Quantum Information and Computation, Vol. 23, No. 1&2 (2023) 0001–0015

NEW QUANTUM CODES DERIVED FROM THE IMAGE OF CONSTACYCLIC CODES

LIQI WANG
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
liqiwangg@163.com

XIUJING ZHENG
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
xiujingzheng99@163.com

SHIXIN ZHU
School of Mathematics, Hefei University of Technology
Hefei 230601, Anhui, P. R. China
zhushixin@hfut.edu.cn

Received September 4, 2022
Revised October 27, 2022

Assume that q is a prime power and $m \geq 2$ is a positive integer. Cyclic codes over $\mathbb{F}_{q^{2m}}$ of length $n = \frac{q^{2m}-1}{\rho}$ with $\rho \mid (q-1)$, and constacyclic codes over $\mathbb{F}_{q^{2m}}$ of length $n = \frac{q^{2m}-1}{\rho}$ with $\rho \mid (q+1)$ are considered in this paper, respectively. Two classes of quantum codes are derived from the images of these codes by the Hermitian construction. Compared with the previously known quantum codes, the quantum codes in our scheme have better parameters.

Keywords: Constacyclic codes, Cyclic codes, Hermitian self-orthogonal codes, Quantum codes

1 Introduction

The research of quantum information science based on quantum mechanics originated in the 1970s and attracted extensive attention until the mid-1990s. In the process of quantum information processing, the interaction between the quantum system and the external environment is inevitable, which leads to the serious attenuation of the coherence of the quantum system and finally degenerates from the coherent superposition state to the mixed state, resulting in quantum decoherence. Research shows that quantum codes can not only protect the stored quantum information, but also realize fault-tolerant quantum gate operation, fault-tolerant quantum state preparation and fault-tolerant quantum measurement, so that quantum information processing can be carried out reliably in noisy environment. After Calderbank et al.[1] gave the connection between quantum codes and classical codes, the research of quantum codes has made rapidly progress. A number of binary quantum codes with good parameters were constructed from classical self-orthogonal codes over $\mathbb{F}_2$ or $\mathbb{F}_4$ (see[2, 3, 4] and the references therein). Non-binary quantum codes also attracted scholars' attention due to the fact that they can be used in the realization of fault-tolerant quantum computation, and lots of non-binary quantum codes were obtained (see [5, 6, 7, 8, 9, 10, 11] and the references therein).

Let q be a prime power. A linear code $\mathcal{C}$ of length n with dimension k and minimum distance d over $\mathbb{F}_q$, denoted as $[n, k, d]_q$, is a k-dimensional subspace of $\mathbb{F}_q^n$. A q-ary quantum code of length n with size K is a K-dimensional subspace of a q^n-dimensional Hilbert space $(\mathbb{C}^q)^{\otimes n} \cong \mathbb{C}^{q^n}$. If a quantum code has minimum distance d, then it can correct up to $\lfloor \frac{d-1}{2} \rfloor$ quantum errors. Let $k = log_q K$. We use $[[n, k, d]]_q$ to denote a q-ary quantun code of length n with size q^k and minimum distance d. The parameters of an $[n, k, d]_q$ linear code and an $[[n, k, d]]_q$ quantum code must satisfy the following well-known bound, respectively.

Theorem 1: [12](Singleton bound) A linear code with parameters $[n, k, d]_q$ must satisfy

$$k \leq n - d + 1.$$

Theorem 2: [13](Quantum Singleton bound) A quantum code with parameters $[[n, k, d]]_q$ must satisfy

$$k \leq n - 2d + 2.$$

If these bounds are achieved, it is called a maximum-distance-separable (MDS) code and a quantum maximum-distance-separable (MDS) code, respectively. Constacyclic codes, which contain the well-known classes of cyclic codes and negacyclic codes, have good algebraic structural properties. They are naturally considered to construct quantum codes. Kai and Zhu [14] constructed two classes of new quantum MDS codes using negacyclic codes. Subsequently, Kai et al. [15] gave a necessary and sufficient condition for constacyclic codes to be Hermitian self-orthogonal and constructed some quantum MDS codes through the Hermitian construction. After that, quantum codes with good parameters, especically, quantum MDS codes have been derived from constacyclic codes([16, 17, 18, 19, 20, 21, 22, 23, 24] and the references therein). In recent years, the images of constacyclic codes also have been used to the construction of quantum codes. Grassl et al.[25] constructed some quantum codes with good parameters from the binary images of Reed-Solomon codes over $\mathbb{F}_{2^k}$ using the concatenated method. Tangataj and McLaughlin[26] obtained some new quantum codes with good parameters from Hermitian self-orthogonal codes, which can be seen as the images of cyclic codes over $\mathbb{F}_{4^m}$. Sundeep and Tangataj[27] generalized the results of [26], and got the self-orthogonality of q-ary images of q^m-ary codes. Some new quantum codes were also constructed from the images of cyclic codes over $\mathbb{F}_{4^m}$. Recently, Kai et al.[28] gave a sufficient condition for the q^2-ary images of constacyclic codes over $\mathbb{F}_{q^{2m}}$ to be Hermitian self-orthogonal and two classes of quantum codes were constructed. Very recently, Zhu et al.[29] derived three classes of quantum codes from the q^2-ary images of cyclic codes over $\mathbb{F}_{q^{2m}}$.

Going on the line of the above work, we study the images of cyclic codes over $\mathbb{F}_{q^{2m}}$ of length $\frac{q^{2m}-1}{\rho}$ with $\rho \mid (q-1)$, and the images of constacyclic codes over $\mathbb{F}_{q^{2m}}$ of length $\frac{q^{2m}-1}{\rho}$ with

$\rho \mid (q+1)$, respectively. New quantum codes with better parameters are constructed through the images of these codes. The paper is organized as follows: In Sect.2, some background and basic results about constacyclic codes and quantum codes are reviewed. In Sect.3, the maximum nonzero set to make the images of constacyclic codes (including cyclic codes) to be Hermitian self-orthogonal is determined. New quantum codes are constructed. Compared with the ones available in the literature, these quantum codes have better parameters. Section 4 gives a conclusion.

