we omit it. □

Theorem 3.1: Let $n=\frac{q^2-1}{a}$, where q is a prime power of the form $q=am+\ell$, $a=\frac{\ell^2-1}{3}$, $\ell=6t+2$, and t,m are positive integers. Then there exist EAQMDS codes with the following parameters:
(1) $[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$, where $2\leq d\leq(2t+1)m+1$;
(2) $[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$, where $(2t+1)m+2\leq d\leq(4t+2)m+2$;
(3) $[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$, where $(4t+1)m+3\leq d\leq(6t+2)m+3$;
(4) $[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$, where $(6t+2)m+4\leq d\leq(8t+2)m+4$.

Proof: Let q be a prime power of the form $q=am+\ell$, $a=\frac{\ell^2-1}{3}$, and $\ell=6t+2$. Considering the cyclic code $\mathcal{C}$ of length $n=\frac{q^2-1}{a}$ over $\mathbb{F}_{q^2}$ with defining set

$$Z=\begin{cases}\bigcup_{j=0}^{\delta}C_{1+j}, & 0\leq\delta\leq(4t+2)m;\\ \bigcup_{j=m}^{\delta'}C_{1+j}, & m\leq\delta'\leq(8t+3)m+2.\end{cases}$$

(1) If $Z=\bigcup_{j=0}^{\delta}C_{1+j}$, where $0\leq\delta\leq(4t+2)m$. By Lemma 3.3, we get that $|Z\cap Z^{-q}|=0$ if $0\leq\delta\leq(2t+1)m-1$, and $|Z\cap Z^{-q}|=1$ if $(2t+1)m\leq\delta\leq(4t+2)m$. Since every q^2-cyclotomic coset $C_x=\{x\}$ and x is an integer, then one obtains that Z consists of $\delta+1$ consecutive integers

$$\{1,2,3,\ldots,\delta+1\},$$

which implies that $\mathcal{C}$ has minimum distance at least $\delta+2$. Hence, $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n,n-\delta-1,\geq\delta+2]$. Combining Theorem 2.2 with the EA-quantum Singleton bound, there are q-ary EAQMDS codes with parameters as desired.

(2) If $Z=\bigcup_{j=m}^{\delta'}C_{1+j}$, where $m\leq\delta'\leq(8t+3)m+2$. By Lemma 3.3, we obtain that $|Z\cap Z^{-q}|=2$ if $(4t+2)m+1\leq\delta'\leq(6t+3)m+1$, and $|Z\cap Z^{-q}|=3$ if $(6t+3)m+2\leq\delta'\leq(8t+3)m+2$. Since every q^2-cyclotomic coset $C_x=\{x\}$ and x is an integer, then one can get that Z consists of $\delta'-m+1$ consecutive integers

$$\{1+m,2+m,3+m,\ldots,\delta'+1\},$$

which implies that $\mathcal{C}$ has minimum distance at least $\delta'-m+2$. Hence, $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n,n-\delta'+m-1,\geq\delta'-m+2]$. Combining Theorem 2.2 with the EA-quantum Singleton bound, there are q-ary EAQMDS codes with parameters as desired. The result follows. □

Now we consider the case $\ell\equiv4(\text{mod }6)$ and a useful lemma is given in the following.

Lemma 3.4: Let $n=\frac{q^2-1}{a}$, where q is a prime power of the form $q=am+\ell$, $a=\frac{\ell^2-1}{3}$, $\ell=6t+4$, and t,m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set

$$Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j},$$

where $(2t+2)m+1 \leq \delta \leq (6t+5)m+1$, then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof: According to Lemma 2.1, we only need to proof that $Z \cap Z^{-q} = \emptyset$. Suppose that $Z \cap Z^{-q} \neq \emptyset$, then there exist two integers i and j, where $(2t+2)m+1 \leq i, j \leq (6t+5)m+1$ such that

$$1 + i \equiv -q(1+j) (\mathrm{mod}\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0 (\mathrm{mod}\ n).$$

Since $\ell = 6t+4, \quad a = \frac{\ell^2-1}{3}$, then

$$q = am + \ell = (12t^2 + 16t + 5)m + 6t + 4,$$

$$n = \frac{q^2-1}{a} = (12t^2 + 16t + 5)m^2 + (12t+8)m + 3.$$

Due to $(2t+2)m+1 \leq j \leq (6t+5)m+1$, we now divide into the following subcases.

(i) If $km + 1 \leq j \leq (k+1)m$, where $2t+2 \leq k \leq 4t+3$. Then we have

$$\begin{aligned}&kn + [(24t^2 + 34t + 12) - k(6t+4)]m + 12t + 10 - 3k \\ &\leq 1 + q + i + qj \leq \\ &(1+k)n - [(1+k)(6t+4) - (12t^2 + 22t + 10)]m - 3(k - 2t - 1),\end{aligned}$$

which implies that $kn < 1 + q + i + qj < (1+k)n$. It is a contradiction.

(ii) If $km + 2 \leq j \leq (k+1)m + 1$, where $4t+4 \leq k \leq 6t+4$. Then we have

$$\begin{aligned}&kn + [(36t^2 + 50t + 17) - k(6t+4)]m + 18t + 14 - 3k \\ &\leq 1 + q + i + qj \leq \\ &(1+k)n - [(1+k)(6t+4) - (24t^2 + 32t + 10)]m + 12t - 3k + 7,\end{aligned}$$

which implies that $kn < 1 + q + i + qj < (1+k)n$. It is also a contradiction.

(iii) Note that

$$-q[(4t+4)m + 1] \equiv (12t^2 + 24t + 11)m + 6t + 8\ (\mathrm{mod}\ n),$$

and $C_{(12t^2+24t+11)m+6t+8} \notin Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j}$, where $(2t+2)m+1 \leq \delta \leq (6t+5)m+1$,.

Therefore, we conclude that $Z \cap Z^{-q} = \emptyset$ as desired. □

Lemma 3.5: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. Then

(1) $-qC_{(6t+5)m+3} = C_{(6t+5)m+3}$;

(2) $-qC_{(8t+6)m+4} = C_{(4t+4)m+2}$;

(3) $-qC_{(10t+7)m+5} = C_{(2t+3)m+1}$.

