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PARTITION GHZ SLOCC CLASS OF THREE QUBITS INTO TEN FAMILIES UNDER LU

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In [Science 340:1205, (2013)], via entanglement polytopes Michael Walter et al. obtained a finite yet systematic classification of multi-particle entanglement. It is well known that under SLOCC, pure states of three (four) qubits are partitioned into six (nine) families. Acín et al. proposed the generalized Schmidt decomposition for three qubits and partitioned pure states of three qubits into five types. In this paper,we present a LU invariant and an entanglement measures for the GHZ SLOCC class of three qubits, and partition states of the GHZ SLOCC class of three qubits into ten families and each family into two subfamilies under LU. We give a necessary and sufficient condition for the uniqueness of the generalized Schmidt decomposition for the GHZ SLOCC class.

Keywords:

1 Introduction

Quantum entanglement is considered as a key quantum mechanical resource in quantum information and computation such as quantum teleportation, quantum cryptography, quantum metrology, and quantum key distribution [1]. To understand entanglement, lots of efforts have contributed to study the convertibility of two states under local unitary operators (LU), local operations and classical communication (LOCC), and Stochastic LOCC (SLOCC).

Two pure n-qubit states $|\psi'\rangle$ and $|\psi\rangle$ are LU (SLOCC) equivalent if the two states satisfy the following equation,

$$|\psi'\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n|\psi\rangle, \tag{1}$$

where 2 by 2 matrices A_i are unitary (invertible).

Under SLOCC, pure states of three qubits were distinguished into six equivalence classes GHZ, W, A-BC, B-AC, C-AB, and A-B-C [2], and pure states of four qubits were partitioned into nine families or more [3, 4]. Classification of multipartite entangled states by multidimensional determinants were investigated [5].

Under LU, Acín et al. divided pure states of three qubits into five types [6]. It is known that if two states are LU equivalent, then they have the same amount of entanglement and can do the same tasks in quantum information theory [2, 3, 7, 8].

Lots of efforts have devoted to studying the characterization, the quantification, and the classification of the entanglement via Schmidt decomposition [6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28, 29]. Acín et al. proposed the generalized Schmidt decomposition of three qubits [6].

Carteret et al. discuss the Schmidt decomposition for the multipartite system [15]. Kraus proved that two states are LU equivalent if and only if they have the same standard forms [7, 8]. Vicente et al. derived a new decomposition for pure states of three qubits, which is characterized by five parameters (up to local unitary operations) [10]. Liu et al. proposed a practical entanglement classification scheme for pure states of general multipartite for arbitrary dimensions under LU [20]. Li and Qiao proposed a practical method for finding the canonical forms for pure and mixed states of arbitrary dimensional multipartite systems under LU [21]. Via the generalized Schmidt decomposition of three qubits, Kumari and Adhikari partitioned positive states (i.e. the states with the phase factor $\theta = 0$) of the GHZ SLOCC class into four subclasses and proposed the witness operator for the classification [28].

The canonical form and the entanglement measure and classification for three qubits have been widely studied topics [6, 10, 22, 23, 24, 25, 26, 27, 28, 29]. The SLOCC entanglement classification of three and four qubits has many applications. For example, a one-to-one correspondence between the SLOCC entanglement classification of three and four qubits and the classification of the extremal black holes was established [30]. LU classification of black holes corresponding to GHZ SLOCC class is studied in [31].

In this paper, we present a LU invariant for the GHZ SLOCC class of three qubits, and partition the GHZ SLOCC class of three qubits into ten families and each family into two subfamilies under LU. Thus, the infinite LU equivalence classes of the GHZ SLOCC classes are partitioned into 20 subfamilies.

2 Parameters ϱ and ι for the GHZ SLOCC class

By means of LU transformations, any pure state of three qubits can be written as

$$\begin{aligned} |\psi\rangle &= \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle \\ &\quad +\lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \tag{2}$$

where $\lambda_i \geq 0$, $\sum_{i=0}^{4} \lambda_i^2 = 1$, $0 \leq \varphi \leq \pi$, φ is referred to as the phase of $|\psi\rangle$ [6]. In this paper, φ is limited to $[0, 2\pi)$. Eq. (2) is referred to as Acín et al.'s Schmidt Decomposition (ASD) of the state $|\psi\rangle$.

For simplicity, $|\psi\rangle$ is written as

$$|\psi\rangle = (\lambda_0, \lambda_1 e^{i\varphi}, \lambda_2, \lambda_3, \lambda_4), \tag{3}$$

which is the set of the coefficients of the five LBPS (local bases product states). The set of the coefficients in Eq. (3) is called Acín et al.'s Schmidt coefficients (ASC) of $|\psi\rangle$.

A state is referred as to an i-LBPS state if the state has just i non-vanishing Schmidt coefficients.

Motivation 1. In [6], the authors partitioned pure states of three qubits into five types: Types 1, 2 (2a and 2b), 3 (3a and 3b), 4 (4a, 4b, and 4c), and 5. Specially, states of the GHZ SLOCC class were partitioned into Types 2b, 3b, 4b, 4c, and 5 [6].

The Schmidt Decomposition for three qubits and the LU classification of three qubits have had a significant impact on QIC. It is well known that in several aspects the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ can be regarded as the maximally entangled state of three qubits [2]. Recently, Kumari and Adhikari divided positive states (i.e. the states with the vanishing phases) of the GHZ SLOCC class into four subclasses 1, 2, 3, 4 and compared the maximal teleportation fidelities, the entanglement, and the tangle in the four subclasses.

Of course, it is fundamental to partition the GHZ SLOCC class completely under LU.

Motivation 2. We want to find a criterion to determine what states have the unique Schmidt decomposition or not. For the states which have the unique Schmidt decomposition, one can see that subjected to local random unitary noise, their ASDs do not change.

