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SYMMETRIC STATES AND DYNAMICS OF THREE QUANTUM BITS

FRANCESCA ALBERTINI
Dipartimento di Tecnica e Gestione del Sistemi Industriali
Universita' di Padova, Vicenza, Italy
albertin@math.unipd.it

DOMENICO D'ALESSANDRO
Department of Mathematics, Iowa State University
Ames IA 50011, U.S.A.
daless@iastate.edu

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The unitary group acting on the Hilbert space $\mathcal{H} := (C^2)^{\otimes 3}$ of three quantum bits admits a Lie subgroup, $U^{S_3}(8)$, of elements which permute with the symmetric group of permutations of three objects. Under the action of such a Lie subgroup, the Hilbert space $\mathcal{H}$ splits into three invariant subspaces of dimensions 4, 2 and 2 respectively, each corresponding to an irreducible representation of $su(2)$. The subspace of dimension 4 is uniquely determined and corresponds to states that are themselves invariant under the action of the symmetric group. This is the so called *symmetric sector*. The subspaces of dimension two are not uniquely determined and we parametrize them all. We provide an analysis of pure states that are in the subspaces invariant under $U^{S_3}(8)$. This concerns their entanglement properties, separability criteria and dynamics under the Lie subgroup $U^{S_3}(8)$. As a physical motivation for the states and dynamics we study, we propose a physical set-up which consists of a symmetric network of three spin $\frac{1}{2}$ particles under a common driving electro-magnetic field. For such system, we solve the control theoretic problem of driving a separable state to a state with maximal distributed entanglement.

Keywords: Quantum entanglement, symmetric states, quantum symmetric evolution, spin networks, quantum control.

1 Introduction

The study of quantum states is a current line of research in quantum physics (see, e.g., [3]), in particular for what concerns their entanglement properties. Entanglement is considered a resource in quantum information processing, and classifying states according to the amount and type of entanglement is a problem of both fundamental and practical importance. A related problem is to study how quantum dynamical evolution changes the entanglement of states (see,e.g., [22] for the two qubits case) and, in the quantum control setting [5], how to induce such dynamics for a specific physical setup. The two qubits case is fairly well understood, while the case of three qubits requires further exploration. In particular, three qubits are the simplest type of system which displays two types of entanglement: a *pairwise entanglement* quantifying the entanglement between pairs of qubits and a *distributed* *entanglement* [4]. These types of entanglement are constrained by inequalities often referred to as *monogamy relations* [23].

In this paper we are concerned with systems of three qubits whose dynamics are subject to a permutation symmetry among the three qubits. The possible unitary evolutions on the Hilbert space $\mathcal{H} := (C^2)^{\otimes 3}$ of the system consists of unitaries which commute with the permutation group S_3.[a] Such a Lie subgroup of $U(8)$, which we denote by $U^{S_3}(8)$, has dimension 20 [1]. Its Lie algebra, $u^{S_3}(8)$, is spanned by the matrices

$$i\Pi(\sigma_1 \otimes \sigma_2 \otimes \sigma_3). \tag{1}$$

where Π denotes the *symmetrization operator*, $\Pi := \frac{1}{3!}\sum_{P \in S_3} P$, and $\sigma_{1,2,3}$ are chosen to be the 2×2 identity or one of the Pauli matrices,

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The number of the matrices (1) is equal to the way of choosing the number of occurrences of the identity and $\sigma_{x,y,z}$ out of three positions, which is equal to 20.[b]

The three qubit space $\mathcal{H}$ under the action of the Lie algebra $u^{S_3}(8)$, or of the Lie group $U^{S_3}(8)$, splits into three invariant subspaces two of which have dimension 2 and one of which has dimension 4. These correspond to irreducible representations of $su(2)$ [1], [6]. The subspace of dimension 4 is uniquely determined. It is the so-called *symmetric sector* [17], that is, the space of the states that do not change under permutation of the qubits. For the symmetric sector, we shall use the orthogonal basis, (not normalized)

$$\begin{aligned} \phi_0 &:= |000\rangle, \\ \phi_1 &:= |100\rangle + |010\rangle + |001\rangle, \\ \phi_2 &:= |011\rangle + |101\rangle + |110\rangle, \\ \phi_3 &:= |111\rangle. \end{aligned} \tag{3}$$

This notation follows the number of 1's that appear in each state, in the sense that ϕ_j is the sum of the states in the computational basis which have j 1's. States in the symmetric sector where studied in [17] and a complete list of invariants under local unitary and symmetric transformations was given there.