Proof: (1) As $q = (12t^2+16t+5)m+6t+4$, and $n = (12t^2+16t+5)m^2+(12t+8)m+3$, then

$$\begin{aligned} -q[(6t+5)m+3] &= -[(12t^2+16t+5)m+6t+4][(6t+5)m+3] \\ &= -(6t+5)n+(6t+5)m+3 \\ &\equiv (6t+5)m+3 \pmod{n}, \end{aligned}$$

which implies that $-qC_{(6t+5)m+3} = C_{(6t+5)m+3}$.
(2) $-qC_{(8t+6)m+4} = C_{(4t+4)m+2}$ holds for the following reason

$$\begin{aligned} -q[(8t+6)m+4] &= -[(12t^2+16t+5)m+6t+4][(8t+6)m+4] \\ &= -(8t+6)n+(4t+4)m+2 \\ &\equiv (4t+4)m+2 \pmod{n}. \end{aligned}$$

(3) $-qC_{(10t+7)m+5} = C_{(2t+3)m+1}$ also holds for the following reason

$$\begin{aligned} -q[(10t+7)m+5] &= -[(12t^2+16t+5)m+6t+4][(10t+7)m+5] \\ &= -(10t+7)n+(2t+3)m+1 \\ &\equiv (2t+3)m+1 \pmod{n}. \end{aligned}$$

□

Lemma 3.6: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am+\ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set $Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j}$, then

$$|Z \cap Z^{-q}| = \begin{cases} 0, & (2t+2)m+1 \le \delta \le (6t+5)m+1; \\ 1, & (6t+5)m+2 \le \delta \le (8t+6)m+2; \\ 3, & (8t+6)m+3 \le \delta \le (10t+7)m+3. \end{cases}$$

Proof: (1) Let $Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j}$, where $(2t+2)m+1 \le \delta \le (6t+5)m+1$. Then $|Z \cap Z^{-q}| = 0$ follows from Lemma 3.4.
(2) Let

$$Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j} = Z_1 \cup Z_2 \cup C_{(6t+5)m+3},$$

where $Z_1 = \bigcup_{j=(2t+2)m+1}^{(6t+5)m+1} C_{1+j}$, $Z_2 = \bigcup_{j=(6t+5)m+3}^{\delta} C_{1+j}$ and $(6t+5)m+3 \le \delta \le (8t+6)m+2$.

Then

$$\begin{aligned}
Z^{-q} \cap Z = & (Z_1{}^{-q} \cup Z_2{}^{-q} \cup -qC_{(6t+5)m+3}) \cap (Z_1 \cup Z_2 \cup C_{(6t+5)m+3}) \\
= & (Z_1{}^{-q} \cap Z_1) \cup (Z_1{}^{-q} \cap Z_2) \cup (Z_1{}^{-q} \cap C_{(6t+5)m+3}) \cup \\
& (Z_2{}^{-q} \cap Z_1) \cup (Z_2{}^{-q} \cap Z_2) \cup (Z_2{}^{-q} \cap C_{(6t+5)m+3}) \cup \\
& (-qC_{(6t+5)m+3} \cap Z_1) \cup (-qC_{(6t+5)m+3} \cap Z_2) \cup \\
& (-qC_{(6t+5)m+3} \cap C_{(6t+5)m+3})
\end{aligned}$$

According to Lemma 3.4, $Z_1 \cap Z_1^{-q} = \emptyset$. It follows from Lemma 3.5, one can get

$$\begin{aligned}
-qC_{(6t+5)m+3} \cap Z_1 &= \emptyset, \\
-qC_{(6t+5)m+3} \cap Z_2 &= \emptyset, \\
Z_1{}^{-q} \cap C_{(6t+5)m+3} &= \emptyset, \\
Z_2{}^{-q} \cap C_{(6t+5)m+3} &= \emptyset, \\
-qC_{(6t+5)m+3} \cap C_{(6t+5)m+3} &= C_{(6t+5)m+3}.
\end{aligned}$$

Now we only have to proof that $Z_1{}^{-q} \cap Z_2 = Z_2{}^{-q} \cap Z_1 = \emptyset$, and $Z_2{}^{-q} \cap Z_2 = \emptyset$.

Suppose that $Z_2{}^{-q} \cap Z_1 \neq \emptyset$, then there exist two integers i and j, where $(2t+2)m+1 \leq i \leq (6t+5)m+1$ and $(6t+5)m+3 \leq j \leq (8t+6)m+2$, such that

$$1 + i \equiv -q(1+j) (\text{mod } n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0 \ (\text{mod } n).$$

We seek a contradiction as follows.

As $(2t+2)m+1 \leq i \leq (6t+5)m+1$. If $km+3 \leq j \leq (k+1)m+2$, where $6t+5 \leq k \leq 8t+5$. Then we have

$$\begin{aligned}
& kn + [48t^2 + 66t + 22 - k(6t+4)]m + 24t + 18 - 3k \\
& \leq 1 + q + i + qj \leq \\
& (1+k)n - [(1+k)(6t+4) - 36t^2 - 54t - 20]m - (3k - 18t - 11),
\end{aligned}$$

Hence,

$$kn < 1 + q + i + qj < (1+k)n.$$

It is a contradiction. Therefore, $Z_2{}^{-q} \cap Z_1 = \emptyset$, and $Z_1{}^{-q} \cap Z_2 = (Z_2{}^{-q} \cap Z_1)^{-q} = \emptyset$.

Finally, suppose that $Z_2{}^{-q} \cap Z_2 \neq \emptyset$, then there exist two integers i and j, where $(6t+5)m+3 \leq i, j \leq (8t+6)m+2$, such that

$$1 + q + i + qj \equiv 0 (\text{mod } n).$$

Going on the line of the proofs similar to the above cases, one can get such case is impossible either.

Therefore,

$$Z^{-q} \cap Z = C_{(6t+5)m+3} = \{(6t+5)m+3\},$$

which means that $|Z \cap Z^{-q}| = 1$.

(3) Let

$$Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j} = Z_1 \cup Z_2 \cup Z_3 \cup C_{(6t+5)m+3} \cup C_{(8t+6)m+4},$$

where $Z_1 = \bigcup_{j=(2t+2)m+1}^{(6t+5)m+1} C_{1+j}$, $Z_2 = \bigcup_{j=(6t+5)m+3}^{(8t+6)m+2} C_{1+j}$, $Z_3 = \bigcup_{j=(8t+6)m+4}^{\delta} C_{1+j}$ and $(8t+6)m+4 \leq \delta \leq (10t+7)m+3$. Then it can be proved by using the same method, we omit it here for simplification. □

Theorem 3.2: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. Then there exist EAQMDS codes with the following parameters:

(1) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 2, d]]_q$, where $2 \leq d \leq (4t+3)m+2$;

(2) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 3, d; 1]]_q$, where $(4t+3)m+3 \leq d \leq (6t+4)m+3$;

(3) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 5, d; 3]]_q$, where $(6t+4)m+4 \leq d \leq (8t+5)m+4$.