Motivation 3. It is interesting to study the classification for back holes via the LU entanglement classification of the GHZ SLOCC class [31]. Lot of research had been done on the relation between the SLOCC entanglement classification of three and four qubits and the classification of the extremal black holes [30].

Kallosh and Linde investigated the black holes with 4 non-vanishing integer charges q_0, p^1, p^2, p^3, which correspond to the following states [32].

$$-p^1|001\rangle - p^2|010\rangle - p^3|100\rangle + q_0|111\rangle. \tag{4}$$

The states in Eq. (4) belong to the GHZ SLOCC class. It is potential to establish a relation between the classification of the black holes with 4 non-vanishing charges q_0, p^1, p^2, p^3 and the LU classification of the GHZ SLOCC class [31].

We next discuss LU classification of the GHZ SLOCC equivalence class via ASD. $|\psi\rangle$ in Eq. (2) belongs to the GHZ SLOCC class if and only if $\lambda_0\lambda_4 \neq 0$ [22].

Let

$$\gamma = \lambda_1\lambda_4 e^{i\varphi} - \lambda_2\lambda_3. \tag{5}$$

The parameters ϱ and ι were defined for the states with $\gamma \neq 0$ of the GHZ SLOCC class [23]. Here, we define the parameters ϱ and ι for the whole GHZ SLOCC class, then show that $|\ln \varrho|$ is a LU invariant for the whole GHZ SLOCC class below.

For $|\psi\rangle$ in Eq. (2), when $\lambda_0\lambda_4 \neq 0$, we define

$$\begin{aligned} \varrho(|\psi\rangle) &= \sqrt{J_4 + J_1}/\sqrt{(\lambda_2^2+\lambda_4^2)(\lambda_3^2+\lambda_4^2)}, & (6)\\ \iota(|\psi\rangle) &= (\lambda_2\lambda_3 + \gamma^*/\varrho^2)/\lambda_4, & (7) \end{aligned}$$

where γ^* is the complex conjugate of γ, $J_1 = |\gamma|^2$, and $J_4 = (\lambda_0\lambda_4)^2$. J_1 and J_4 are LU invariants [6]. Clearly, $\varrho(|\psi\rangle) > 0$.

When it is clear from the context, we write ϱ and ι for $\varrho(|\psi\rangle)$ and $\iota(|\psi\rangle)$, respectively. ϱ and ι are used to describe the LU equivalence of two ASD states and partition ASD states of the GHZ SLOCC class under LU below.

Let us consider the state

$$|\psi_{\varrho,\iota}\rangle = ((1/\varrho)\lambda_0, \varrho\iota, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4). \tag{8}$$

We say that $|\psi_{\varrho,\iota}\rangle$ is obtained by applying $\varrho - \iota$ transformation to $|\psi\rangle$. Note that the phase of the complex number ι is just the phase of $|\psi_{\varrho,\iota}\rangle$. Let $\varrho' = \varrho(|\psi_{\varrho,\iota}\rangle)$ and $\iota' = \iota(|\psi_{\varrho,\iota}\rangle)$. Then, a calculation yields

$$\varrho' = 1/\varrho. \tag{9}$$

That is,

$$\varrho'\varrho = 1. \tag{10}$$

Via Eqs. (7, 9),

$$\iota' = \varrho\lambda_1 e^{i\varphi}. \tag{11}$$

We next show that $|\psi\rangle$ can also be obtained by applying $\varrho' - \iota'$ transformation to $|\psi_{\varrho,\iota}\rangle$ in Eq. (8). From Eqs. (8, 9, 11), a calculation yields that

$$\begin{aligned} & ((1/\varrho')((1/\varrho)\lambda_0), \varrho'\iota', \varrho'(\varrho\lambda_2), \varrho'(\varrho\lambda_3), \varrho'(\varrho\lambda_4)) \\ = \;& (\lambda_0, \lambda_1 e^{i\varphi}, \lambda_2, \lambda_3, \lambda_4) = |\psi\rangle. \end{aligned} \tag{12}$$

Therefore, if $|\psi\rangle$ can be $\varrho - \iota$ transformed into $|\psi_{\varrho,\iota}\rangle$, then $|\psi_{\varrho,\iota}\rangle$ can also be $\varrho' - \iota'$ transformed into $|\psi\rangle$.

3 LU Partition of the GHZ SLOCC class via ϱ, ι, and γ

It is known that $\lambda_0\lambda_4 \neq 0$ for the GHZ SLOCC class. Here, the states with non-negative (real and complex) coefficients are called positive (real and complex) states.

3.1 LU classification of positive states with $\gamma = 0$

3.1.1 Calculating ϱ, ι, and $|\psi_{\varrho,\iota}\rangle$

When $\gamma = 0$, $|\psi\rangle$ in Eq. (3) can be written as

$$|\psi\rangle = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{13}$$

where $\lambda_1\lambda_4 = \lambda_2\lambda_3$ [23]. For $|\psi\rangle$ in Eq. (13), a calculation yields that

$$\varrho = \lambda_0/\sqrt{1-\lambda_0^2}, \tag{14}$$

$$\iota = \lambda_2\lambda_3/\lambda_4 = \lambda_1\lambda_4/\lambda_4 = \lambda_1, \tag{15}$$

$$|\psi_{\varrho,\iota}\rangle = ((1/\varrho)\lambda_0, \varrho\lambda_1, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4). \tag{16}$$

Specially, when $\varrho = 1$, from Eqs. (14, 16), we obtain

$$\lambda_0 = \frac{1}{\sqrt{2}}, \tag{17}$$

$$|\psi_{\varrho,\iota}\rangle = |\psi\rangle. \tag{18}$$

Result 1. In light of Proposition 2 in [23], one can know that $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ with $\gamma = 0$ if and only if $|\psi'\rangle = |\psi_{\varrho,\iota}\rangle$ in Eq. (16).