We will write a general state in the symmetric sector as

$$\psi := c_0\phi_0 + c_1\phi_1 + c_2\phi_2 + c_3\phi_3, \tag{4}$$

for complex coefficients c_0, c_1, c_2, c_3, with $|c_0|^2 + 3|c_1|^2 + 3|c_2|^2 + |c_3|^2 = 1$. This can also be seen as a four level system which can be used to implement two qubits.

The subspaces of dimension 2, which represent isomorphic representations of $su(2)$, are *not* uniquely determined.

The results presented in this paper and a plan for the following sections is as follows. In section 2 we perform the decomposition of the Hilbert space $\mathcal{H} := (C^2)^{\otimes 3}$ into invariant

[a]The group of permutation of n objects, also called the symmetric group, is denoted by S_n.
[b]The general argument for n qubit is presented in [1].

subspaces for $u^{S_3}(8)$. We see that while the four dimensional invariant subspace is uniquely determined (it is the symmetric sector) the pair of 2-dimensional invariant subspaces is not. We obtain a parametrization of all the possible decompositions and therefore of all the possible 2-dimensional invariant subspaces. In section 3 we recall the general measures of entanglement for three qubits introduced in [4], in particular the *pairwise entanglement* (concurrence), quantifying the entanglement between two qubits when the third one is traced out and the *distributed entanglement*. These measures are invariant under local unitary transformations. In section 4, we calculate the entanglement measures for states in the symmetric sector and give conditions of separability. The expressions we find complement the ones found in [17] which are based on the Majorana polynomial representations of states [14]. We also briefly recall such a representation which has an elegant geometric interpretation and provide a complete set of local invariants for these states. In section 5 we analyze the entanglement of the states in the two dimensional invariant subspaces. We find that the distributed entanglement is zero on each of these invariant subspaces and it is at most $\frac{1}{3}$ for states in the direct sum of these subspaces, that is, states in the orthogonal complement of the symmetric sector. We then turn our attention to the *dynamics*, in particular the ones given by the Lie subgroup of the unitary group $U(8)$ which permutes with the symmetric group, i.e., $U^{S_3}(8)$. In section 6, we study such dynamics on the invariant subspaces. In particular, for the symmetric sector, we prove that the group of local (symmetric) unitary transformations is a maximal Lie subgroup of $U^{S_3}(8)$ which leaves the adopted measures of entanglement unchanged. We then study in general how the elements of the group $U^{S_3}(8)$ change the entanglement in this subspace. We give a factorization of possible unitary evolutions on the symmetric sector in evolutions that modify the entanglement and evolutions that do not. In section 7, we give a physical application of the analysis described in the previous sections. We consider a symmetric network of three spin $\frac{1}{2}$ particles coupled via identical Ising interaction and driven by a common electro-magnetic field. The dynamics of this model satisfies the symmetry assumptions considered in this paper. We propose an algorithm to drive such a system from a separable state (with zero entanglement) to a state with maximum distributed entanglement. A summary of the results is given in section 8.

As we have mentioned, the three qubit case is of interest because it is the simplest case where one can define distributed entanglement. It is also the simplest case where the decomposition in invariant subspaces[c] is not unique. In the two qubit case, the decomposition into invariant subspaces is $\mathcal{H} = \mathcal{S} \oplus \mathcal{A}$ where $\mathcal{S}$ is the two qubits symmetric sector spanned by (cf. (3)) $\left\{|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)\right\}$, and $\mathcal{A}$ is the one dimensional subspace spanned by the antisymmetric state, $\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$. States with maximum entanglement occur (in particular the Bell states) in both invariant subspaces. We shall see that for the three qubit case the distributed entanglement is zero in the two dimensional invariant subspaces and can take any value in $[0, 1]$ in the four dimensional symmetric sector.

We envision a possible extension of the results presented here to n qubits at the price of higher computational complexity in the calculations and resulting formulas. In particular, the method to obtain all possible decompositions we will describe in the next section extends in principle to more than three qubits. The extension of the entanglement analysis we will

[c] We mean here invariant subspaces of $U^{S_n}(2^n)$, that is, the subgroup of $U(2^n)$ invariant under the permutation group of n objects.

present will require the use of generalizations of measures of distributed entanglement such as the one obtained in [16].