2 Preliminaries

Throughout this paper, let $\mathbb{F}_{q^2}$ be the finite field with q^2 elements, where q is a prime power. A q^2-ary linear code $\mathcal{C}$ of length n with dimension k, denoted by $[n,k]_{q^2}$, is a linear subspace of $\mathbb{F}_{q^2}^n$ with dimension k. The number of nonzero components of $\mathbf{c} \in \mathcal{C}$ is said to be the weight $\mathrm{wt}(\mathbf{c})$ of the codeword $\mathbf{c}$. The minimum nonzero weight of all codewords in $\mathcal{C}$ is said to be the minimum distance of $\mathcal{C}$, which is denoted by $d(\mathcal{C})$. An $[n,k]_{q^2}$ linear code with minimum distance d is denoted by $[n,k,d]_{q^2}$. Let $\mathbb{F}_{q^2}^*$ be the multiplicative group of $\mathbb{F}_{q^2}$. Suppose that $\eta \in \mathbb{F}_{q^2}^*$, a linear code $\mathcal{C}$ of length n over $\mathbb{F}_{q^2}$ is called an η-constacyclic code if for each codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, then $(\eta c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}$. As we know, the case $\eta = 1$ is the so-called cyclic code and $\eta = -1$ yields the negacyclic code. Generally, each codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ is identified with its polymonial representation $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. An η-constacyclic code $\mathcal{C}$ over $\mathbb{F}_{q^2}$ of length n can be viewed as a principal ideal $\langle g(x)\rangle$ in the quotient ring $\mathbb{F}_{q^2}[x]/\langle x^n - \eta\rangle$, where $g(x)$ is a monic factor of $x^n - \eta$. The polynomial $g(x)$ is called the generator polynomial of $\mathcal{C}$, and the polynomial $h(x) = (x^n - \eta)/g(x)$ is referred to as the parity-check polynomial of $\mathcal{C}$. The dimension k of $\mathcal{C}$ is $k = n - \deg(g(x)) = \deg(h(x))$.

Assume that p is the characteristic of $\mathbb{F}_{q^2}$, then p is a prime. Suppose that n and p are coprime, i.e., gcd(n, p)=1, so the polynomial $x^n - \eta$ over $\mathbb{F}_{q^2}$ does not have repeated roots. Let $\eta \in \mathbb{F}_{q^2}^*$ with order r, then η is called a primitive r-th root of unity. Let the multiplicative order of q^2 modulo nr be m, i.e., $ord_{nr}(q^2) = m$, then there is a primitive rn-th root of unity α in $\mathbb{F}_{q^{2m}}$ such that $\alpha^n = \eta$. Let $\xi = \alpha^r$, so ξ is a primitive n-th root of unity. Then the roots of $x^n - \eta$ are $\alpha\xi^i = \alpha^{1+ri}$, where $0 \le i \le n-1$. Hence,

$$x^n - \eta = \prod_{i=0}^{n-1}(x - \alpha^{1+ri}).$$

For each $s \in \Omega = \{1 + ri \mid 0 \le i \le n-1\}$, the q^2-cyclotomic coset modulo nr containing s, denoted by $C_{q^2}[s, nr]$, is defined as

$$C_{q^2}[s, nr] = \{sq^{2l} (\mathrm{mod}\ nr) \mid 0 \le l \le m_s - 1\},$$

where m_s is the smallest positive integer such that $sq^{2m_s} \equiv s \pmod{nr}$.

The zero set Z of the η-constacyclic code $\mathcal{C} = \langle g(x)\rangle$ is defined as

$$Z = \{j \in \Omega \mid g(\alpha^j) = 0\},$$

which is a union of some q^2-cyclotomic cosets, and the nonzero set T is defined as

$$T = \Omega \setminus Z = \{j \in \Omega \mid h(\alpha^j) = 0\}.$$

It is well-known that the minimum distance of $\mathcal{C}$ obeys the following bound.

Theorem 3: [15, 30](Constacyclic BCH bound) Suppose that $gcd(n,q)=1$. Let $\mathcal{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$. If there exist integers b and δ such that the generator polynomial of $\mathcal{C}$ has elements $\{\alpha^{1+rj}|b\leq j\leq b+\delta-2\}$ as its zeros, then the minimum distance of $\mathcal{C}$ is at least δ.

For two vectors $\mathbf{u}=(u_0,u_1,\ldots,u_{n-1})$ and $\mathbf{v}=(v_0,v_1,\ldots,v_{n-1})$ belong to $\mathbb{F}_{q^2}^n$, their Hermitian inner product is defined as

$$\langle\mathbf{u},\mathbf{v}\rangle_h=u_0v_0^q+u_1v_1^q+\cdots+u_{n-1}v_{n-1}^q.$$

The Hermitian dual code of an $[n,k]_{q^2}$ linear code $\mathcal{C}$ is a linear code with dimension $n-k$ and is defined by

$$\mathcal{C}^{\perp_h}=\{\mathbf{u}\in\mathbb{F}_{q^2}^n|\langle\mathbf{u},\mathbf{v}\rangle_h=0, for\ all\ \mathbf{v}\in\mathcal{C}\}.$$

If $\mathcal{C}\subseteq\mathcal{C}^{\perp_h}$, then an $[n,k]_{q^2}$ linear code is said to be Hermitian self-orthogonal.

The following theorem presents a connection between quantum codes and classical linear codes.

Theorem 4: [5](Hermitian Construction) Suppose that $\mathcal{C}$ is an $[n,k,d]_{q^2}$ linear code with $\mathcal{C}\subseteq\mathcal{C}^{\perp_h}$, then there exists an $[[n,n-2k,\geq d]]_q$ quantum code.

As we know, $\mathbb{F}_{q^{2m}}$ can be viewed as a vector space over $\mathbb{F}_{q^2}$. Let

$$\mathcal{A}=\{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$$

be a basis of $\mathbb{F}_{q^{2m}}$ over $\mathbb{F}_{q^2}$. For any $\chi=(x_0,x_1,\ldots,x_{n-1})\in\mathbb{F}_{q^{2m}}^n$, each entry of χ can be expressed as $x_i=\sum_{j=0}^{m-1}x_{ij}\alpha_j$, where $x_{ij}\in\mathbb{F}_{q^2}$. Defining a map

$$\mathcal{L}_A:\mathbb{F}_{q^{2m}}^n\longmapsto\mathbb{F}_{q^2}^{nm}$$

$$\mathcal{L}_A(x_0,x_1,\ldots,x_{n-1})=(x_{00},\ldots,x_{n-1,0},x_{0,1},\ldots,x_{n-1,1},x_{0,m-1},\ldots,x_{n-1,m-1})$$

Assume that $\mathcal{D}$ is an $[n,k,d]$ linear code over $\mathbb{F}_{q^{2m}}$. Let the q^2-ary image of $\mathcal{D}$ with respect to the basis $\mathcal{A}$ be $\mathcal{L}_{\mathcal{A}}(\mathcal{D})=\{\mathcal{L}_{\mathcal{A}}(\mathbf{d})\mid\mathbf{d}\in\mathcal{D}\}$. Then $\mathcal{L}_{\mathcal{A}}(\mathcal{D})$ is an $[mn,km,\geq d]$ linear code over $\mathbb{F}_{q^2}$.