From Result 1, we have the following corollary 1.1.

Corollary 1.1. If $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ with $\gamma = 0$, then $|\ln\varrho| = |\ln\varrho'|$, and $|\psi'\rangle$ and $|\psi\rangle$ both are positive and have the same kinds of LBPS.

3.1.2 LU classification of positive states with $\gamma = 0$

The states with $\gamma = 0$ are partitioned into four positive families P_i, $i = 1, \cdots, 4$. Ref. Table 1.

We next argue that P_i, $i = 1, \cdots, 4$, are LU inequivalent.

In light of Result 1 and via Eqs. (13, 16), one can see that λ_i and $\varrho\lambda_i$, $i = 1, 2, 3$, both vanish or neither does. It guarantees that P_i, $i = 1, \cdots, 4$, are LU inequivalent. For example, $\lambda_1 \neq 0$ for P_1 while $\lambda_1 = 0$ for P_i, $i = 2, 3, 4$. Therefore, P_1 is LU inequivalent to P_i, $i = 2, 3, 4$.

Again, P_i is divided into two subfamilies P_i' (states with $\varrho = 1$) and P_i'' (states with $\varrho \neq 1$). Ref. Table 1. Corollary 1.1 implies that P_i' and P_i'', $i = 1, 2, 3, 4$, are LU inequivalent. Clearly, each LU class of P_i' is a singleton, and each LU class of P_i'' consists of only two states $|\psi\rangle$ and $|\psi_{\varrho,\iota}\rangle$.

Via Eq. (14), a calculation yields that $\lambda_0 = \frac{1}{\sqrt{2}}$ if and only if $\varrho = 1$. In light of Result 1 and Corollary 1.1, we have the following Corollary 1.2.

Corollary 1.2. ASD of a positive state with $\gamma = 0$ is unique if and only if $\varrho = 1$. In other words, ASD of a positive state with $\gamma = 0$ is unique if and only if $\lambda_0 = \frac{1}{\sqrt{2}}$.

The contrapositive version of Corollary 1.2 leads to the following. ASD of a positive state with $\gamma = 0$ is not unique if and only if $\varrho \neq 1$ (in other words, $\lambda_0 \neq \frac{1}{\sqrt{2}}$).

Table 1. Positive families P_i, $i = 1, \cdots, 4$, for which $\gamma = 0$

	$\varrho = 1$	$\varrho \neq 1$
P_1; $\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$‡	P_1'; $\lambda_0 = \frac{1}{\sqrt{2}}$	P_1''
P_2; $\{(\lambda_0, 0, 0, \lambda_3, \lambda_4)\}$	P_2'; $\lambda_0 = \frac{1}{\sqrt{2}}$	P_2''
P_3;$\{(\lambda_0, 0, \lambda_2, 0, \lambda_4)\}$	P_3'; $\lambda_0 = \frac{1}{\sqrt{2}}$	P_3''
P_4; $\{(\lambda_0, 0, 0, 0, \lambda_4)\}$	P_4' ={GHZ}	P_4''

‡ Each state of P_1 satisfies that $\lambda_1\lambda_4 = \lambda_2\lambda_3 \neq 0$.

3.2 LU classification of real states with $\gamma\lambda_2\lambda_3 \neq 0$

Let

$$|\psi\rangle = (\lambda_0, \delta\lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{19}$$

where $\delta = \pm 1$, $\gamma \neq 0$, $\lambda_2\lambda_3 \neq 0$, but λ_1 may vanish.

3.2.1 Calculating ϱ, ι, and $|\psi_{\varrho,\iota}\rangle$

A calculation yields that

$$\begin{aligned} \gamma &= \delta\lambda_1\lambda_4 - \lambda_2\lambda_3, & (20)\\ \varrho &= \sqrt{J_4 + J_1}/\sqrt{(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)} & (21)\\ \iota &= (\lambda_2\lambda_3 + \gamma/\varrho^2)/\lambda_4, & (22)\\ |\psi_{\varrho,\iota}\rangle &= ((1/\varrho)\lambda_0, \varrho\iota, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4). & (23) \end{aligned}$$

One can know that ι is real and $|\psi_{\varrho,\iota}\rangle$ is a real state.

When $\varrho = 1$, from Eqs. (21, 22, 23), a calculation yields that

$$\begin{aligned} \iota &= \delta\lambda_1, & (24)\\ \lambda_0^2 + \lambda_1^2 &= \frac{1}{2} + \frac{\delta\lambda_1\lambda_2\lambda_3}{\lambda_4}, & (25)\\ |\psi_{\varrho,\iota}\rangle &= |\psi\rangle. & (26) \end{aligned}$$

Conclusion 1. (i). Via Eq. (24), we can conclude that for a real 5-LBPS state with $\gamma \neq 0$, if $\varrho = 1$, then $\iota \neq 0$ because $\lambda_1 \neq 0$.

(ii). The contrapositive version of the above (i) leads to the following. For a real 5-LBPS state with $\gamma \neq 0$, if $\iota = 0$ then $\varrho \neq 1$.