2 Decomposition into Invariant Subspaces

Consider the Lie algebra $su^{S_3}(8)$, that is, the subalgebra of $su(8)$ of matrices which commute with the permutation matrices in S_3. For $su(8)$, we consider the standard representation on $C^8 \simeq \mathcal{H}$ and therefore the matrices in $su^{S_3}(8)$ are also 8×8. In appropriate coordinates, such matrices take a block diagonal form with blocks of dimension 4×4, 2×2 and 2×2, which correspond to invariant subspaces of $\mathcal{H}$ of dimensions 4, 2 and 2, respectively [1]. Such subspaces correspond to irreducible representations of $su(2)$ of dimensions 4, 2 and 2 respectively (the ones of dimension 2 being isomorphic representations). To obtain a basis for such subspaces, in terms of the computational basis $\{|jkl\rangle\ , j,k,l=0,1\}$ one may apply standard methods of the quantum theory of angular momentum (see, e.g., [9], [18]) which overlap with representation theory and the theory of Young tableau and representations of the symmetric group (see, e.g., [8]). For instance, the Clebsch-Gordan coefficients described in [9] (in the table on pg. 375), give one possible change of coordinates to obtain the bases of such invariant subspaces. Another method is given by the use of Young symmetrizers which was reviewed in [6] in the form that uses *Hermitian* Young symmetrizers as described in [2]. Such a method is based on the Schur-Weyl duality, a decomposition of the space $V^{\otimes n}$ into the direct sum of spaces $\hat{S}_\lambda \otimes V_\lambda$ where the V_λ's are irreducible modules of the general linear group $Gl(\dim(V))$ and $\hat{S}_\lambda$ are irreducible modules of the (finite) symmetric group S_n. The Young symmetrizers are a set of projections onto the various invariant subspaces which satisfy certain requirements (completeness, orthogonality (cf., e.g., [6])) but are not uniquely determined. Therefore, the resulting decomposition in invariant subspaces is not unique and it is of interest to find *all* the possible decompositions.

In order to achieve all the possible decompositions, we will use in this section a technique that was described in [5] (see Chapter 4 section 4.3.4). One considers the *commutant* $\mathcal{C}$ of $su^{S_3}(8)$ in $u(8)$ which is a reductive Lie algebra and therefore it admits Cartan subalgebras, i.e., a maximal Abelian subalgebra. The main observation is that, if $W \oplus V_1 \oplus V_2$ (with $\dim(W)=4$ and $\dim(V_1)=\dim(V_2)=2$) is a decomposition in invariant subspaces for $su^{S_3}(8)$, then $\mathcal{C}$ admits a Cartan subalgebra which, in the appropriate coordinates, has a basis given by $A_1 := \texttt{diag}(i\mathbf{1}_4, \mathbf{0}_2, \mathbf{0}_2)$, $A_2 := \texttt{diag}(\mathbf{0}_4, i\mathbf{1}_2, \mathbf{0}_2)$, $A_3 := \texttt{diag}(\mathbf{0}_4, \mathbf{0}_2, i\mathbf{1}_2)$, where $\mathbf{1}_r$ ($\mathbf{0}_r$) is the $r \times r$ identity (zero) matrix, and `diag` refers to block diagonal matrices.[d]Possible decompositions are therefore in correspondence with Cartan subalgebras of the commutant $\mathcal{C}$. Thus, a method to obtain all the possible decompositions is the following algorithm.

1. Compute the commutant $\mathcal{C}$ of $su^{S_3}(8)$ in $u(8)$.

2. Find all possible Cartan subalgebras of $\mathcal{C}$ which (in this case) all have dimension 3. The following steps refers to the a Cartan subalgebra $\mathcal{A}$. In order to deal with Hermitian matrices rather than skew-Hermitian ones, we consider $i\mathcal{A}$.

3. Take a basis of $i\mathcal{A}$ and (orthogonally) diagonalize its elements simultaneously (this is possible since these are mutually commuting Hermitian matrices).

[d]Sometime we omit the index r, in $\mathbf{1}_r$ or $\mathbf{0}_r$ when the dimension is obvious from the context.

4. Place the elements on the diagonal in three row vectors so as to form a 3×8 matrix, which we shall denote by M.

5. Perform a Gaussian row reduction algorithm to place the matrix in a Reduced Row Echelon Form (see, e.g., [13]). This corresponds to taking linear combinations of the matrices in the basis of $i\mathcal{A}$ so as to obtain a new basis of elements which only have eigenvalues 1 and 0. Call these elements (in the original coordinates) $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ with $\tilde{A}_1$ having eigenvalue 1 with multiplicity 4, $\tilde{A}_2$ and $\tilde{A}_3$ having eigenvalue 1 with multiplicity 2

6. W is the eigenspace of $\tilde{A}_1$ corresponding to eigenvalue 1. V_1 and V_2 are the eigenspaces of $\tilde{A}_2$ and $\tilde{A}_3$, respectively, corresponding to eigenvalue 1. Notice that once we know W and V_1, the subspace V_2 is simply the orthogonal complement of $W \oplus V_1$.