Lemma 5:[28] Assume that $\mathcal{A}=\{\alpha_0,\alpha_1,\ldots,\alpha_{m-1}\}$ is a basis of $\mathbb{F}_{q^{2m}}$ over $\mathbb{F}_{q^2}$, and $\mathcal{B}=\{\beta_0,\beta_1,\ldots,\beta_{m-1}\}$ is its Hermitian dual basis. Let $\mathcal{D}$ be an $[n,k,d]$ linear code over $\mathbb{F}_{q^{2m}}$ and $\mathcal{D}^{\perp_h}$ be the Hermitian dual code of $\mathcal{D}$. If m is odd, then $\mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}=\mathcal{L}_{\mathcal{B}}(\mathcal{D}^{\perp_h})$. If m is even, then $\mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}=\mathcal{L}_{\mathcal{B}}(\mathcal{D}^{\perp_h})^q$.

Assume that $\eta\in\mathbb{F}_{q^2}^*$, and its order is r. Note that $\mathbb{F}_{q^2}^*$ is a subgroup of $\mathbb{F}_{q^{2m}}^*$. Hence, $\eta\in\mathbb{F}_{q^{2m}}^*$ and has order r. An η-constacyclic code $\mathcal{D}=\langle\tilde{g}(x)\rangle$ of length n over $\mathbb{F}_{q^{2m}}$ is an ideal in $\mathbb{F}_{q^{2m}}[x]/\langle x^n-\eta\rangle$. The zero set of $\mathcal{D}$ is

$$Z_{2m}=\{j\in\Omega\mid\tilde{g}(\alpha^j)=0\},$$

which is a union of some q^{2m}-cyclotomic cosets modulo nr, and the nonzero set of $\mathcal{D}$ is

$$T_{2m} = \{j \in \Omega \mid \tilde{h}(\alpha^j) = 0\},$$

where $\tilde{h}(x) = (x^n - \eta)/\tilde{g}(x)$. Let $r \mid (q+1)$. There is a sufficient condition for $\mathcal{L}_A(\mathcal{D})$ to be Hermitian self-orthogonal.

Lemma 6:[28] Let $\mathcal{D}$ be an η-constacyclic code in $\mathbb{F}_{q^{2m}}[x]/\langle x^n - \eta\rangle$ with nonzero set T_{2m}, If $(1+ri)q^{2l+1} + (1+rj) \not\equiv 0 \pmod{nr}$ for any $1+ri, 1+rj \in T_{2m}$ and any nonnegative integer l. Then $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$.

Remark 7: If $r = 1$, i.e., $\mathcal{D}$ is a cyclic code, then the above sufficient condition is becoming $aq^{2l+1} + b \not\equiv 0 \pmod n$ for any $a, b \in T_{2m}$ and any nonnegative integer l.

Finally, we give the following famous result, which will be used in the sequel.

Theorem 8:[31] If there exists an $[[n, k, d]]$ quantum code over $\mathbb{F}_q$, then there is an $[[n+1, k, d]]$ quantum code over $\mathbb{F}_q$.

3 New quantum codes

In this section, some new quantum codes with better parameters than the known ones are derived from cyclic MDS codes and constacyclic MDS codes over $\mathbb{F}_{q^{2m}}$, respectively.

3.1 Construction I

Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q-1)$, q is a prime power, and $m \geq 2$ is a positive integer. We first consider the cyclic codes of length n over $\mathbb{F}_{q^{2m}}$, It is obvious that n is coprime to q^{2m} and all the q^{2m}-cyclotomic cosets modulo n are given by $C_{q^{2m}}[i, n] = \{i\}$, for $0 \leq i \leq n-1$. Define

$$\Theta_{max} = \begin{cases} \frac{q^3-q^2+q-1}{\rho} - 1, & m = 2, \\ \frac{q^{m+1}-q^2}{\rho}, & m = 2k \geq 4, \\ \frac{q^m-\rho-1}{\rho}, & m = 2k+1 \geq 3. \end{cases} \tag{1}$$

Lemma 9: Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q-1)$, q is a prime power, and $m \geq 2$ is a positive integer. If $\mathcal{D}$ is the cyclic code of length n over $\mathbb{F}_{q^{2m}}$ with nonzero set $T_{2m} = \bigcup_{i=1}^{\theta} C_{q^{2m}}[i, n]$, where $1 \leq \theta \leq \Theta_{max}$, then $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$.

Proof: According to Lemma 6, we only need to proof that for any $x, y \in T_{2m}$,

$$xq^{2l+1} + y \not\equiv 0 \pmod n.$$

Assume that there exist $x, y \in T_{2m}$ such that

$$xq^{2l+1} + y \equiv 0 \pmod n,$$

where $0 \leq l \leq 2m-1$. Observe that

$$x + yq^{2(2m-l-1)+1} \equiv 0 \pmod{n},$$

hence, when $m \leq l \leq 2m-1$, we have $0 \leq 2m-l-1 \leq m-1$, so we only seek contradictions for the case $0 \leq l \leq m-1$. Let $T_{2m}^* = \bigcup_{i=1}^{\theta} C_{q^{2m}}[i,n]$, where $1 \leq \theta \leq \Theta^*$.