When $\lambda_1 = 0$, from Eqs. (19, 20, 21, 22, 23), we obtain

$$
\begin{aligned}
|\psi\rangle &= (\lambda_0, 0, \lambda_2, \lambda_3, \lambda_4), && (27)\\
\gamma &= -\lambda_2\lambda_3, && (28)\\
\iota &= \lambda_2\lambda_3(1 - 1/\varrho^2)/\lambda_4, && (29)\\
\varrho &= \sqrt{(\lambda_0\lambda_4)^2 + (\lambda_2\lambda_3)^2}/\sqrt{(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)} && (30)\\
|\psi_{\varrho,\iota}\rangle &= ((1/\varrho)\lambda_0, \varrho\iota, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4), && (31)
\end{aligned}
$$

Conclusion 2. (i). For the 4-LBPS state $|\psi\rangle$ with $\lambda_1 = 0$, a calculation yields that $\iota = 0$ if and only if $\varrho = 1$ if and only if $\lambda_0 = 1/\sqrt{2}$ (i.e. $|\psi\rangle$ is of the form $\left(1/\sqrt{2}, 0, \lambda_2, \lambda_3, \lambda_4\right)$) [23].

(ii). The contrapositive version of the above (i) leads to the following. For the 4-LBPS state $|\psi\rangle$ with $\lambda_1 = 0$, $\iota \neq 0$ if and only if $\varrho \neq 1$ if and only if $\lambda_0 \neq 1/\sqrt{2}$ (i.e. $|\psi\rangle = \left(\lambda_0(\neq 1/\sqrt{2}), 0, \lambda_2, \lambda_3, \lambda_4\right)$).

Result 2. In light of (i) of Proposition 3 in [23], $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ in Eq. (19) if and only if $|\psi'\rangle = |\psi_{\varrho,\iota}\rangle$ in Eq. (23).

From Result 2, we have the following corollary 2.1.

Corollary 2.1. If $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ in Eq. (19), then $|\psi'\rangle$ is also real because ι is real and $|\ln\varrho| = |\ln\varrho'|$.

3.2.2 Two states with different number of LBPS may be LU equivalent

When $\varrho \neq 1$, then ι in Eq. (29) is a non-zero real number and $|\psi_{\varrho,\iota}\rangle$ in Eq. (31) is a real 5-LBPS state. Thus, the 4-LBPS state $\left(\lambda_0(\neq 1/\sqrt{2}), 0, \lambda_2, \lambda_3, \lambda_4\right)$ is LU equivalent to a real 5-LBPS state $|\psi_{\varrho,\iota}\rangle$. It means that the number of LBPS is not a LU invariant.

For example, let

$$
\begin{aligned}
|\phi\rangle &= (1/2)(1, 0, 1, 1, 1), && (32)\\
|\phi'\rangle &= (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}). && (33)
\end{aligned}
$$

Clearly, $|\phi\rangle$ has four LBPS and $|\phi'\rangle$ has five LBPS.

Moreover, a calculation yields that

$$|\phi'\rangle = H \otimes H \otimes H|\phi\rangle, \tag{34}$$

where H is the Hadmard matrix. It is well known that the Hadmard matrix is a unitary matrix. Therefore, $|\phi'\rangle$ and $|\phi\rangle$ are LU equivalent though they have different number of LBPS.

3.2.3 LU classification of real states with $\gamma\lambda_2\lambda_3 \neq 0$

All the real states with $\gamma\lambda_2\lambda_3 \neq 0$ are partitioned into the real families R_1 and R_2. Let R_1 be the family consisting of the 5-LBPS real states with $\gamma \neq 0$ and $\iota \neq 0$. Let R_2 be the family consisting of the 4-LBPS real states of the form $(\lambda_0, 0, \lambda_2, \lambda_3, \lambda_4)$ (of course, $\gamma \neq 0$) and the 5-LBPS real states with $\gamma \neq 0$ and $\iota = 0$. Ref. Table 2.

We next argue that R_1 and R_2 are LU inequivalent.

Let $|\psi\rangle$ be a state in R_1. Then, $|\psi\rangle$ is a 5-LBPS state with $\gamma\iota \neq 0$. In light of Result 2, if $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$, then $|\psi'\rangle = |\psi_{\varrho,\iota}\rangle$ in Eq. (23). Thus, $|\psi'\rangle$ is also a real 5-LBPS state with $\iota' = \delta\varrho\lambda_1 \neq 0$ (ref. Eq. (11)). Then, it is not hard to see that R_1 and R_2 are LU inequivalent.

Table 2. Real families R_1 and R_2 for which $\gamma \neq 0$ and $\lambda_2\lambda_3 \neq 0$

R_1	5-LBPS real states with $\gamma \neq 0$ and $\iota \neq 0$
R_2	4-LBPS real states with $\lambda_1 = 0$
	and 5-LBPS real states with $\gamma \neq 0$ and $\iota = 0$

Family R_i is divided into two subfamilies R_i' (consisting of the states with $\varrho = 1$) and R_i'' (consisting of the states with $\varrho \neq 1$). Ref. Table 3. Corollary 2.1 implies that R_i' and R_i'' are LU inequivalent. One can know that each LU class of R_i' is a singleton and each LU class of R_i'' consists of only two states $|\psi\rangle$ and $|\psi_{\varrho,\iota}\rangle$.

Table 3. Family R_i is divided into two subfamilies R_i' (the states with $\varrho = 1$) and R_i'' (the states with $\varrho \neq 1$).

$\varrho = 1$	$\varrho \neq 1$,
$R_1' = \{(\lambda_0, \delta\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$	$R_1'' = \{|\psi\rangle, |\psi_{\varrho,\iota}\rangle\}$
$R_2' = \{(\frac{1}{\sqrt{2}}, 0, \lambda_2, \lambda_3, \lambda_4)\}$	$R_2'' = \{|\psi\rangle, |\psi_{\varrho,\iota}\rangle\}$

In light of Result 2 and Corollary 2.1, we have the following Corollary 2.2.