Proof: Let $\mathcal{C}$ be a cyclic code of length $n = \frac{q^2-1}{a}$ over $\mathbb{F}_{q^2}$ with defining set $Z = \bigcup_{j=(2t+2)m+1}^{\delta} C_{1+j}$, where $(2t+2)m+1 \leq \delta \leq (10t+7)m+3$.

By Lemma 3.6, we obtain that $|Z \cap Z^{-q}| = 0$ if $(2t+2)m+1 \leq \delta \leq (6t+5)m+1$, $|Z \cap Z^{-q}| = 1$ if $(6t+5)m+2 \leq \delta \leq (8t+6)m+2$, and $|Z \cap Z^{-q}| = 3$ if $(8t+6)m+3 \leq \delta \leq (10t+7)m+3$. Since every q^2-cyclotomic coset $C_x = \{x\}$ and x is an integer, then one can get that Z consists of $\delta - (2t+2)m$ consecutive integers

$$\{(2t+2)m+2, (2t+2)m+3, (2t+2)m+4, \ldots, \delta, \delta+1\},$$

which implies that $\mathcal{C}$ has minimum distance at least $\delta - (2t+2)m + 1$. Hence, $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n, n-\delta+(2t+2)m, \geq \delta-(2t+2)m+1]$. Combining Theorem 2.2 with the EA-quantum Singleton bound, there are q-ary EAQMDS codes with parameters as desired. The result follows. □

Example 3.1: In Table 1, we list some new EAQMDS codes of length $\frac{q^2-1}{a}$ obtained from Theorems 3.1 and 3.2, where q is a prime power of the form $q = am + \ell$, $a = \frac{\ell^2-1}{3}$ is an odd integer, and m is a positive integer.

3.2 The Case $q = am - \ell$

In this subsection, we consider the case q is a prime power of the form $q = am - \ell$, where $a = \frac{\ell^2-1}{3}$, $\ell \equiv 2 \pmod 6$ or $\ell \equiv 4 \pmod 6$, and ℓ is a positive integer. We first consider the case $\ell \equiv 2 \pmod 6$ and a useful lemma is given in the following.

Lemma 3.7: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set

$$Z = \bigcup_{j=2tm-1}^{\delta} C_{1+j},$$

Table 1. New EAQMDS codes of length $n = \frac{q^2-1}{a}$ with $a = \frac{\ell^2-1}{3}$ odd

ℓ	m	$q = am + \ell$	$[[n,k,d;c]]_q$	d
4	1	9	$[[16, 18-2d, d]]_9$	$2 \le d \le 5$
			$[[16, 19-2d, d; 1]]_9$	$6 \le d \le 7$
			$[[16, 21-2d, d; 3]]_9$	$8 \le d \le 9$
	3	19	$[[72, 74-2d, d]]_{19}$	$2 \le d \le 11$
			$[[72, 75-2d, d; 1]]_{19}$	$12 \le d \le 15$
			$[[72, 77-2d, d; 3]]_{19}$	$16 \le d \le 19$
	5	29	$[[168, 170-2d, d]]_{29}$	$2 \le d \le 17$
			$[[168, 171-2d, d; 1]]_{29}$	$18 \le d \le 23$
			$[[168, 173-2d, d; 3]]_{29}$	$24 \le d \le 29$
8	1	29	$[[40, 42-2d, d]]_{29}$	$2 \le d \le 4$
			$[[40, 43-2d, d; 1]]_{29}$	$5 \le d \le 8$
			$[[40, 44-2d, d; 2]]_{29}$	$8 \le d \le 11$
			$[[40, 45-2d, d; 3]]_{29}$	$12 \le d \le 14$
	3	71	$[[240, 242-2d, d]]_{71}$	$2 \le d \le 10$
			$[[240, 243-2d, d; 1]]_{71}$	$11 \le d \le 20$
			$[[240, 244-2d, d; 2]]_{71}$	$18 \le d \le 27$
			$[[240, 245-2d, d; 3]]_{71}$	$28 \le d \le 34$
10	1	43	$[[56, 58-2d, d]]_{43}$	$2 \le d \le 9$
			$[[56, 59-2d, d; 1]]_{43}$	$10 \le d \le 13$
			$[[56, 61-2d, d; 3]]_{43}$	$14 \le d \le 17$
	3	109	$[[360, 362-2d, d]]_{109}$	$2 \le d \le 23$
			$[[360, 363-2d, d; 1]]_{109}$	$24 \le d \le 33$
			$[[360, 365-2d, d; 3]]_{109}$	$34 \le d \le 43$
14	1	79	$[[96, 98-2d, d]]_{79}$	$2 \le d \le 6$
			$[[96, 99-2d, d; 1]]_{79}$	$7 \le d \le 12$
			$[[96, 100-2d, d; 2]]_{79}$	$12 \le d \le 17$
			$[[96, 101-2d, d; 3]]_{79}$	$18 \le d \le 22$

where $2tm-1 \leq \delta \leq (6t+1)m-5$, then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof: Due to Lemma 2.1, we have to proof that $Z \cap Z^{-q} = \emptyset$. Assume that $Z \cap Z^{-q} \neq \emptyset$, then there exist two integers i and j, where $2tm-1 \leq i, j \leq (6t+1)m-5$, such that

$$1+i \equiv -q(1+j)(\mathrm{mod}\ n),$$

which is equivalent to

$$1+q+i+qj \equiv 0(\mathrm{mod}\ n).$$

Since $\ell = 6t+2, \quad a = \frac{\ell^2-1}{3}$, then

$$q = am - \ell = (12t^2+8t+1)m-6t-2,$$

$$n = \frac{q^2-1}{a} = (12t^2+8t+1)m^2-(12t+4)m+3.$$

We now divide into the following subcases to seek some contradictions.

(i) If $km-2 \leq j \leq (k+1)m-3$, where $2t+1 \leq k \leq 4t-1$. Then we have

$$\begin{aligned}&kn+[k(6t+2)-12t^2-6t-1]m+6t+2-3k\\&\leq 1+q+i+qj \leq\\&(1+k)n-[24t^2+10t+1-(k+1)(6t+2)]m+12t-3k-3.\end{aligned}$$

Hence,

$$kn < 1+q+i+qj < (1+k)n.$$

It is a contradiction.