Let us carry out the above program for our example. The commutant $\mathcal{C}$ is found by solving the linear system of equations $[\mathcal{C}, B_j] = 0$, where $\{B_j\}$ is a basis of $su^{S_3}(8)$. In fact, since the matrices iH_x, iH_y and iH_{zz}, with (cf. (2))

$$H_{x,y,z} := \sigma_{x,y,z} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_{x,y,z} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma_{x,y,z}, \tag{5}$$

$$H_{zz} := \sigma_z \otimes \sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z \otimes \sigma_z + \sigma_z \otimes \mathbf{1} \otimes \sigma_z, \tag{6}$$

generate all of $su^{S_3}(8)$ [1], it is enough to solve

$$[\mathcal{C}, H_x] = 0, \qquad [\mathcal{C}, H_y] = 0, \qquad [\mathcal{C}, H_{zz}] = 0.$$

This computation, which was done in [5], leads to the basis $\{E_1, E_2, E_3, E_4, E_5\}$ for $\mathcal{C}$, with

$$iE_1 := \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1},$$

$$iE_2 := \sigma_x \otimes \mathbf{1} \otimes \sigma_x + \sigma_y \otimes \mathbf{1} \otimes \sigma_y + \sigma_z \otimes \mathbf{1} \otimes \sigma_z,$$

$$iE_3 = \sigma_x \otimes \sigma_x \otimes \mathbf{1} + \sigma_y \otimes \sigma_y \otimes \mathbf{1} + \sigma_z \otimes \sigma_z \otimes \mathbf{1},$$

$$iE_4 = \mathbf{1} \otimes \sigma_x \otimes \sigma_x + \mathbf{1} \otimes \sigma_y \otimes \sigma_y + \mathbf{1} \otimes \sigma_z \otimes \sigma_z,$$

$$iE_5 = \sigma_x \otimes (\sigma_y \otimes \sigma_z - \sigma_z \otimes \sigma_y) + \sigma_y \otimes (\sigma_z \otimes \sigma_x - \sigma_x \otimes \sigma_z) + \sigma_z \otimes (\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x).$$

An analysis of the Lie algebra $\mathcal{C}$ shows that it is the direct sum of one two dimensional Abelian Lie algebra spanned by E_1 and $E_1 + E_2 + E_3$ and a three dimensional Lie algebra isomorphic to $su(2)$ and spanned by $\{E_5, (E_2 - E_3), (E_2 - E_4)\}$. The Lie algebra $su(2)$ has a one dimensional Cartan subalgebra which may be spanned by any non zero element. Therefore a general Cartan subalgebra $\mathcal{A}$ of $\mathcal{C}$ is such that an orthogonal basis of $i\mathcal{A}$ is given by $\{F_1, F_2, F_3\}$, with

$$F_1 := iE_1, \qquad F_2 := i(E_2 + E_3 + E_4), \qquad F_3 := iaE_5 + ib(E_2 - E_3) + ic(E_2 - E_4), \tag{7}$$

for any, not simultaneously zero, real parameters (a, b, c).

We now proceed to step 3 of the above algorithm. Here and in the following we shall use the standard notation, $\vec{e}_j$, $j = 1, ..., 8$, for the elements of the standard basis in C^8, $\vec{e}_j$ being the vector of all zeros except in the j-th position occupied by 1. The matrix F_1 is just the identity matrix which is diagonal in every basis. The matrix F_2 has eigenvalue $\lambda_1 = 3$ with eigenspace Q_3 spanned by $\{\vec{e}_1, \vec{e}_2 + \vec{e}_3 + \vec{e}_5, \vec{e}_4 + \vec{e}_6 + \vec{e}_7, \vec{e}_8\}$ and eigenvalue $\lambda_2 = -3$ with eigenspace Q_{-3} spanned by $\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4\}$, with $\vec{f}_1 := \vec{e}_3 - \vec{e}_2$, $\vec{f}_2 := \vec{e}_2 + \vec{e}_3 - 2\vec{e}_5$, $\vec{f}_3 := \vec{e}_6 - \vec{e}_4$, $\vec{f}_4 := -2\vec{e}_7 + \vec{e}_6 + \vec{e}_4$. Consider now F_3 acting on Q_3 and Q_{-3}. Direct verification shows that F_3 is zero on Q_3. On Q_{-3} we have