Corollary 2.2. ASD of a real state with $\gamma\lambda_2\lambda_3 \neq 0$ is unique if and only if $\varrho = 1$. In other words, via Eq. (25), ASD of a real state with $\gamma\lambda_2\lambda_3 \neq 0$ is unique if and only if $\lambda_0^2 + \lambda_1^2 = \frac{1}{2} + \frac{\delta\lambda_1\lambda_2\lambda_3}{\lambda_4}$.

The contrapositive version of Corollary 2.2 leads to the following. ASD of a real state with $\gamma\lambda_2\lambda_3 \neq 0$ is not unique if and only if $\varrho \neq 1$. In other words, ASD of a real state with $\gamma\lambda_2\lambda_3 \neq 0$ is not unique if and only if $\lambda_0^2 + \lambda_1^2 \neq \frac{1}{2} + \frac{\delta\lambda_1\lambda_2\lambda_3}{\lambda_4}$.

3.2.4 *The number of LBPS is not LU invariant for R_2''*

In light of (ii) of Conclusion 1, the 5-LBPS real states with $\gamma \neq 0$ and $\iota = 0$ belong to R_2''.

In light of (i) of Conclusion 2, R_2' consists of the states of only the form $(\frac{1}{\sqrt{2}}, 0, \lambda_2, \lambda_3, \lambda_4)$.

In light of (ii) of Conclusion 2, the states of the form $(\lambda_0(\neq 1/\sqrt{2}), 0, \lambda_2, \lambda_3, \lambda_4)$ belong to R_2''.

In light of Conclusions 1 and 2, each LU class of R_2'' is a pair of a 4-LBPS state $|\psi\rangle$ of the form $(\lambda_0(\neq 1/\sqrt{2}), 0, \lambda_2, \lambda_3, \lambda_4)$ and a real 5-LBPS state with $\gamma \neq 0$ and $\iota = 0$ $(= |\psi_{\varrho,\iota}\rangle)$. For example, $(|\phi\rangle, |\phi'\rangle)$ is a LU class of R_2''.

Therefore, the number of LBPS is not LU invariant for R_2''.

3.3 *LU classification of complex states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$*

Let $|\psi\rangle$ be the state with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$. Then, clearly $\lambda_1 \neq 0$ for the states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$.

3.3.1 *Calculating ϱ, ι, and $|\psi_{\varrho,\iota}\rangle$*

From Appendix A, when $\lambda_1 \neq 0$ and $\lambda_2\lambda_3 = 0$, the following two states are LU equivalent for any φ and χ.

$$|\psi\rangle = (\lambda_0, \lambda_1 e^{i\varphi}, \lambda_2, \lambda_3, \lambda_4), \tag{35}$$

$$|\varpi\rangle = (\lambda_0, \lambda_1 e^{i\chi}, \lambda_2, \lambda_3, \lambda_4), \tag{36}$$

Note that $\varrho(|\psi\rangle) = \varrho(|\varpi\rangle)$. It implies that a state with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ has infinite ASD. Appendix A tells us that we don't need to consider the phases when determining if two states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ are LU equivalent. That is, we only need to consider the following states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$.

$$|\psi\rangle = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{37}$$

For $|\psi\rangle$ in Eq. (37) with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$, a calculation yields that

$$\iota = \lambda_1/\varrho^2, \tag{38}$$

$$\varrho = \sqrt{\lambda_0^2 + \lambda_1^2}/\sqrt{1 - \lambda_0^2 - \lambda_1^2}, \tag{39}$$

$$|\psi_{\varrho,\iota}\rangle = ((1/\varrho)\lambda_0, \lambda_1/\varrho, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4). \tag{40}$$

When $\varrho = 1$, from Eqs. (38, 39, 40), obtain

$$\iota = \lambda_1, \tag{41}$$

$$\lambda_0^2 + \lambda_1^2 = 1/2, \tag{42}$$

$$|\psi_{\varrho,\iota}\rangle = |\psi\rangle. \tag{43}$$

Result 3. In light of (ii) of Proposition 3 in [23], one can know that $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ if and only if $|\psi'\rangle = |\psi_{\varrho,\iota}\rangle$ ignoring the phases.

From Result 3, we have the following corollary 3.1.

Corollary 3.1. If $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$ with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$, then $|\ln\varrho| = |\ln\varrho'|$ and $|\psi'\rangle$ and $|\psi\rangle$ have the same kinds of LBPS.

3.3.2 *LU classification of complex states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$*

In Table 4, we partition the complex states with $\gamma \neq 0$, and $\lambda_2\lambda_3 = 0$ into three families C_i, $i = 1, 2, 3$.

We next argue that C_i, $i = 1, 2, 3$, are LU inequivalent.

In light of Result 3 and via Eqs. (37, 40), one can know that λ_i and $\varrho\lambda_i$, $i = 2, 3$, both vanish or neither does. But, $\lambda_2 = 0$ for C_1 and C_3 while $\lambda_2 \neq 0$ for C_2, and $\lambda_3 = 0$ for C_2 and C_3 while $\lambda_3 \neq 0$ for C_1. Therefore, C_i, $i = 1, 2, 3$, are LU inequivalent.

Each complex Family C_i is divided into two subfamilies C_i' (states with $\varrho = 1$) and C_i'' (states with $\varrho \neq 1$). Note that each LU class includes infinite states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$. After ignoring phase, each LU class of C_i' is a singleton and each LU class of C_i'' consists of only two states $|\psi\rangle$ and $|\psi_{\varrho,\iota}\rangle$. Ref. Table 4. Corollary 3.1 implies that C_i' and C_i'', $i = 1, 2, 3$, are LU inequivalent.

In light of Result 3 and Corollary 3.1, we have the following Corollary 3.2.

Corollary 3.2. Ignoring phases, ASD of a complex state with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ is unique if and only if $\varrho = 1$. In other words, via Eq. (42), ASD of a complex state with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ is unique if and only if $\lambda_0^2 + \lambda_1^2 = 1/2$ ignoring phases.