(i) If $m = 2$, then $l = 0, 1$.
If $l = 0$, then we have

$$q + 1 \leq xq + y \leq (\frac{q^3 - q^2 + q - 1}{\rho} - 1)(q+1) < \frac{q^4 - 1}{\rho} = n,$$

which is a contradiction.
If $l = 1$, then we have

$$q + 1 \leq x + yq \leq (\frac{q^3 - q^2 + q - 1}{\rho} - 1)(q+1) < \frac{q^4 - 1}{\rho} = n,$$

which is also a contradiction.
Moreover, if $\Theta^* = \Theta_{max} + 1$, then there exist $x = y = \frac{q^3-q^2+q-1}{\rho} \in T_{2m}^*$ such that

$$xq + y = \frac{q^3 - q^2 + q - 1}{\rho}(q+1) \equiv 0 \pmod{n}.$$

(ii) $m = 2k \geq 4$:
If $0 \leq l \leq \frac{m-2}{2}$, then

$$q + 1 \leq xq^{2l+1} + y \leq \frac{q^{m+1} - q^2}{\rho}(q^{m-1} + 1) = \frac{q^{2m} - q^2}{\rho} < n.$$

This is a contradiction.
If $\frac{m}{2} \leq l \leq m-1$, note that $q^{2m} \equiv 1 \pmod{n}$, the above congruence is equivalent to $x + yq^{2(m-l-1)+1} \equiv 0 \pmod{n}$. One can easily get $0 \leq m-l-1 \leq \frac{m-2}{2}$, so we can get a contradiction similar to the above case.
Moreover, if $\Theta^* = \Theta_{max} + 1$, one can take $x = \Theta^* = \frac{q^{m+1}-q^2}{\rho} + 1$ and $l = \frac{m-2}{2}$, then

$$-q^{m-1}(\frac{q^{m+1} - q^2}{\rho} + 1) \equiv \frac{q^{m+1} - \rho q^{m-1} - 1}{\rho} \pmod{n}.$$

Namely, there exist $x = \Theta^*, y = \frac{q^{m+1}-\rho q^{m-1}-1}{\rho} \in T_{2m}^*$ and $l = \frac{m-2}{2}$ such that

$$xq^{2l+1} + y \equiv 0 \pmod{n}.$$

(iii) $m = 2k+1 \geq 3$:
If $0 \leq l \leq \frac{m-1}{2}$, then

$$q + 1 \leq xq^{2l+1} + y \leq (\frac{q^m - 1}{\rho} - 1)(q^m + 1) < n.$$

It is a contradiction.

If $\frac{m+1}{2} \leq l \leq m-1$, then $0 \leq m-l-1 \leq \frac{m-3}{2}$. One can get

$$q+1 \leq x + yq^{2(m-l-1)+1} \leq (\frac{q^m-1}{\rho}-1)(q^{m-2}+1) < n.$$

It is also a contradiction.

Moreover, if $\Theta^* = \Theta_{max}+1$, then there exist $x = y = \frac{q^m-1}{\rho} \in T^*_{2m}$, and $l = \frac{m-1}{2}$ such that

$$xq^{2l+1} + y = \frac{q^m-1}{\rho}q^m + \frac{q^m-1}{\rho} = \frac{q^{2m}-1}{\rho} \equiv 0 \pmod{n}. \square$$

Theorem 10: Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q-1)$, q is a prime power and $m \geq 2$ is a positive integer. Then there exists a q-ary quantum code with parameters $[[mn, mn-2m\theta, \geq \theta+1]]$, where $1 \leq \theta \leq \Theta_{max}$.

Proof: Let $\mathcal{D}$ be the cyclic code over $\mathbb{F}_{q^{2m}}$ of length n with nonzero set

$$T_{2m} = \bigcup_{i=1}^{\theta} C_{q^{2m}}[i,n],$$

where $1 \leq \theta \leq \Theta_{max}$. By Lemma 9, we have $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$. The zero set of $\mathcal{D}$ is

$$Z_{2m} = C_{q^{2m}}[0,n] \bigcup \bigcup_{i=\theta+1}^{n-1} C_{q^{2m}}[i,n].$$

It can be easily obtained that the zero set Z_{2m} has $n-\theta$ consecutive roots $\alpha^{\theta+1}, \ldots, \alpha^{n-1}, \alpha^n$, where α is a primitive n-th root of unity. By Theorem 3, we have $d \geq n-\theta+1$ and according to Theorem 1, $d \leq n-\theta+1$. Hence, $\mathcal{D}$ is an $[n, \theta, n-\theta+1]$ MDS code over $\mathbb{F}_{q^{2m}}$, and $\mathcal{D}^{\perp_h}$ is also an MDS code, which has parameters $[n, n-\theta, \theta+1]_{q^{2m}}$. By Lemma 5, the minimum distance of $\mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$ is $d^* \geq \theta+1$. It follows from $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$ that $\mathcal{L}_{\mathcal{A}}(\mathcal{D})$ has parameters $[mn, m\theta, \geq \theta+1]_{q^2}$. According to Theorem 4, there exists a q-ary quantum code with parameters $[[mn, mn-2m\theta, \geq \theta+1]]$. $\square$

Remark 11: If $\rho = 1$, then $n = q^{2m}-1$, quantum codes of length mn had been constructed in [29]. It can be easily seen that our results coincide with theirs within such case. Therefore, our results can be seen as a generalization of theirs.

Example 12: Let $q = 4$ and $m = 2$, then ρ can be 1 and 3. Hence, $n = 255$, $\Theta_{max} = 50$ and $n = 85$, $\Theta_{max} = 16$, respectively. According to Theorem 10, there exist quantum codes of lengths 510 and 170 over $\mathbb{F}_4$. By Theorem 8, we can also get quantum codes of lengths 511 and 171. The quantum codes obtained here have larger code rate than the quantum twisted codes shown in [32]. The results are listed in Table 1.