(ii) If $km-3 \leq j \leq (k+1)m-4$, where $4t+1 \leq k \leq 6t-1$. Then we have

$$\begin{aligned}&kn+[k(6t+2)-24t^2-14t-2]m+12t+4-3k\\&\leq 1+q+i+qj \leq\\&(1+k)n-[36t^2+18t+2-(k+1)(6t+2)]m+18t-3k-1.\end{aligned}$$

Hence,

$$kn < 1+q+i+qj < (1+k)n.$$

It is a contradiction.

(iii) If $2tm-1 \leq j \leq (2t+1)m-3$, then

$$2tn+(12t^2+6t)m-6t \leq 1+q+i+qj \leq (2t+1)n-(12t^2-1)m+6t-3,$$

which implies that $2tn < 1+q+i+qj < (2t+1)n$. Hence, it is a contradiction.

(iv) If $4tm-2 \leq j \leq (4t+1)m-4$, then

$$4tn+(12t^2+2t-1)m-6t+2 \leq 1+q+i+qj \leq (4t+1)n-(12t^2+4t)m+6t-1,$$

which implies that $4tn < 1+q+i+qj < (4t+1)n$. Hence, it is a contradiction.

(v) If $6tm-3 \leq j \leq (6t+1)m-5$, then

$$6tn+(12t^2-2t-2)m-6t+4 \leq 1+q+i+qj \leq (6t+1)n-q-1,$$

which implies that $6tn < 1 + q + i + qj < (6t+1)n$. Hence, it is a contradiction.

Therefore, we conclude that $Z \cap Z^{-q} = \emptyset$ as desired, which implied that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$. $\square$

Lemma 3.8: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. Then

(1) $-qC_{(6t+1)m-3} = C_{(6t+1)m-3}$;

(2) $-qC_{(8t+2)m-4} = C_{4tm-2}$;

(3) $-qC_{(10t+2)m-5} = C_{(8t+1)m-4}$.

Proof: (1) As $q = (12t^2+8t+1)m - (6t+2)$ and $n = (12t^2+8t+1)m^2 - (12t+4)m + 3$, then we have

$$\begin{aligned} -q[(6t+1)m-3] &= -[(12t^2+8t+1)m-(6t+2)][(6t+1)m-3] \\ &= -(6t+1)n + (6t+1)m - 3 \\ &\equiv (6t+1)m - 3 \pmod{n}, \end{aligned}$$

which implies that $-qC_{(6t+1)m-3} = C_{(6t+1)m-3}$.
(2) $-qC_{(8t+2)m-4} = C_{4tm-2}$ holds for the following reason

$$\begin{aligned} -q[(8t+2)m-4] &= -[(12t^2+8t+1)m-(6t+2)][(8t+2)m-4] \\ &= -(8t+2)n + 4tm - 2 \\ &\equiv 4tm - 2 \pmod{n}. \end{aligned}$$

(3) $-qC_{(10t+2)m-5} = C_{(8t+1)m-4}$ also holds for the following reason

$$\begin{aligned} -q[(10t+2)m-5] &= -[(12t^2+8t+1)m-(6t+2)][(10t+2)m-5] \\ &= -(10t+2)n + (8t+1)m - 4 \\ &\equiv (8t+1)m - 4 \pmod{n}. \end{aligned}$$

$\square$

Lemma 3.9: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set $Z = \bigcup_{j=2tm-1}^{\delta} C_{1+j}$, then

$$|Z \cap Z^{-q}| = \begin{cases} 0, & 2tm-1 \le \delta \le (6t+1)m-5; \\ 1, & (6t+1)m-4 \le \delta \le (8t+2)m-6; \\ 3, & (8t+2)m-5 \le \delta \le (10t+2)m-7. \end{cases}$$

Proof: (1) Let $Z = \bigcup_{j=2tm-1}^{\delta} C_{1+j}$, where $2tm-1 \le \delta \le (6t+1)m-5$. Then $|Z \cap Z^{-q}| = 0$ follows from Lemma 3.7.
(2) Let

$$Z = \bigcup_{j=2tm-1}^{\delta} C_{1+j} = Z_1 \cup Z_2 \cup C_{(6t+1)m-3},$$

where $Z_1 = \bigcup_{j=2tm-1}^{(6t+1)m-5} C_{1+j}$, $Z_2 = \bigcup_{j=(6t+1)m-3}^{\delta} C_{1+j}$ and $(6t+1)m-3 \le \delta \le (8t+2)m-6$. Then

$$\begin{aligned} Z^{-q} \cap Z &= ({Z_1}^{-q} \cup {Z_2}^{-q} \cup -qC_{(6t+1)m-3}) \cap (Z_1 \cup Z_2 \cup C_{(6t+1)m-3}) \\ &= ({Z_1}^{-q} \cap Z_1) \cup ({Z_1}^{-q} \cap Z_2) \cup ({Z_1}^{-q} \cap C_{(6t+1)m-3}) \cup \\ &\quad ({Z_2}^{-q} \cap Z_1) \cup ({Z_2}^{-q} \cap Z_2) \cup ({Z_2}^{-q} \cap C_{(6t+1)m-3}) \cup \\ &\quad (-qC_{(6t+1)m-3} \cap Z_1) \cup (-qC_{(6t+1)m-3} \cap Z_2) \cup \\ &\quad (-qC_{(6t+1)m-3} \cap C_{(6t+1)m-3}) \end{aligned}$$

By Lemma 3.7, $Z_1 \cap Z_1^{-q} = \emptyset$. It follows from Lemma 3.8, one can get

$$\begin{aligned} -qC_{(6t+1)m-3} \cap Z_1 &= \emptyset, \\ -qC_{(6t+1)m-3} \cap Z_2 &= \emptyset, \\ {Z_1}^{-q} \cap C_{(6t+1)m-3} &= \emptyset, \\ {Z_2}^{-q} \cap C_{(6t+1)m-3} &= \emptyset, \\ -qC_{(6t+1)m-3} \cap C_{(6t+1)m-3} &= C_{(6t+1)m-3}. \end{aligned}$$

Now we only have to verify that ${Z_1}^{-q} \cap Z_2 = {Z_2}^{-q} \cap Z_1 = \emptyset$, and ${Z_2}^{-q} \cap Z_2 = \emptyset$.