The contrapositive version of Corollary 3.2 leads to the following. ASD of a complex state with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ is not unique if and only if $\varrho \neq 1$ (in other words, $\lambda_0^2 + \lambda_1^2 \neq 1/2$) ignoring phases.

Table 4. Complex families C_1, C_2,and C_3 for which $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$

$\gamma \neq 0, \lambda_2\lambda_3 = 0$	$\varrho = 1$	$\varrho \neq 1$
C_1;$\{(\lambda_0, \lambda_1 e^{i\varphi}, 0, \lambda_3, \lambda_4)\}$	C_1';$\lhd$	C_1'';$\{\lvert\psi\rangle, \lvert\psi_{\varrho,\iota}\rangle\}$
C_2;$\{(\lambda_0, \lambda_1 e^{i\varphi}, \lambda_2, 0, \lambda_4)\}$	C_2';$\lhd$	C_2'';$\{\lvert\psi\rangle, \lvert\psi_{\varrho,\iota}\rangle\}$
C_3;$\{(\lambda_0, \lambda_1 e^{i\varphi}, 0, 0, \lambda_4)\}$	C_2';$\lhd$	C_2'';$\{\lvert\psi\rangle, \lvert\psi_{\varrho,\iota}\rangle\}$

$\lhd$ $\lambda_0^2 + \lambda_1^2 = 1/2$.

3.4 LU classification of complex 5-LBPS states with the phases $\varphi \neq 0$ or π

Let $|\psi\rangle$ be a complex 5-LBPS state with the phases $\varphi \neq 0$ or π. We write $|\psi\rangle$ as follows.

$$|\psi\rangle = (\lambda_0, \lambda_1 e^{i\varphi}, \lambda_2, \lambda_3, \lambda_4), \tag{44}$$

3.4.1 Calculating ϱ, ι, and the state $|\psi_{\varrho,\iota}\rangle$

For the complex 5-LBPS states with the phases $\varphi \neq 0$ or π, via Eqs. (5, 6, 7, 8), we have the following

$$\gamma = \lambda_1\lambda_4 e^{i\varphi} - \lambda_2\lambda_3, \tag{45}$$
$$\iota = (\lambda_2\lambda_3 + \gamma^*/\varrho^2)/\lambda_4, \tag{46}$$
$$\varrho = \sqrt{J_4 + J_1}/\sqrt{(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)}, \tag{47}$$
$$|\psi_{\varrho,\iota}\rangle = ((1/\varrho)\lambda_0, \varrho\iota, \varrho\lambda_2, \varrho\lambda_3, \varrho\lambda_4). \tag{48}$$

It is not hard to see that $\gamma \neq 0$ and the imaginary part of ι does not vanish. Thus, $|\psi_{\varrho,\iota}\rangle$ is also a complex 5-LBPS state whose phase is not 0 or π.

When $\varrho = 1$, from the above equations a calculation yields that

$$\iota = \lambda_1 e^{-i\varphi}, \tag{49}$$
$$|\psi_{\varrho,\iota}\rangle = (\lambda_0, \lambda_1 e^{-i\varphi}, \lambda_2, \lambda_3, \lambda_4) \tag{50}$$
$$= |\psi^*\rangle \tag{51}$$
$$\neq |\psi\rangle. \tag{52}$$

where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$.

Result 4.

In light of (i) of Proposition 3 in [23], $|\psi'\rangle$ ($\neq |\psi\rangle$) is LU equivalent to the complex 5-LBPS state $|\psi\rangle$ with the phase $\varphi \neq 0$ or π if and only if $|\psi'\rangle = |\psi_{\varrho,\iota}\rangle$ in Eq. (48).

From Result 4, we have the following Corollaries 4.1.

Corollary 4.1. If $|\psi'\rangle$ is LU equivalent to the complex 5-LBPS state $|\psi\rangle$ with the phase $\varphi \neq 0$ or π, then $|\ln \varrho| = |\ln \varrho'|$ and $|\psi'\rangle$ is also a complex 5-LBPS state whose phase is not 0 or π.

3.4.2 LU classification of complex 5-LBPS states with the phases $\varphi \neq 0$ or π

Let Family C_4 consist of complex 5-LBPS states with the phases $\varphi \neq 0$ or π. It implies that for those states, $\gamma \neq 0$ and $\iota \neq 0$.

We next argue that C_4 is LU inequivalent to C_i, $i = 1, 2, 3$.

In light of Corollary 4.1, if $|\psi'\rangle$ is LU equivalent to the complex 5-LBPS state $|\psi\rangle$ with the phase $\varphi \neq 0$ or π, then $|\psi'\rangle$ is also a complex 5-LBPS state. From Table 4, one can see that C_i, $i = 1, 2$, consist of 4-LBPS states and C_3 consists of 3-LBPS states. Therefore, C_4 is LU inequivalent to C_i, $i = 1, 2, 3$.

C_4 is divided into two subfamilies C_4' (states with $\varrho = 1$) and C_4'' (states with $\varrho \neq 1$). Ref. Table 5. Via Eq. (51) and in light of Result 4, each LU class of C_4' consists of a state and its complex conjugate, while each LU class of C_4'' consists of only two states $|\psi\rangle$ and $|\psi_{\varrho,\iota}\rangle$, where $|\psi_{\varrho,\iota}\rangle \neq |\psi^*\rangle$.

In light of Result 4 and Corollary 4.1, we have the following Corollary 4.2.

Corollary 4.2. Considering $|\psi\rangle$ and its complex conjugate $|\psi^*\rangle$ to be the same, ASD of a complex 5-LBPS state with the phases $\varphi \neq 0$ or π is unique if and only if $\varrho = 1$.