Table 1. Code comparison

New quantum codes	Lengthened codes	Quantum twisted codes in [32]
$[[170, 162, \geq 3]]_4$	$[[171, 162, \geq 3]]_4$	$[[171, 160, 3]]_4$
$[[170, 158, \geq 4]]_4$	$[[171, 158, \geq 4]]_4$	$[[171, 151, 4]]_4$
$[[170, 154, \geq 5]]_4$	$[[171, 154, \geq 5]]_4$	$[[171, 142, 5]]_4$
$[[170, 150, \geq 6]]_4$	$[[171, 150, \geq 6]]_4$	$[[171, 133, 6]]_4$
$[[170, 146, \geq 7]]_4$	$[[171, 146, \geq 7]]_4$	$--$
$[[170, 142, \geq 8]]_4$	$[[171, 142, \geq 8]]_4$	$[[171, 124, 8]]_4$
$[[170, 138, \geq 9]]_4$	$[[171, 138, \geq 9]]_4$	$[[171, 102, 9]]_4$
$[[170, 134, \geq 10]]_4$	$[[171, 134, \geq 10]]_4$	$[[171, 115, 10]]_4$
$[[170, 130, \geq 11]]_4$	$[[171, 130, \geq 11]]_4$	$[[171, 106, 11]]_4$
$[[170, 126, \geq 12]]_4$	$[[171, 126, \geq 12]]_4$	$[[171, 97, 12]]_4$
$[[170, 122, \geq 13]]_4$	$[[171, 122, \geq 13]]_4$	$--$
$[[170, 118, \geq 14]]_4$	$[[171, 118, \geq 14]]_4$	$[[171, 88, 14]]_4$
$[[170, 114, \geq 15]]_4$	$[[171, 114, \geq 15]]_4$	$[[171, 79, 15]]_4$
$[[170, 110, \geq 16]]_4$	$[[171, 110, \geq 16]]_4$	$[[171, 70, 16]]_4$
$[[170, 106, \geq 17]]_4$	$--$	$--$
$[[510, 502, \geq 3]]_4$	$[[511, 502, \geq 3]]_4$	$[[511, 499, 3]]_4$
$[[510, 498, \geq 4]]_4$	$[[511, 498, \geq 4]]_4$	$[[511, 490, 4]]_4$
$[[510, 494, \geq 5]]_4$	$[[511, 494, \geq 5]]_4$	$[[511, 484, 5]]_4$
$[[510, 490, \geq 6]]_4$	$[[511, 490, \geq 6]]_4$	$[[511, 472, 6]]_4$
$[[510, 486, \geq 7]]_4$	$[[511, 486, \geq 7]]_4$	$[[511, 466, 7]]_4$
$[[510, 482, \geq 8]]_4$	$[[511, 482, \geq 8]]_4$	$[[511, 454, 8]]_4$
$[[510, 478, \geq 9]]_4$	$[[511, 478, \geq 9]]_4$	$[[511, 457, 9]]_4$
$[[510, 474, \geq 10]]_4$	$[[511, 474, \geq 10]]_4$	$[[511, 448, 10]]_4$
$[[510, 470, \geq 11]]_4$	$[[511, 470, \geq 11]]_4$	$[[511, 439, 11]]_4$
$[[510, 466, \geq 12]]_4$	$[[511, 466, \geq 12]]_4$	$[[511, 427, 12]]_4$
$[[510, 462, \geq 13]]_4$	$[[511, 462, \geq 13]]_4$	$[[511, 430, 13]]_4$
$[[510, 458, \geq 14]]_4$	$[[511, 458, \geq 14]]_4$	$[[511, 421, 14]]_4$
$\cdots$	$\cdots$	$\cdots$
$[[510, 406, \geq 27]]_4$	$[[511, 406, \geq 27]]_4$	$[[511, 331, 27]]_4$
$[[510, 402, \geq 28]]_4$	$[[511, 402, \geq 28]]_4$	$[[511, 271, 28]]_4$
$[[510, 398, \geq 29]]_4$	$[[511, 398, \geq 29]]_4$	$[[511, 322, 29]]_4$
$[[510, 394, \geq 30]]_4$	$[[511, 394, \geq 30]]_4$	$[[511, 313, 30]]_4$
$[[510, 390, \geq 31]]_4$	$[[511, 390, \geq 31]]_4$	$[[511, 304, 31]]_4$
$[[510, 386, \geq 32]]_4$	$--$	$--$
$\cdots$	$--$	$--$
$[[510, 305, \geq 51]]_4$	$--$	$--$

3.2 Construction II

Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q+1)$, q is a prime power, and $m \geq 2$ is a positive integer. In this subsection, we consider the η-constacyclic codes over $\mathbb{F}_{q^{2m}}$ of length n. Let ord(η)$= r = \rho$. Obviously, the length n is coprime to q^{2m} and all the q^{2m}-cyclotomic cosets modulo nr are given by $C_{q^{2m}}[i, nr] = \{i\}$, for $0 \leq i \leq n-1$. If $\rho = r = 1$, then $n = q^{2m} - 1$ and the η-constacyclic code is indeed the cyclic code, which is included in Construction I, so we assume that $\rho \neq 1$ here.

If $m = 2k \geq 2$, define

$$\Theta_{max} = \begin{cases} \frac{q^3-q^2+2}{\rho} + \lambda, & m = 2 \text{ } and \text{ } q \geq 4, \\ \frac{q^{m+1}-q^2+2-\rho}{\rho} - [\rho = 2], & m = 2k \geq 4 \text{ } or \text{ } q \leq 3. \end{cases} \tag{2}$$

and $\lambda = \lceil \frac{q-4}{\rho} \rceil - 1$.

If $m = 2k+1 \geq 3$, define

$$\Theta_{max} = \begin{cases} q^m - 2, & \rho = 2, \\ \frac{(q^m+1)(\rho - [\rho \text{ } odd])}{2\rho} - 2, & otherwise. \end{cases} \tag{3}$$

$\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ means the ceiling function and floor function, respectively. If the statement is true, then [statement]=1, otherwise, [statement]=0.

Lemma 13: Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q+1)$, $\rho \neq 1$, q is a prime power and $m \geq 2$ is a positive integer. If $\mathcal{D}$ is an η-constacyclic code over $\mathbb{F}_{q^{2m}}$ of length n with nonzero set $T_{2m} = \bigcup_{i=0}^{\theta} C_{q^{2m}}[i, nr]$, where $0 \leq \theta \leq \Theta_{max}$, then $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$.

Proof: From Lemma 6, we only need to proof that for any $1+\rho x$, $1+\rho y \in T_{2m}$,

$$(1+\rho x)q^{2l+1} + (1+\rho y) \not\equiv 0 \pmod{\rho n}.$$

Assume that there exist $x, y \in T_{2m}$ such that

$$(1+\rho x)q^{2l+1} + (1+\rho y) \equiv 0 \pmod{\rho n},$$

where $0 \leq l \leq 2m-1$. Observe that

$$(1+\rho x) + (1+\rho y)q^{2(2m-l-1)+1} \equiv 0 \pmod{\rho n},$$

hence, when $m \leq l \leq 2m-1$, we have $0 \leq 2m-l-1 \leq m-1$, so we only seek contradictions for the case $0 \leq l \leq m-1$. Let $T_{2m}^* = \bigcup_{i=0}^{\theta} C_{q^{2m}}[i, nr]$, where $0 \leq \theta \leq \Theta^*$.