Assume that ${Z_2}^{-q} \cap Z_1 \neq \emptyset$, then there exist two integers i and j, where $2tm-1 \le i \le (6t+1)m-5$ and $(6t+1)m-3 \le j \le (8t+2)m-6$, such that

$$1 + i \equiv -q(1+j) (\text{mod } n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0 \ (\text{mod } n).$$

We seek contradictions by dividing into the following subcases.

(i) If $km-4 \le j \le (k+1)m-5$, where $6t+2 \le k \le 8t$. Then we have

$$\begin{aligned} &kn + [k(6t+2) - 36t^2 - 22t - 3]m + 18t + 6 - 3k \\ &\le 1 + q + i + qj \le \\ &(1+k)n - [48t^2 + 26t + 3 - (1+k)(6t+2)]m + 24t - 3k + 1, \end{aligned}$$

Hence,

$$kn < 1 + q + i + qj < (1+k)n.$$

It is a contradiction.

(ii) If $(6t+1)m-3 \le j \le (6t+2)m-5$, then

$$(6t+1)n + (12t^2+4t)m - 6t + 1 \le 1 + q + i + qj \le (6t+2)n - (12t^2+2t-1)m + 6t - 2,$$

which implies that $(6t+1)n < 1 + q + i + qj < (6t+2)n$. It is a contradiction.

(iii) If $(8t+1)m-4 \le j \le (8t+2)m-6$, then

$$(8t+1)n + (12t^2-1)m - 6t + 3 \le 1 + q + i + qj \le (8t+2)n - (12t^2+6t)m + 6t,$$

which implies that $(8t+1)n < 1 + q + i + qj < (8t+2)n$. It is a contradiction.

Therefore, $Z_2{}^{-q} \cap Z_1 = \emptyset$, and $Z_1{}^{-q} \cap Z_2 = (Z_2{}^{-q} \cap Z_1)^{-q} = \emptyset$.

Finally, suppose that $Z_2{}^{-q} \cap Z_2 \neq \emptyset$, then there exist two integers i and j, where $(6t+1)m-3 \leq i, j \leq (8t+2)m-6$ such that

$$1+q+i+qj \equiv 0 (\bmod\ n).$$

Going on the line of the proofs similar to the above cases, one can get such case is impossible either.

Therefore,

$$Z^{-q} \cap Z = C_{(6t+1)m-3} = \{(6t+1)m-3\},$$

which means that $|Z \cap Z^{-q}| = 1$.

(3) The remaining case can be proved by using the same method, we omit it here for simplification. □

Theorem 3.3: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+2$, and t, m are positive integers. Then there exist EAQMDS codes with the following parameters:
(1) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 2, d]]_q$, where $2 \leq d \leq (4t+1)m-2$;
(2) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 3, d; 1]]_q$, where $(4t+1)m-1 \leq d \leq (6t+2)m-3$;
(3) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 5, d; 3]]_q$, where $(6t+2)m-2 \leq d \leq (8t+2)m-4$.

Proof: Let $\mathcal{C}$ be a cyclic code of length $n = \frac{q^2-1}{a}$ over $\mathbb{F}_{q^2}$ with defining set $Z = \bigcup_{j=2tm-1}^{\delta} C_{1+j}$, where $2tm-1 \leq \delta \leq (10t+2)m-7$.

By Lemma 3.9, we obtain that $|Z \cap Z^{-q}| = 0$ if $2tm-1 \leq \delta \leq (6t+1)m-5$, $|Z \cap Z^{-q}| = 1$ if $(6t+1)m-4 \leq \delta \leq (8t+2)m-6$, and $|Z \cap Z^{-q}| = 3$ if $(8t+2)m-5 \leq \delta \leq (10t+2)m-7$. Since every q^2-cyclotomic coset $C_x = \{x\}$ and x is an integer, then one can get that Z consists of $\delta - 2tm + 2$ consecutive integers

$$\{2tm, 2tm+1, 2tm+2, \ldots, \delta, \delta+1\},$$

which implies that $\mathcal{C}$ has minimum distance at least $\delta - 2tm + 3$. Hence, $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n, n-\delta+2tm-2, \geq \delta-2tm+3]$. Combining Theorem 2.2 with the EA-quantum Singleton bound, there are q-ary EAQMDS codes with parameters as desired. The result follows. □

Remark 3.1: Let $t = 1$, then EAQMDS codes of length $\frac{q^2-1}{21}$ with $q = 21m - 8$ have been constructed. Actually, EAQMDS codes of length $\frac{q^2-1}{21}$ with $q = 42m + 13$ have also been constructed in [31] with different c from ours. Within the same $c = 3$, one can see that our codes have larger minimum distances than theirs. For example, if $q = 97$, we get EAQMDS codes with parameters $[[448, 453-2d, d; 3]]_{97}$, where $38 \leq d \leq 46$, while the EAQMDS codes constructed in [31] have parameters $[[448, 453-2d, d; 3]]_{97}$, where $32 \leq d \leq 45$.

Now we consider the case $\ell \equiv 4 \pmod 6$ and a useful lemma is shown in the following.

Lemma 3.10: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set

$$Z = \bigcup_{j=0}^{\delta} C_{1+j},$$

where $0 \leq \delta \leq (2t+1)m-3$, then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof: Due to Lemma 2.1, we have to proof that $Z \cap Z^{-q} = \emptyset$. Assume that $Z \cap Z^{-q} \neq \emptyset$, then there exist two integers i and j, where $0 \leq i, j \leq (2t+1)m-3$, such that

$$1 + i \equiv -q(1+j) (\mathrm{mod}\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0 (\mathrm{mod}\ n).$$

Since $\ell = 6t+4, \quad a = \frac{\ell^2-1}{3}$, then

$$q = am - \ell = (12t^2 + 16t + 5)m - 6t - 4,$$

$$n = \frac{q^2-1}{a} = (12t^2 + 16t + 5)m^2 - (12t+8)m + 3.$$

We now divide into the following subcases to seek some contradictions.

(i) If $km - 1 \leq j \leq (k+1)m - 2$, where $1 \leq k \leq 2t-1$. Then we have

$$kn + k(6t+4)m - 3k + 1 \leq 1 + q + i + qj \leq (1+k)n - [12t^2 + 14t + 4 - (k+1)(6t+4)]m + 6t - 3k - 1.$$

Hence,

$$kn < 1 + q + i + qj < (1+k)n.$$

It is a contradiction.