The contrapositive version of Corollary 4.2 leads to the following. Considering $|\psi\rangle$ and its complex conjugate $|\psi^*\rangle$ to be the same, ASD of a complex 5-LBPS state with the phases $\varphi \neq 0$ or π is not unique if and only if $\varrho \neq 1$.

Table 5. Complex Family C_4 (5-LBPS states with $\varphi \neq 0$ or π).

C_4: 5-LBPS states with $\varphi \neq 0,\pi$	each LU class
$C_4' =\{$ states with $\varrho = 1\}$	$=\{|\psi\rangle, |\psi^*\rangle\}$
$C_4'' =\{$states with $\varrho \neq 1\}$	$=\{|\psi\rangle, |\psi_{\varrho,\iota}\rangle\}$

3.4.3 C_4' (C_4'') is the set of states being (not being) LU equivalent to their complex conjugates

For the 5-LBPS state $|\psi\rangle$ with $\varrho(|\psi\rangle) \neq 1$, one can also verify that $|\psi_{\varrho,\iota}\rangle$ is not $|\psi^*\rangle$ as follows. Clearly, $\varrho(|\psi\rangle) = \varrho(|\psi^*\rangle)$, thus $\varrho(|\psi\rangle)\varrho(|\psi^*\rangle) \neq 1$ when $\varrho(|\psi\rangle) \neq 1$. Via Eq. (10), $|\psi_{\varrho,\iota}\rangle$ is not $|\psi^*\rangle$. Thus, each state of C_4'' is LU inequivalent to its complex conjugate.

A twelfth degree complex polynomial invariant I_6, introduced by Grassl [14], can distinguish among two complex conjugate ASD states which are LU inequivalent [9]. The bipartite operational measure E_1 is also used to determine if a state is LU equivalent to its complex conjugate [10]. For a complex 5-LBPS state $|\psi\rangle$ with the phases $\varphi \neq 0$ or π, the value of $\varrho(|\psi\rangle)$ determines whether or not $|\psi\rangle$ is LU equivalent to its complex conjugate $|\psi^*\rangle$. Note that ϱ is positive and simpler than I_6 and E_1.

It is known that $|\psi\rangle$ and $|\psi^*\rangle$ possess the same entanglement properties [7]. The class of states not being LU equivalent to their complex conjugates is referred to as NCLU [10]. The existence of NCLU states is a surprising property of multipartite system which does not exist for the bipartite system [10]. Clearly, one can see that C_4'' is just NCLU, and it is easy to find C_4''.

Remark 1: 5-LBPS states are partitioned into four families : the positive family P_1, the real families R_1 and R_2, and the complex family C_4. Furthermore, each family is divided into two subfamilies. Thus, 5-LBPS states are partitioned into seven subfamilies: P_1', P_1'', R_1', R_1'', R_2'', C_4' and C_4''.

Remark 2. The number of LBPS is a LU invariant for the GHZ SLOCC class except for only R_2''. Thus, two states of the GHZ SLOCC class except for R_2'' with different number of LBPS are LU inequivalent.

Remark 3. For any state of the GHZ SLOCC class, when $\varrho \neq 1$ then its ASD is not unique, while $\varrho = 1$, its ASD is unique for the families $P_i, i = 1, 2, 3, 4$, R_1, and R_2, for the families C_i, $i = 1, 2, 3$, ignoring the phases, and for the family C_4 considering the state and its complex conjugate to be the same. Therefore, for the GHZ SLOCC class, ASD is unique if and only if $\varrho = 1$. Thus, we can call ϱ the uniqueness parameter. When $\varrho \neq 1$, from $|\psi\rangle$ and $|\psi_{\varrho,\iota}\rangle$, we choose the one with $\varrho < 1$ as the canonical ASD.

3.5 The argument for the complete LU classification of the GHZ SLOCC class

We partition the positive states with $\gamma = 0$ into four positive families $P_i, i = 1, 2, 3, 4$, the real states with $\gamma \neq 0$ and $\lambda_2\lambda_3 \neq 0$ into two real families R_1 and R_2, the complex states with $\gamma \neq 0$ and $\lambda_2\lambda_3 = 0$ into three complex families C_i, $i = 1, 2, 3$, and let C_4 include the complex 5-LBPS states with $\varphi \neq 0$ or π. Note that for 5-LBPS states with $\varphi \neq 0$ or π, $\gamma \neq 0$ and $\iota \neq 0$.

In total, we partition the GHZ SLOCC class of three qubits into 10 families. Each family is partitioned into two subfamilies one of which has $\varrho = 1$ while the other one has $\varrho \neq 1$.

(i). Since J_1 is LU invariant, where $J_1 = |\gamma|^2$, and $\gamma = 0$ for P_1, P_2, P_3, and P_4 while $\gamma \neq 0$ for R_1, R_2, C_1, C_2, C_3, and C_4, the positive families P_1, P_2, P_3, and P_4 are LU inequivalent to R_1, R_2, C_1, C_2, C_3, and C_4.

(ii). In light of Results 2 and 3, the real families R_1 and R_2 are LU inequivalent to the complex families C_i, $i = 1, 2, 3$. For any state in R_1 and R_2, the phase is 0 or π, while for any state of C_4, the phase is neither 0 nor π. In light of Results 3 and 4, C_4 is LU inequivalent to the real families R_1 and R_2.

3.6 A LU invariant for the GHZ SLOCC class

In the above section, for the positive states (resp. the real states and the complex states) of the GHZ SLOCC class, we show that $|\ln \varrho|$ is a LU invariant. Thus, the state with $\varrho = 1$ and the state with $\varrho \neq 1$ are LU inequivalent. Then, we can conclude that

(1). $|\ln \varrho|$ is a LU invariant for the whole GHZ SLOCC class.