(i) If $m = 2$ and $q \geq 4$, then $l = 0$ and 1. Since $\lambda = \lceil \frac{q-4}{\rho} \rceil - 1$, $\frac{q-4}{\rho} - 1 \leq \lambda < \frac{q-4}{\rho}$, namely, $q - 4 - \rho \leq \lambda\rho < q - 4$.

If $l = 0$, then

$$q + 1 \leq (1+\rho x)q + (1+\rho y) \leq [1 + \rho(\frac{q^3-q^2+2}{\rho} + \lambda)](q+1) < q^4 - 1 = n.$$

It is a contradiction.

If $l=1$, we have the following congruence because of $q^4\equiv 1 \pmod{\rho n}$,

$$(1+\rho x)q^3+(1+\rho y)\equiv(1+\rho x)+(1+\rho y)q \pmod{\rho n}.$$

So we have the same contradiction as the above case.
Moreover, if $\Theta^*=\Theta_{max}+1=\frac{q^3-q^2+2}{\rho}+\lambda+1$, then we have $1+\rho\Theta^*=1+q^3-q^2+2+\rho(\lambda+1)\geq q^3-q^2+q-1$. Note that

$$-q(1+\rho\Theta^*)\equiv q^3-3q-1-\rho q(\lambda+1) \pmod{\rho n}.$$

Obviously, $q^3-3q-1-\rho q(\lambda+1)\leq q^3-q^2+q-1$, so we can select $x=\Theta^*$, $y=\frac{q^3-3q-2-\rho q(\lambda+1)}{\rho}\in T^*_{2m}$ and $l=0$ such that

$$(1+\rho x)q^{2l+1}+(1+\rho y)\equiv 0 \pmod{\rho n}.$$

(ii) $m=2k\geq 4$ or $q\leq 3$:
If $0\leq l\leq\frac{m-2}{2}$, then

$$q+1\leq(1+\rho x)q^{2l+1}+(1+\rho y)\leq(q^{m+1}-q^2+3-\rho-[\rho=2])(q^{m-1}+1)<\rho n.$$

It is a contradiction.
If $\frac{m}{2}\leq l\leq m-1$, then $0\leq m-l-1\leq\frac{m-2}{2}$. Note that $q^{2m}\equiv 1 \pmod{\rho n}$, the above congruence is equivalent to

$$(1+\rho x)+(1+\rho y)q^{2(m-l-1)+1}\equiv 0 \pmod{\rho n}.$$

So we have a similar contradiction.
Moreover, if $\Theta^*=\Theta_{max}+1$, then $1+\rho\Theta^*=q^{m+1}-q^2+3-[\rho=2]\rho$. Note that

$$-q^{m-1}(1+\rho\Theta^*)\equiv q^{m+1}-3q^{m-1}-1+[\rho=2]\rho q^{m-2} \pmod{\rho n}.$$

Obviously,

$$q^{m+1}-3q^{m-1}-1+[\rho=2]\rho q^{m-2}\leq q^{m+1}-q^2+3-[\rho=2]\rho,$$

then we can select $x=\Theta^*$, $y=\frac{q^{m+1}-3q^{m-1}-2+[\rho=2]\rho q^{m-2}}{\rho}\in T^*_{2m}$ and $l=\frac{m-2}{2}$ such that

$$(1+\rho x)q^{2l+1}+(1+\rho y)\equiv 0 \pmod{\rho n}.$$

(iii) $m=2k+1\geq 3$ and $\rho=2$:
If $0\leq l\leq\frac{m-3}{2}$, then

$$q+1\leq(1+\rho x)q^{2l+1}+(1+\rho y)\leq[1+\rho(q^m-2)](q^{m-2}+1)<\rho n,$$

which is a contradiction.
If $\frac{m+1}{2}\leq l\leq m-1$, one can get the same contradiction as above.
If $l=\frac{m-1}{2}$, then

$$q^m(1+2x)+(1+2y)\equiv 0 \pmod{q^{2m}-1}.$$

Therefore,

$$q+1 \leq q^m(1+2x)+(1+2y) \leq 2q^{2m}-q^m-3 < 2(q^{2m}-1),$$

so it must be

$$q^m(1+2x)+(1+2y)=q^{2m}-1.$$

Hence, we can get $y=-q^m x+\frac{q^{2m}-q^m-2}{2}$.
If $0 \leq x \leq \frac{q^m-3}{2}$, then $y \geq q^m-1$, which contradicts to the fact that $y \leq q^m-2=\Theta_{max}$.
If $\frac{q^m-1}{2} \leq x \leq \Theta_{max}$, then $y<0$, which contradicts to the fact that $y \geq 0$.
Moreover, if $\Theta^*=\Theta_{max}+1$, then there exist $x=q^m-1$, $y=\frac{q^m-3}{2} \in T^*_{2m}$ and $l=\frac{m-1}{2}$ such that

$$q^m[1+2(q^m-1)]+(1+2\frac{q^m-3}{2}) \equiv 0 \pmod{q^{2m}-1}.$$

(iv) $m=2k+1 \geq 3$ and $\rho \neq 2$:
If $0 \leq l \leq \frac{m-3}{2}$, then

$$q+1 \leq (1+\rho x)q^{2l+1}+(1+\rho y) \leq (1+\rho\Theta_{max})(q^{m-2}+1) < \rho n,$$

which is a contradiction.
If $\frac{m+1}{2} \leq l \leq m-1$, one can get the same contradiction as above.
If $l=\frac{m-1}{2}$, then

$$q^m(1+\rho x)+(1+\rho y) \equiv 0 \pmod{q^{2m}-1}.$$

Hence, we have the following congruence

$$q^m(1+\rho x)+(1+\rho y) \equiv 0 \pmod{q^m+1},$$

which is equivalent to $\rho(y-x) \equiv 0 \pmod{q^m+1}$. Then we can get $y=x+\mu\frac{q^m+1}{\rho}$, where $-\frac{\rho-[\rho\ odd]}{2}+1 \leq \mu \leq \frac{\rho-[\rho\ odd]}{2}-1$.
Substituting y into $q^m(1+\rho x)+(1+\rho y) \equiv 0 \pmod{q^{2m}-1}$, one can get