(ii) If $2tm - 1 \leq j \leq (2t+1)m - 3$, then

$$2tn < 2tn + 2t(6t+4)m - 6t + 1 \leq 1 + q + i + qj \leq (2t+1)n - q - 1 < (2t+1)n,$$

which is a contradiction.

(iii) If $0 \leq j \leq m-2$, then

$$0 < 1 + q \leq 1 + q + i + qj \leq n - (12t^2 + 8t)m + 6t - 1 < n,$$

which is also a contradiction.

Therefore, we conclude that $Z \cap Z^{-q} = \emptyset$ as desired, which implied that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$. $\square$

Lemma 3.11: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. Then for $1 \leq i \leq 4$, we have $-qC_{i[(2t+1)m-1]} = C_{i[(2t+1)m-1]}$.

Proof: As $q = (12t^2 + 16t + 5)m - (6t+4)$ and $n = (12t^2 + 16t + 5)m^2 - (12t+8)m + 3$, then we have

$$\begin{aligned} -qi[(2t+1)m-1] &= -i[(12t^2 + 16t + 5)m - (6t+4)][(2t+1)m-1] \\ &= i[-(2t+1)n + (2t+1)m - 1] \\ &\equiv i[(2t+1)m-1] \ (\mathrm{mod}\ n), \end{aligned}$$

which implies that $-qC_{i[(2t+1)m-1]} = C_{i[(2t+1)m-1]}$, where $1 \le i \le 4$. $\square$

Lemma 3.12: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. If $\mathcal{C}$ is a q^2-ary cyclic code of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{1+j}$, then

$$|Z \cap Z^{-q}| = \begin{cases} 0, & 0 \le \delta \le (2t+1)m-3; \\ 1, & (2t+1)m-2 \le \delta \le (4t+2)m-4; \\ 2, & (4t+2)m-3 \le \delta \le (6t+3)m-5; \\ 3, & (6t+3)m-4 \le \delta \le (8t+4)m-6. \end{cases}$$

Proof: (1) Let $Z = \bigcup_{j=0}^{\delta} C_{1+j}$, where $0 \le \delta \le (2t+1)m-3$. Then $|Z \cap Z^{-q}| = 0$ follows from Lemma 3.10.
(2) Let

$$Z = \bigcup_{j=0}^{\delta} C_{1+j} = Z_1 \cup Z_2 \cup C_{(2t+1)m-1},$$

where $Z_1 = \bigcup_{j=0}^{(2t+1)m-3} C_{1+j}$, $Z_2 = \bigcup_{j=(2t+1)m-1}^{\delta} C_{1+j}$ and $(2t+1)m-1 \le \delta \le (4t+2)m-4$. Then

$$\begin{aligned} Z^{-q} \cap Z &= ({Z_1}^{-q} \cup {Z_2}^{-q} \cup -qC_{(2t+1)m-1}) \cap (Z_1 \cup Z_2 \cup C_{(2t+1)m-1}) \\ &= ({Z_1}^{-q} \cap Z_1) \cup ({Z_1}^{-q} \cap Z_2) \cup ({Z_1}^{-q} \cap C_{(2t+1)m-1}) \cup \\ &\quad ({Z_2}^{-q} \cap Z_1) \cup ({Z_2}^{-q} \cap Z_2) \cup ({Z_2}^{-q} \cap C_{(2t+1)m-1}) \cup \\ &\quad (-qC_{(2t+1)m-1} \cap Z_1) \cup (-qC_{(2t+1)m-1} \cap Z_2) \cup \\ &\quad (-qC_{(2t+1)m-1} \cap C_{(2t+1)m-1}) \end{aligned}$$

According to Lemma 3.10, $Z_1 \cap Z_1^{-q} = \emptyset$. It follows from Lemma 3.11, one can get

$$\begin{aligned} -qC_{(2t+1)m-1} \cap Z_1 &= \emptyset, \\ -qC_{(2t+1)m-1} \cap Z_2 &= \emptyset, \\ {Z_1}^{-q} \cap C_{(2t+1)m-1} &= \emptyset, \\ {Z_2}^{-q} \cap C_{(2t+1)m-1} &= \emptyset, \\ -qC_{(2t+1)m-1} \cap C_{(2t+1)m-1} &= C_{(2t+1)m-1}. \end{aligned}$$

Now we only have to verify that ${Z_1}^{-q} \cap Z_2 = {Z_2}^{-q} \cap Z_1 = \emptyset$, ${Z_2}^{-q} \cap Z_2 = \emptyset$.

Assume that ${Z_2}^{-q} \cap Z_1 \neq \emptyset$, then there exist two integers i and j, where $0 \le i \le (2t+1)m-3$ and $(2t+1)m-1 \le j \le (4t+2)m-4$ such that

$$1 + i \equiv -q(1+j) (\mathrm{mod}\ n),$$

which is equivalent to

$$1 + q + i + qj \equiv 0 (\mathrm{mod}\ n).$$

We now divide into the following subcases to seek some contradictions.

(i) If $(2t+1)m-1 \le j \le (2t+2)m-3$, then

$$\begin{aligned}&(2t+1)n+(12t^2+14t+4)m-6t-2\\&\le 1+q+i+qj\le\\&(2t+2)n-(12t^2+10t+1)m+6t,\end{aligned}$$

which implies that $(2t+1)n < 1+q+i+qj < (2t+2)n$. It is a contradiction.

(ii) If $km-2 \le j \le (k+1)m-3$, where $2t+2 \le k \le 4t$. Then we have

$$\begin{aligned}&kn+[k(6t+4)-(12t^2+16t+5)]m+6t+5-3k\\&\le 1+q+i+qj\le\\&(1+k)n-[24t^2+30t+9-(1+k)(6t+4)]m+12t-3k+3.\end{aligned}$$

Hence,

$$kn < 1+q+i+qj < (1+k)n,$$

which is a contradiction.

(iii) If $(4t+1)m-2 \le j \le (4t+2)m-4$, then

$$\begin{aligned}&(4t+1)n+(12t^2+6t-1)m-6t+2\\&\le 1+q+i+qj\le\\&(4t+2)n-(12t^2+18t+6)m+6t+4,\end{aligned}$$

which implies that $(4t+1)n < 1+q+i+qj < (4t+2)n$. It is a contradiction.

Therefore, $Z_2{}^{-q} \cap Z_1 = \emptyset$, and $Z_1{}^{-q} \cap Z_2 = (Z_2{}^{-q} \cap Z_1)^{-q} = \emptyset$.