$$F_3\vec{f}_1 = 3c\vec{f}_1 + (2ia + 2b + c)\vec{f}_2,$$

$$F_3\vec{f}_2 = (3c + 6b + 6ia)\vec{f}_1 - 3c\vec{f}_2,$$

$$F_3\vec{f}_3 = 3b\vec{f}_3 + (b + 2c + 2ia)\vec{f}_4,$$

$$F_3\vec{f}_4 = (3b + 6c - 6ia)\vec{f}_3 - 3b\vec{f}_4.$$

This shows that the subspace Q_{-3} splits into two invariant subspaces for F_3 spanned by $\{\vec{f}_1, \vec{f}_2\}$ and $\{\vec{f}_3, \vec{f}_4\}$ respectively. Calculating the spectrum of F_3 on such subspaces we find that F_3 has eigenvalues $\pm\lambda$ on both subspaces where

$$\lambda := 2\sqrt{3c^2 + 3cb + 3b^2 + 3a^2}. \tag{8}$$

Notice that λ is never zero otherwise we would have $a = b = c = 0$ which we have excluded. Listing the eigenvalues of F_1, F_2 and F_3 and constructing the matrix M of the above algorithm, we have

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & \lambda & \lambda & -\lambda & -\lambda \end{pmatrix}.$$

Row reduction to transform this matrix in its Reduced Row Echelon Form, which is

$$RREF(M) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

corresponds to multiplication of M on the left by the matrix

$$R := \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{\lambda} \\ \frac{1}{2} & -\frac{1}{6} & -\frac{1}{\lambda} \end{pmatrix}.$$

The rows of the matrix R give the coefficients for the linear combinations of $\{F_1, F_2, F_3\}$ whose eigenspaces are the sought vector spaces. In particular consider

$$\Pi_1 := \frac{1}{2}\left(F_1 + \frac{1}{3}F_2\right), \tag{9}$$

$$\Pi_2 := \frac{1}{2}\left(\frac{1}{2}F_1 - \frac{1}{6}F_2 + \frac{1}{\lambda}F_3\right), \tag{10}$$

$$\Pi_3 := \frac{1}{2}\left(\frac{1}{2}F_1 - \frac{1}{6}F_2 - \frac{1}{\lambda}F_3\right). \tag{11}$$

Independently of the values of a, b, and c, these projections form a complete set of, symmetric, mutually orthogonal idempotents, (which are also called *generalized Young symmetrizers* [6]). In fact, one can verify directly that

$$\Pi_1 + \Pi_2 + \Pi_3 = \mathbf{1}, \tag{12}$$

$$\Pi_j \Pi_k = \delta_{j,k} \Pi_j. \tag{13}$$

Here $\delta_{j,k} = 0$ if $j \neq k$ and $\delta_{j,j} = 1$.The eigenspaces corresponding to the eigenvalue 1 of these matrices are the spaces W (for Π_1), V_1 (for Π_2), V_2 (for Π_3). They coincide with the images of these matrices. The result is formally given in the following theorem.

Theorem 1 *Every decomposition of $\mathcal{H} := (C^2)^{\otimes 3} = W \oplus V_1 \oplus V_2$ in invariant subspaces for $su^{S_3}(8)$ ($SU^{S_3}(8)$) or $u^{S_3}(8)$ ($U^{S_3}(8)$) corresponds to a triple $(a, b, c) \neq (0, 0, 0)$. An orthogonal basis of W is given by*

$$\mathcal{W} := \{\vec{e}_1, \vec{e}_2 + \vec{e}_3 + \vec{e}_5, \vec{e}_4 + \vec{e}_6 + \vec{e}_7, \vec{e}_8\}, \tag{14}$$

which coincides with the basis $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ in (3) and uniquely determines W, the symmetric sector. An orthogonal basis of V_1 is given by $\{|v_1\rangle, |w_1\rangle\}$ with

$$\begin{array}{rcl} |v_1\rangle &=& x_2|001\rangle + x_3|010\rangle + x_5|100\rangle \\ |w_1\rangle &=& x_4|011\rangle + x_6|101\rangle + x_7|110\rangle \end{array} \tag{15}$$

with $x_2 = x_7 = \frac{1}{\sqrt{2}}$, $x_3 = x_6 = -\frac{1}{\sqrt{2}}$, and $x_5 = x_4 = 0$ if $a = 0$, $c = -2b$ and $b < 0$, with $x_2 = x_7 = -\lambda + 6ia + 6b + 6c$, $x_3 = x_6 := -\lambda - 6ia - 6b$, $x_5 = x_4 = 2\lambda - 6c$, in all the other cases. An orthogonal basis of V_2 is given by

$$\begin{array}{rcl} |v_2\rangle &=& y_2|001\rangle + y_3|010\rangle + y_5|100\rangle \\ |w_2\rangle &=& y_4|011\rangle + y_6|101\rangle + y_7|110\rangle \end{array} \tag{16}$$

with $y_2 = y_7 = \frac{1}{\sqrt{2}}$, $y_3 = y_6 = -\frac{1}{\sqrt{2}}$, and $y_5 = y_4 = 0$ if $a = 0$, $c = -2b$ and $b > 0$, with $y_2 = y_7 = -\lambda - 6ia - 6b - 6c$, $y_3 = y_6 := -\lambda + 6ia + 6b$, $y_5 = y_4 = 2\lambda + 6c$, in all the other cases.