$$(q^m+1)(1+\rho x+\mu) \equiv 0 \pmod{q^{2m}-1},$$

which implies that $1+\rho x+\mu \equiv 0 \pmod{q^m-1}$. Hence, $x=\frac{\nu(q^m-1)-1-\mu}{\rho}$. Since $\rho \mid (q+1)$ and m is odd, $q^m-1 \equiv -2 \pmod{\rho}$, and then $2\nu+\mu+1=0 \pmod{\rho}$.
If $\nu \leq -1$ or $\nu \geq \frac{\rho-[\rho\ odd]}{2}$, then $x<0$ or $x \geq \frac{\rho-[\rho\ odd]}{2\rho}(q^m-2) > \Theta_{max}$, which contradicts to the fact that $x \in T_{2m}$. So $0 \leq \nu \leq \frac{\rho-[\rho\ odd]}{2}-1$.
Substituting $x=\frac{\nu(q^m-1)-1-\mu}{\rho}$ into $y=x+\mu\frac{q^m+1}{\rho}$, one can get $y=\frac{(\mu+\nu)q^m-\nu-1}{\rho}$.
If $\mu+\nu \leq 0$ or $\mu+\nu \geq \frac{\rho-[\rho\ odd]}{2}$, then $y<0$ or $y \geq \frac{\rho-[\rho\ odd]}{2\rho}(q^m-1) > \Theta_{max}$, which contradicts to the fact that $y \in T_{2m}$. So $1 \leq \mu+\nu \leq \frac{\rho-[\rho\ odd]}{2}-1$. Hence, $2\nu+\mu+1 \leq \rho-[\rho\ odd]-1 < \rho$, which is a contradiction.
Moverover, if $\Theta^*=\Theta_{max}+1$, then there exist $x=\Theta^*=\frac{(q^m+1)(\rho-[\rho\ odd])}{2\rho}-1$ and $y=\frac{(q^m+1)(\rho+[\rho\ odd]-2)-2\rho}{2\rho} \in T^*_{2m}$ such that

$$q^m(1+\rho x)+(1+\rho y) \equiv 0 \pmod{\rho n}. \square$$

Theorem 14: Let $n = \frac{q^{2m}-1}{\rho}$, where $\rho \mid (q+1)$, $\rho \neq 1$, q is a prime power, and $m \geq 2$ is a positive integer. Then there exists a q-ary quantum code with parameters $[[mn, mn-2m(\theta+1), \geq \theta+2]]$, where $0 \leq \theta \leq \Theta_{max}$.

Proof: Let $\mathcal{D}$ be the η-constacyclic code over $\mathbb{F}_{q^{2m}}$ of length n with nonzero set

$$T_{2m} = \bigcup_{i=0}^{\theta} C_{q^{2m}}[i, nr],$$

where $0 \leq \theta \leq \Theta_{max}$. By Lemma 13, $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$. The zero set of $\mathcal{D}$ is

$$Z_{2m} = \bigcup_{i=\theta+1}^{n-1} C_{q^{2m}}[i, nr].$$

It can be easily obtained that the zero set Z_{2m} has $n-\theta-1$ consecutive roots $\alpha^{\theta+1}, \ldots, \alpha^{n-1}$, where α is a primitive n-th root of unity. From Theorem 3, we have $d \geq n-\theta$ and by Theorem 1, $d \leq n-\theta$. Hence, $\mathcal{D}$ is an $[n, \theta+1, n-\theta]$ MDS code over $\mathbb{F}_{q^{2m}}$. $\mathcal{D}^{\perp_h}$ is also an MDS code, which has parameters $[n, n-\theta-1, \theta+2]_{q^{2m}}$. Due to Lemma 5, the minimum distance of $\mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$ is $d^* \geq \theta+2$. It follows that $\mathcal{L}_{\mathcal{A}}(\mathcal{D})$ has parameters $[mn, m(\theta+1), \geq \theta+2]_{q^2}$ because of $\mathcal{L}_{\mathcal{A}}(\mathcal{D}) \subseteq \mathcal{L}_{\mathcal{A}}(\mathcal{D})^{\perp_h}$. According to Theorem 4, there exists a q-ary quantum code with parameters $[[mn, mn-2m(\theta+1), \geq \theta+2]]$. □

Remark 15: If $\rho = q+1$, then $n = \frac{q^{2m}-1}{q+1}$, quantum codes of length mn had been extensively studied in [29] using cyclic codes. The concrete parameters are in the following:

- $[[mn, mn-2mu, \geq u+1]]$, $q \equiv 1 (\mathrm{mod}\ 4)$, $m \geq 2$ and $1 \leq u \leq U_1$,
- $[[mn, mn-2mu, \geq u+1]]$, $q \equiv 3 (\mathrm{mod}\ 4)$, $m \geq 2$ and $1 \leq u \leq U_2$,

where

$$U_1 = \begin{cases} \frac{q^{m+1}-q^2-q+1}{q+1}, & m = 2k \geq 2, k = 1, 2, \ldots \\ \frac{q^m-3}{2}, & m = 2k+1 \geq 3, k = 1, 2, \ldots \end{cases} \tag{4}$$

$$U_2 = \begin{cases} \frac{q^{m+1}-q^2-q+1}{q+1}, & m = 2k \geq 2, k = 1, 2, \ldots \\ \frac{2q^m-q+1}{q+1}, & m = 2k+1 \geq 3, k = 1, 2, \ldots \end{cases} \tag{5}$$

Comparing our results with theirs within such length, one can see that our quantum codes have minimum distances larger than or equal to theirs. Hence, our results are better and more general.

Example 16: Let $q = 5$ and $m = 2$, then ρ can be 2, 3 and 6(except $\rho = 1$). If $\rho = 3$, then $n = 208$ and $\Theta_{max} = 34$. According to Theorem 14, there exist quantum codes of length 416 over $\mathbb{F}_5$. By Theorem 8, we can get quantum codes of length 417. The quantum codes obtained by this method have larger code rate than the quantum twisted codes shown in [32]. The results are listed in Table 2.