Finally, suppose that $Z_2{}^{-q} \cap Z_2 \ne \emptyset$, then there exist two integers i and j, where $(2t+1)m-1 \le i,j \le (4t+2)m-4$, such that

$$1+q+i+qj \equiv 0 (\text{mod } n).$$

Going on the line of the proofs similar to the above cases, such case is impossible either. Therefore,

$$Z^{-q} \cap Z = C_{(2t+1)m-1} = \{(2t+1)m-1\},$$

which means that $|Z \cap Z^{-q}| = 1$.

(3) The remaining cases can be proved by using the same method, here we omit it. □

Theorem 3.4: Let $n = \frac{q^2-1}{a}$, where q is a prime power of the form $q = am-\ell$, $a = \frac{\ell^2-1}{3}$, $\ell = 6t+4$, and t, m are positive integers. Then there exist EAQMDS codes with the following parameters:
(1) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a}-2d+2, d]]_q$, where $2 \le d \le (2t+1)m-1$;
(2) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a}-2d+3, d; 1]]_q$, where $(2t+1)m \le d \le (4t+2)m-2$;
(3) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a}-2d+4, d; 2]]_q$, where $(4t+2)m-1 \le d \le (6t+3)m-3$;
(4) $[[\frac{q^2-1}{a}, \frac{q^2-1}{a}-2d+5, d; 3]]_q$, where $(6t+3)m-2 \le d \le (8t+4)m-4$.

Proof: Let $\mathcal{C}$ be a cyclic code of length $n = \frac{q^2-1}{a}$ over $\mathbb{F}_{q^2}$ with defining set $Z = \bigcup_{j=0}^{\delta} C_{1+j}$, where $0 \le \delta \le (8t+4)m-4$.

According to Lemma 3.12, we obtain that

$$c = |Z \cap Z^{-q}| = \begin{cases} 0, & 0 \le \delta \le (2t+1)m-3; \\ 1, & (2t+1)m-2 \le \delta \le (4t+2)m-4; \\ 2, & (4t+2)m-3 \le \delta \le (6t+3)m-5; \\ 3, & (6t+3)m-4 \le \delta \le (8t+4)m-6. \end{cases}$$

Since every q^2-cyclotomic coset $C_x = \{x\}$ and x is an integer, then one can get that Z consists of $\delta+1$ consecutive integers

$$\{1, 2, 3, \ldots, \delta, \delta+1\},$$

which implies that $\mathcal{C}$ has minimum distance at least $\delta+2$. Hence, $\mathcal{C}$ is a q^2-ary cyclic code with parameters $[n, n-\delta-1, \ge \delta+2]$. Combining Theorem 2.2 with the EA-quantum Singleton bound, there are q-ary EAQMDS codes with parameters as desired. The result follows. □

Remark 3.2: Let $t = 0$ in Theorems 3.2 and 3.4, then EAQMDS codes of length $\frac{q^2-1}{5}$ have been constructed, where $q = 5m \pm 4$. Actually, EAQMDS codes of length $\frac{q^2-1}{5}$ have also been constructed in [29] and [30]. However, their q is different from ours.

Example 3.2: In Table 2, we list some new EAQMDS codes of length $\frac{q^2-1}{a}$ obtained from Theorems 3.3 and 3.4, where q is a prime power of the form $q = am - \ell$, $a = \frac{\ell^2-1}{3}$ is an odd integer, and m is a positive integer.

4 Conclusion

EAQMDS codes with parameters $[[\frac{q^2-1}{a}, \frac{q^2-1}{a} - 2d + 2 + c, d; c]]_q$ were constructed by exploiting less pre-shared maximally entangled states c, where q is a prime power with the form $q = am \pm \ell$, $a = \frac{\ell^2-1}{3}$ is an odd integer, $\ell \equiv 2 \pmod 6$ or $\ell \equiv 4 \pmod 6$, and m is a positive integer. As said in [45], EAQMDS codes of length $\frac{q^2-1}{a}$ with a either divides $q+1$ or divides $q-1$ had been extensively studied. However, our a either divides $q+\ell$ or divides $q-\ell$.

In [1], maximal-entanglement EAQMDS codes, i.e., $c = n-k$, were constructed. It is easy to see that our EAQMDS codes are not maximal-entanglement ones.

In [5], the author presented that an EAQEC code with parameters $[[2n, n-h, d, n-h]]_q$ can be derived from an Hermitian self-dual code with parameters $[2n, n, d]_{q^2}$, while in [4], some families of EAQMDS codes with flexible parameters were also constructed via Hermitian self-dual codes and three of them with the similar lengths to ours are listed below:

- $[[\frac{q^2-1}{m}, \frac{q^2-1}{m} - w - h, w+1, w-h]]_q$, where $q \ge 3$ is a prime power, $m = 2k+1$ is an odd divisor of $q+1$, w is a positive integer satisfying $w < \frac{(k+1)(q-1)}{2k+1}$, and h is a nonnegative integer satisfying $0 \le h \le w$.
- $[[\frac{q^2-1}{m_1} + \frac{q^2-1}{m_2} - \frac{q^2-1}{m_1 m_2}, \frac{q^2-1}{m_1} + \frac{q^2-1}{m_2} - \frac{q^2-1}{m_1 m_2} - k - h, k+1, k-h]]_q$, where $q \ge 3$ is a prime power, m_1 and m_2 are odd divisors of $q+1$ satisfying $\gcd(m_1, m_2) = 1$, k is a positive integer satisfying $1 \le k \le \frac{q-1}{2}$, and h is a nonnegative integer satisfying $0 \le h \le k$.