A direct computation using (8) shows that the decomposition is orthogonal. Furthermore we also have

$$x_2 + x_3 + x_5 = 0, \qquad x_4 + x_6 + x_7 = 0; \tag{17}$$

$$y_2 + y_3 + y_5 = 0, \qquad y_4 + y_6 + y_7 = 0. \tag{18}$$

Proof. The theorem follows by explicitly writing the matrices Π_1, Π_2 and Π_3 in (9), (10) (11).

The matrix Π_1 is the following:

$$\Pi_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the orthogonal basis for the subspace W, is the one given by equation (14).

The work for the matrices Π_2 and Π_3 which depend on the parameters a, b, and c, requires some extra considerations. Let us consider the discussion for Π_2 and V_1. The matrix Π_2 in (10) is $\Pi_2 := \frac{1}{12\lambda}(\Pi_{2,1}, \Pi_{2,2})$ with

$$\Pi_{2,1} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4\lambda - 12b & -2\lambda - 12ia - 12c & 0 \\ 0 & -2\lambda + 12ia - 12c & 4\lambda + 12b + 12c & 0 \\ 0 & 0 & 0 & 4\lambda - 12c \\ 0 & -2\lambda - 12ia + 12b + 12c & -2\lambda + 12ia - 12b & 0 \\ 0 & 0 & 0 & -2\lambda - 12ia - 12b \\ 0 & 0 & 0 & -2\lambda + 12ia + 12b + 12c \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_{2,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2\lambda + 12ia + 12b + 12c & 0 & 0 & 0 \\ -2\lambda - 12ia - 12b & 0 & 0 & 0 \\ 0 & -2\lambda + 12ia - 12b & -2\lambda - 12ia + 12b + 12c & 0 \\ 4\lambda - 12c & 0 & 0 & 0 \\ 0 & 4\lambda + 12b + 12c & -2\lambda + 12ia - 12c & 0 \\ 0 & -2\lambda - 12ia - 12c & 4\lambda - 12b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Considering the columns $2, 3$, and 5 of Π_2, one sees that the sum of second, third and fifth row is zero. Therefore at the most two of these columns are linearly independent. In fact, using the definition of λ in (8), it follows that only one column is linearly independent.

Taking the 5^{th} column divided by two, one obtains the first element in the (orthogonal) basis of V_1, when $a \neq 0$ or $c \neq -2b$, or $b \geq 0$. If $a = 0$, $c = -2b$ and $b < 0$, then the 5^{th} column is zero, but the second column is proportional to the vector $\frac{1}{\sqrt{2}}|001\rangle - \frac{1}{\sqrt{2}}|010\rangle$.

Analogously one obtains the second vector of the basis, considering the 4^{th}, 6^{th} and 7^{th} column of Π_2.

The discussion of Π_3 and V_2 is analogous. In fact, an explicit calculation shows that Π_3 can be obtained from Π_2 with the exchanges $a \leftrightarrow -a$, $b \leftrightarrow -b$, $c \leftrightarrow -c$. Alternatively one obtains V_2 as the orthogonal complement of the direct sum of W and V_1.

□

The above decomposition includes, as special cases, decompositions found in the standard quantum physics literature. For example, the two dimensional space obtained with the Young symmetrizers in [6], which can also be obtained with the recursive use of the Clebsch-Gordan coefficients [9], is spanned by

$$\hat{\psi}_1 := \frac{1}{\sqrt{2}}|010\rangle - \frac{1}{\sqrt{2}}|100\rangle := \frac{1}{\sqrt{2}}\vec{e}_3 - \frac{1}{\sqrt{2}}\vec{e}_5$$

$$\hat{\psi}_2 := -\frac{1}{\sqrt{2}}|011\rangle + \frac{1}{\sqrt{2}}|101\rangle = -\frac{1}{\sqrt{2}}\vec{e}_4 + \frac{1}{\sqrt{2}}\vec{e}_6,$$

and it is obtained as a special case of V_1. In fact by choosing $a = 0$, $b = \frac{1}{9\sqrt{2}}$, and $c = -\frac{1}{18\sqrt{2}}$, which give $\lambda = \frac{1}{3\sqrt{2}}$, we have that $|v_1\rangle = -\hat{\psi}_1$ and $|w_1\rangle = -\hat{\psi}_2$

We remark that all the triple $(a, b, c) \neq (0, 0, 0)$ are possible although the decompositions are not in one to one correspondence with the set of triples. Different triples can give the same decomposition (for instance all the ones with $a = 0$, $c = -2b$, $b < 0$).