Example 17: Let $q = 8$ and $m = 2$, then ρ can be 3 and 9(except $\rho = 1$). If $\rho = 9$, then $n = 455$, $\Theta_{max} = 50$. According to Theorem 14, there exist quantum codes of length 910 over

$\mathbb{F}_8$. By Theorem 8, we can get quantum codes of lengths 915 and 925. The quantum codes obtained by this method have larger code rate than the quantum twisted codes shown in [32]. The results are listed in Table 3.

4 Conclusion

In this paper, we studied cyclic codes of length $\frac{q^{2m}-1}{\rho}$ with $\rho \mid (q-1)$ over $\mathbb{F}_{q^{2m}}$ and constacyclic codes of length $\frac{q^{2m}-1}{\rho}$ with $\rho \mid (q+1)$ over $\mathbb{F}_{q^{2m}}$. The maximum nonzero sets to make the images of cyclic codes and constacyclic codes to be Hermitian self-orthogonal were given, respectively. Then, these Hermitian self-orthogonal codes were utilized to construct quantum codes, and the resulting quantum codes have better parameters than the previously known ones.

Acknowledgements We are grateful to the anonymous referees and the associate editor for their useful comments and suggestions that improved the presentation and quality of this paper. The work was supported by the National Natural Science Foundation of China (12271137, U21A20428, 61972126), and the Fundamental Research Funds for the Central Universities of China (PA2021KCPY0040).

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Table 2. Code comparison

New quantum codes	Lengthened codes	Quantum twisted codes in [32]
$[[416, 412, \geq 2]]_5$	$--$	$--$ $--$
$[[416, 408, \geq 3]]_5$	$[[417, 408, \geq 3]]_5$	$[[416, 406, 3]]_5$ $[[417, 407, 3]]_5$
$[[416, 404, \geq 4]]_5$	$[[417, 404, \geq 4]]_5$	$[[416, 402, 4]]_5$ $[[417, 403, 4]]_5$
$[[416, 400, \geq 5]]_5$	$[[417, 400, \geq 5]]_5$	$[[416, 394, 5]]_5$ $[[417, 395, 5]]_5$
$[[416, 396, \geq 6]]_5$	$[[417, 396, \geq 6]]_5$	$[[416, 390, 6]]_5$ $[[417, 391, 6]]_5$
$[[416, 392, \geq 7]]_5$	$[[417, 392, \geq 7]]_5$	$[[416, 382, 7]]_5$ $[[417, 383, 7]]_5$
$[[416, 388, \geq 8]]_5$	$[[417, 388, \geq 8]]_5$	$[[416, 378, 8]]_5$ $[[417, 379, 8]]_5$
$\cdots$	$\cdots$	$\cdots$ $\cdots$
$[[416, 332, \geq 22]]_5$	$[[417, 332, \geq 22]]_5$	$[[416, 302, 22]]_5$ $[[417, 303, 22]]_5$
$[[416, 328, \geq 23]]_5$	$[[417, 328, \geq 23]]_5$	$[[416, 294, 23]]_5$ $[[417, 295, 23]]_5$
$[[416, 324, \geq 24]]_5$	$[[417, 324, \geq 24]]_5$	$[[416, 290, 24]]_5$ $[[417, 291, 24]]_5$
$[[416, 320, \geq 25]]_5$	$[[417, 320, \geq 25]]_5$	$[[416, 282, 25]]_5$ $[[417, 283, 25]]_5$
$[[416, 316, \geq 26]]_5$	–	$[[416, 280, 26]]_5$ –
$[[416, 312, \geq 27]]_5$	$[[417, 312, \geq 27]]_5$	$[[416, 278, 27]]_5$ $[[417, 279, 27]]_5$
$[[416, 308, \geq 28]]_5$	$[[417, 308, \geq 28]]_5$	$[[416, 274, 28]]_5$ $[[417, 275, 28]]_5$
$[[416, 304, \geq 29]]_5$	$[[417, 304, \geq 29]]_5$	$[[416, 266, 29]]_5$ $[[417, 267, 29]]_5$
$[[416, 300, \geq 30]]_5$	$[[417, 300, \geq 30]]_5$	$[[416, 262, 30]]_5$ $[[417, 263, 30]]_5$
$[[416, 296, \geq 31]]_5$	$[[417, 296, \geq 31]]_5$	$[[416, 254, 31]]_5$ $[[417, 255, 31]]_5$
$[[416, 292, \geq 32]]_5$	$[[417, 292, \geq 32]]_5$	$[[416, 250, 32]]_5$ $[[417, 251, 32]]_5$
$[[416, 288, \geq 33]]_5$	$[[417, 288, \geq 33]]_5$	$[[416, 242, 33]]_5$ $[[417, 243, 33]]_5$
$[[416, 284, \geq 34]]_5$	$[[417, 284, \geq 34]]_5$	$[[416, 239, 34]]_5$ $[[417, 239, 34]]_5$
$[[416, 280, \geq 35]]_5$	–	$[[416, 232, 35]]_5$ –
$[[416, 276, \geq 36]]_5$	$[[417, 276, \geq 36]]_5$	$[[416, 230, 36]]_5$ $[[417, 231, 36]]_5$

Table 3. Code comparison

New quantum codes	Lengthened codes	Quantum twisted codes in [32]
$[[910, 906, \geq 2]]_8$	$--$	$--$
$[[910, 902, \geq 3]]_8$	$[[915, 902, \geq 3]]_8$	$[[915, 895, 3]]_8$
$[[910, 898, \geq 4]]_8$	$[[915, 898, \geq 4]]_8$	$[[915, 875, 4]]_8$
$[[910, 894, \geq 5]]_8$	$[[915, 894, \geq 5]]_8$	$[[915, 855, 5]]_8$
$[[910, 890, \geq 6]]_8$	$[[915, 890, \geq 6]]_8$	$[[915, 835, 6]]_8$
$[[910, 886, \geq 7]]_8$	$--$	$--$
$[[910, 882, \geq 8]]_8$	$--$	$--$
$[[910, 878, \geq 9]]_8$	$[[925, 878, \geq 9]]_8$	$[[925, 793, 9]]_8$
$[[910, 874, \geq 10]]_8$	$--$	$--$
$[[910, 870, \geq 11]]_8$	$--$	$--$
$[[910, 866, \geq 12]]_8$	$--$	$--$
$\cdots$	$--$	$--$
$[[910, 706, \geq 52]]_8$	$--$	$--$

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