Table 2. New EAQMDS codes of length $n = \frac{q^2-1}{a}$ with $a = \frac{\ell^2-1}{3}$ odd

ℓ	m	$q = am - \ell$	$[[n,k,d;c]]_q$	d
8	5	97	$[[448, 450-2d, d]]_{97}$	$2 \leq d \leq 23$
			$[[448, 451-2d, d; 1]]_{97}$	$24 \leq d \leq 37$
			$[[448, 453-2d, d; 3]]_{97}$	$38 \leq d \leq 46$
10	3	89	$[[240, 242-2d, d]]_{89}$	$2 \leq d \leq 8$
			$[[240, 243-2d, d; 1]]_{89}$	$9 \leq d \leq 16$
			$[[240, 244-2d, d; 2]]_{89}$	$17 \leq d \leq 24$
			$[[240, 245-2d, d; 3]]_{89}$	$25 \leq d \leq 32$
14	3	181	$[[504, 506-2d, d]]_{181}$	$2 \leq d \leq 25$
			$[[504, 507-2d, d; 1]]_{181}$	$26 \leq d \leq 39$
			$[[504, 509-2d, d; 3]]_{181}$	$40 \leq d \leq 50$
16	3	239	$[[672, 674-2d, d]]_{239}$	$2 \leq d \leq 14$
			$[[672, 675-2d, d; 1]]_{239}$	$15 \leq d \leq 28$
			$[[672, 676-2d, d; 2]]_{239}$	$29 \leq d \leq 42$
			$[[672, 677-2d, d; 3]]_{239}$	$43 \leq d \leq 56$
20	1	113	$[[96, 98-2d, d]]_{113}$	$2 \leq d \leq 11$
			$[[96, 99-2d, d; 1]]_{113}$	$12 \leq d \leq 17$
			$[[96, 101-2d, d; 3]]_{113}$	$18 \leq d \leq 22$
22	1	139	$[[120, 122-2d, d]]_{139}$	$2 \leq d \leq 6$
			$[[120, 123-2d, d; 1]]_{139}$	$7 \leq d \leq 12$
			$[[120, 124-2d, d; 2]]_{139}$	$13 \leq d \leq 18$
			$[[120, 125-2d, d; 3]]_{139}$	$19 \leq d \leq 24$

- $[[\frac{q^2-1}{m}, \frac{q^2-1}{m} - k - h, k+1, k-h]]_q$, where $q = 2^h a + 1 \geq 3$ is an odd prime power, a is odd, $m = 2^{h_1} a_1 \geq 6$ is an even divisor of $q-1$, $h_1 \leq h$, a_1 is an odd divisor of a, k is a positive integer satisfying $1 \leq k \leq \frac{q+1}{2} + 2^{h-h_1} \frac{a}{a_1} - 1$, and h is a nonnegative integer satisfying $0 \leq h \leq k$.

One can see that their lengths are different from ours, and the method presented in [4, 5] is also different from ours.

In [6], the author proved that for any given length $n \leq q^2 + 1$ and any given distance $d \leq \frac{n+2}{2}$, there exsits at least one $[[n, k, d, c]]$ EAQMDS code with nonzero c parameter. In one sense, our results proved the rightness of his.

In [36], three new propagation rules for constructing EAQEC codes were introduced:

- $[[n, k, d, c]]_q \longrightarrow [[n, k+i, d, c+i]]_q$;
- $[[n, k, d, c]]_q \longrightarrow [[n+1, k-1, d', c]]_q$, where $d \leq d' \leq d+1$;
- $[[n, k, d, c]]_q \longrightarrow [[n+1, k, d', c-1]]_q$, where $d' \leq d$.

Actually, the idea of the first propagation rule is quite the same as our construction method (Theorem 2.2), and the EAQMDS codes in this paper can't be derived via the last two propagation rules due to the fact that EAQEC codes of length $\frac{q^2-1}{a} - 1$ with $a = \frac{\ell^2-1}{3}$ are unknown either.

Hence, EAQMDS codes obtained in this paper are new in the sense that their parameters are not covered by the codes available in the literature, please see Table 3. The case a being an even integer will be considered later.

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Table 3. New EAQMDS codes

a	q	ℓ	$[[n,k,d;c]]_q$	d	Refs.
ℓ^2-1	$am+\ell$	odd	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le \ell m+1$	[45]
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$(\ell-1)m+2\le d\le (2\ell-1)m+2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(2\ell-1)m+3\le d\le 2\ell m+2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+6,d;4]]_q$	$2\ell m+3\le d\le (3\ell-2)m+3$	
	$am-\ell$		$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le (\ell-1)m-1$	[45]
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$m+1\le d\le \ell m-1$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$\ell m\le d\le (2\ell-1)m-2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(2\ell-1)m-1\le d\le 2\ell m-2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+7,d;5]]_q$	$2\ell m-1\le d\le (3\ell-2)m-3$	
$\frac{\ell^2-1}{2}$	$am+\ell$	$\ell\equiv 1 \pmod 4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le \ell m+2$	[45]
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$(\ell-1)m+3\le d\le \frac{3\ell-1}{2}m+3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+6,d;4]]_q$	$\frac{3\ell-1}{2}m+4\le d\le (2\ell-1)m+4$	
		$\ell\equiv 3 \pmod 4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le \frac{\ell+1}{2}m+1$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$\frac{\ell-1}{2}m+2\le d\le \ell m+2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$\ell m+3\le d\le \frac{3\ell-1}{2}m+3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+6,d;4]]_q$	$\frac{3\ell-1}{2}m+4\le d\le \frac{3\ell+1}{2}m+3$	
	$am-\ell$	$\ell\equiv 1 \pmod 4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le \frac{\ell-1}{2}m-1$	[45]
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$\frac{\ell-1}{2}m\le d\le (\ell-1)m-2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$(\ell-1)m-1\le d\le \frac{3(\ell-1)}{2}m-3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$\frac{3(\ell-1)}{2}m-2\le d\le \frac{3\ell-1}{2}m-3$	
		$\ell\equiv 3 \pmod 4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le \ell m-2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$\ell m-1\le d\le \frac{3\ell-1}{2}m-3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+6,d;4]]_q$	$\frac{3\ell-1}{2}m-2\le d\le (2\ell-1)m-4$	
$\frac{\ell^2-1}{3}$	$am+\ell$	$6t+2$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le (2t+1)m+1$	New
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$(2t+1)m+2\le d\le (4t+2)m+2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$(4t+1)m+3\le d\le (6t+2)m+3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(6t+2)m+4\le d\le (8t+2)m+4$	
		$6t+4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le (4t+3)m+2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$(4t+3)m+3\le d\le (6t+4)m+3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(6t+4)m+4\le d\le (8t+5)m+4$	
	$am-\ell$	$6t+2$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le (4t+1)m-2$	New
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$(4t+1)m-1\le d\le (6t+2)m-3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(6t+2)m-2\le d\le (8t+2)m-4$	
		$6t+4$	$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+2,d]]_q$	$2\le d\le (2t+1)m-1$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+3,d;1]]_q$	$(2t+1)m\le d\le (4t+2)m-2$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+4,d;2]]_q$	$(4t+2)m-1\le d\le (6t+3)m-3$	
			$[[\frac{q^2-1}{a},\frac{q^2-1}{a}-2d+5,d;3]]_q$	$(6t+3)m-2\le d\le (8t+4)m-4$	

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