3 Measures of Entanglement for General Three Qubits States

For a general multi-partite quantum system, a measure of entanglement is a nonnegative real function on the space of density matrices which satisfies certain axioms. In particular it does not increase under local operations and classical communication (LOCC), it is zero on separable states (that is, statistical mixtures of product states), it is unchanged by local unitary operations, and it is usually normalized to one (cf., e.g., [3] and [11] for a detailed introduction to entanglement measures). For the case of two qubits A and B, a very common measure is the *concurrence* [20] whose square is called the 2-tangle, τ_{AB}. This can be defined from the density matrix ρ_{AB}, by calculating the spectrum of $\rho_{AB}\sigma_y \otimes \sigma_y \rho^*_{AB}\sigma_y \otimes \sigma_y$,[e] which can be shown to be made of real and nonnegative values $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq \lambda_4^2$, and τ_{AB} is defined as

$$\tau_{AB} = \left[\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}\right]^2. \tag{19}$$

Consider now a pure state for a system of three qubits, A, B, and C. One can consider, after tracing out C, the entanglement between A and B, τ_{AB}, and analogously τ_{AC} and τ_{BC}. In this case a monogamy relation holds [19] [21]: If A is fully entangled with B, that is, $\tau_{AB} = 1$, then we must have $\tau_{AC} = 0$, that is, the state ρ_{AC} is separable (where ρ_{AC} is the partial trace with respect to subsystem B). In fact, a more refined inequality holds [4]. Consider a pure state ρ and consider the system as a bipartite system $A - (BC)$. Even though BC is four dimensional, it follows from the Schmidt decomposition (see, e.g., [15] pg. 109) that only two (orthogonal) directions are necessary to express the full state. Therefore, we can treat effectively (BC) as a two level system and define the entanglement $\tau_{A(BC)}$ between A and (BC). Then one has the following inequality which was one of the main results of [4]

$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)}. \tag{20}$$

The difference between $\tau_{A(BC)}$ and $\tau_{AB} + \tau_{AC}$ is by definition, the *distributed entanglement* or $3-$tangle, which we denote simply by τ, that is, the amount of entanglement not due to pairwise entanglement between the quantum bits. Explicit formulas were given in [4] for $\tau_{A(BC)}$ and τ. We report them below because we shall use them in our analysis. Let ρ_A be the partial trace with respect to the subsystem (BC).

$$\tau_{A(BC)} = 4\det(\rho_A); \tag{21}$$

$$\tau = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} = 4\left|t_{000}^2 t_{111}^2 + t_{001}^2 t_{110}^2 + t_{010}^2 t_{101}^2 + t_{100}^2 t_{011}^2 - 2d_1 + 4d_2\right|, \tag{22}$$

with

$$d_1 := t_{000}t_{111}t_{011}t_{100} + t_{000}t_{111}t_{101}t_{010} + t_{000}t_{111}t_{110}t_{001} +$$
$$t_{011}t_{100}t_{101}t_{010} + t_{011}t_{100}t_{110}t_{001} + t_{101}t_{010}t_{110}t_{001},$$
$$d_2 := t_{000}t_{110}t_{101}t_{011} + t_{111}t_{001}t_{010}t_{100},$$

for a state

$$|\psi\rangle = \sum_{ijk} t_{ijk}|ijk\rangle,$$

with $i, j, k = 0, 1$.

[e]Given a matrix ρ or a complex constant c, we always denote by ρ^* and by c^* the complex conjugate.

Relations such as (20) are known as *monogamy relations* in quantum information theory and they are valid also for others, but not all, measures of entanglement [23], thus allowing to define a distributed entanglement. We have chosen the (distributed) tangle as our reference measure of entanglement not only because it satisfies the monogamy inequality (20) but also because explicit, relatively simple, expressions exist for its calculation, expressions we will elaborate upon in the following sections.