(2). For any two states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the GHZ SLOCC class, if $\varrho(|\psi_1\rangle)\varrho(|\psi_2\rangle) \neq 1$, then $|\psi_1\rangle$ and $|\psi_2\rangle$ are LU inequivalent.

We propose $\frac{1}{1+|\ln \varrho|}$ as a measure of the entanglement for the GHZ SLOCC class. For the measure, the GHZ state has the maximal entanglement $\frac{1}{1+|\ln \varrho|} = 1$. For $|\phi\rangle$, $\varrho = 1/\sqrt{2}$ and for $|\phi'\rangle$, $\varrho' = \sqrt{2}$. Thus, for $|\phi\rangle$ and $|\phi'\rangle$, $\frac{1}{1+|\ln \varrho|} = \frac{1}{1+\sqrt{2}}$.

3.7 Some states with the unique ASD

For the positive or real state $|\psi\rangle$ with $\varrho = 1$, subjected to local random unitary noise, the ASD of $|\psi\rangle$ does not change. That is, $U_1 \otimes U_2 \otimes U_3|\psi\rangle$ and $|\psi\rangle$ have the same ASD.

We give the following positive states for which $\varrho = 1$.

$$
\begin{aligned}
&(1/\sqrt{2})(|000\rangle + |111\rangle) \\
&(1/\sqrt{2})|000\rangle + (1/2)|101\rangle + (1/2)|111\rangle \\
&(1/\sqrt{2})|000\rangle + (1/2)|110\rangle + (1/2)|111\rangle \\
&\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2\sqrt{2}}|110\rangle + \frac{1}{2}|111\rangle \\
&\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}(|100\rangle + |101\rangle + |110\rangle + |111\rangle)
\end{aligned}
$$

4 Summary

It is well known that pure states of three qubits are partitioned into six SLOCC equivalence classes, two of which are the W SLOCC equivalence class and the GHZ SLOCC equivalence class [2]. Acín et al. partitioned pure states of three qubits into five types [6]. The positive states of the GHZ SLOCC class were partitioned into four subclasses [28].

We propose the LU invariant $|\ln \varrho|$ and the entanglement measure $\frac{1}{1+|\ln \varrho|}$ for the GHZ SLOCC equivalence class of three qubits. Via parameters ϱ, ι, and γ, we partition positive, real, and complex states of the GHZ SLOCC class into ten families, and each family into two subfamilies under LU.

Each LU class of C_4' is a pair of a complex 5-LBPS state with the phases $\varphi \neq 0$ or π and its complex conjugate. C_4'' is the set of states not being LU equivalent to their complex conjugates. But,

C_4'' does not exist for the bipartite system. It is interesting to find criteria to partition C_4' and C_4'' furthermore.

Each LU class of R_2'' is a pair of a 4-LBPS state $|\psi\rangle$ of the form $(\lambda_0(\neq 1/\sqrt{2}), 0, \lambda_2, \lambda_3, \lambda_4)$ and a real 5-LBPS state with $\gamma \neq 0$ and $\iota = 0$ $(= |\psi_{\varrho,\iota}\rangle)$. We show that the number of LBPS is a LU invariant for the GHZ SLOCC class except for only R_2''.

We show that for the GHZ SLOCC class, ASD of a state is unique if and only if $\varrho = 1$. Thus, subjected to local random unitary noise, ASD of a state with $\varrho = 1$ does not change. We give some positive states with $\varrho = 1$.

A. Kumari and S. Adhikari partitioned positive states (i.e. the states with the phase factor $\theta = 0$) of the GHZ SLOCC class into four subclasses. Via this LU classification of the GHZ SLOCC class, it is easy to see that the four subclasses are inequivalent under LU.

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Appendix A LU equivalence of some special ASD states

Let

$$|\psi\rangle = (\lambda_0, \lambda_1 e^{i\omega}, \lambda_2, \lambda_3, \lambda_4), \tag{B1}$$

$$|\psi'\rangle = (\lambda_0, \lambda_1 e^{i\omega'}, \lambda_2, \lambda_3, \lambda_4). \tag{B2}$$

When $\lambda_2\lambda_3 = 0$ and $\lambda_0\lambda_1\lambda_4 \neq 0$, we can show that $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$.

Case 1. $\lambda_3 = 0$ and $\lambda_0\lambda_1\lambda_2\lambda_4 \neq 0$ Let

$$\begin{aligned} U^A &= diag\left(e^{i\phi_1}, e^{i(2\phi_1+\phi_2)}\right), \\ U^B &= diag\left(e^{-i\phi_1}, e^{-i\phi_1}\right), \\ U^C &= diag(1, e^{i(-\phi_2-\phi_1)}). \end{aligned} \tag{B3}$$

where $\phi_2 + \phi_1 = \omega' - \omega$. Via U^A, U^B, and U^C in Eq. (B3), a calculation yields that $|\psi'\rangle = U^A \otimes U^B \otimes U^C|\psi\rangle$. Therefore, $|\psi'\rangle$ is LU equivalent to $|\psi\rangle$.

Case 2. $\lambda_2 = 0$ and $\lambda_0\lambda_1\lambda_3\lambda_4 \neq 0$ or $\lambda_2 = \lambda_3 = 0$ and $\lambda_0\lambda_1\lambda_4 \neq 0$. Let

$$\begin{aligned} U^A &= diag\left(e^{i\alpha}, e^{i\beta}\right), \\ U^B &= diag\left(e^{-i\alpha}, e^{-i\beta}\right), \\ U^C &= I, \\ \beta - \alpha &= \omega' - \omega. \end{aligned}$$

Then, $|\psi'\rangle = U^A \otimes U^B \otimes U^C|\psi\rangle$.