4 States in the Symmetric Sector and their Entanglement

For any state in the symmetric sector, because of symmetry, we have $\rho_A = \rho_B = \rho_C$, so for these states, $\tau_{AB} = \tau_{AC} = \tau_{BC}$ and therefore we have, using (21) (22),

$$\tau_{AB} = \tau_{AC} = \tau_{BC} = \frac{4\det(\rho_A) - \tau}{2}. \tag{23}$$

In order to express the entanglement measures τ and $\tau_{AB} = \tau_{AC} = \tau_{BC}$ in a compact fashion, we introduce an extra piece of notation. Define

$$X_2 := c_0 c_2 - c_1^2, \qquad X_3 := c_0 c_3 - c_1 c_2, \qquad X_4 := c_1 c_3 - c_2^2. \tag{24}$$

The quantities X_2, X_3 and X_4 give a quick test of separability for states in the symmetric sector as described in the following proposition.

Proposition 4.1 A state ψ (4) in the symmetric sector is separable if and only if

$$X_2 = X_3 = X_4 = 0. \tag{25}$$

In this case, ψ is a symmetric product state of the form

$$\psi = \phi \otimes \phi \otimes \phi, \tag{26}$$

with ϕ a one qubit state.

Proof. Assume that ψ in (4) is a product state, i.e.,

$$\psi = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) \otimes (\gamma_0|0\rangle + \gamma_1|1\rangle).$$

Expanding and comparing with (4), we have

$$c_0 = \alpha_0\beta_0\gamma_0,$$

$$c_1 = \alpha_1\beta_0\gamma_0 = \alpha_0\beta_1\gamma_0 = \alpha_0\beta_0\gamma_1,$$
$$c_2 = \alpha_1\beta_1\gamma_0 = \alpha_1\beta_0\gamma_1 = \alpha_0\beta_1\gamma_1,$$
$$c_3 = \alpha_1\beta_1\gamma_1.$$

Using these in (24) one verifies (25). For example, for X_2 we have

$$c_0 c_2 = \alpha_0\beta_0\gamma_0\alpha_1\beta_1\gamma_0 = (\alpha_1\beta_0\gamma_0)(\alpha_0\beta_1\gamma_0) = c_1^2.$$

Viceversa, assume (25) is verified and consider the state (4). If $c_0 = 0$, then, from (25), (24), it follows that $c_1 = 0$ and $c_2 = 0$. Therefore the state coincides with $|111\rangle$ which is separable and of the symmetric form (26). If $c_0 \neq 0$ we can assume $c_0 = 1$, without loss of generality keeping the state not normalized. Conditions (24) (25) give $c_2 = c_1^2$, $c_3 = c_1^3$. Therefore, the state ψ in (4) is of the form (26) with $\phi = |0\rangle + c_1|1\rangle$. □

With the notation (24), the entanglement measures τ and $\tau_{AB} = \tau_{AC} = \tau_{BC}$ take a compact form as described in the following two propositions.

Proposition 4.2 The distributed entanglement τ on the symmetric sector is given by

$$\tau = 4|X_3^2 - 4X_2X_4|. \tag{27}$$

Proof. Applying formula (22) we obtain

$$\tau = 4|c_0^2c_3^2 - 3c_1^2c_2^2 - 6c_0c_1c_2c_3 + 4c_0c_2^3 + 4c_3c_1^3|. \tag{28}$$

Direct verification using formulas (24) in (27) shows that τ in (27) coincides with (28). □

Proposition 4.3 The pairwise entanglement $\tau_{AB} = \tau_{AC} = \tau_{BC}$ is given by

$$\tau_{AB} = \tau_{AC} = \tau_{BC} = 2\left(\det(\rho_A) - |X_3^2 - 4X_2X_4|\right), \tag{29}$$

where

$$\det(\rho_A) = |X_3|^2 + 2|X_2|^2 + 2|X_4|^2. \tag{30}$$

Therefore the expression for the pairwise entanglement is

$$\tau_{AB} = \tau_{AC} = \tau_{BC} = 2\left(|X_3|^2 + 2|X_2|^2 + 2|X_4|^2 - |X_3^2 - 4X_2X_4|\right). \tag{31}$$

Proof. We explicitly write the state ψ in (4) as $\psi = (c_0, c_1, c_1, c_2, c_1, c_2, c_2, c_3)^T$ and the associated density matrix $\rho = \psi\psi^\dagger$. By taking the partial trace with respect to B and C, we obtain,

$$\rho_A := \begin{pmatrix} |c_0|^2 + 2|c_1|^2 + |c_2|^2 & c_0c_1^* + 2c_1c_2^* + c_2c_3^* \\ c_0^*c_1 + 2c_1^*c_2 + c_2^*c_3 & |c_1|^2 + 2|c_2|^2 + |c_3|^2 \end{pmatrix}, \tag{32}$$

and, after simplifications,

$$\det(\rho_A) = 2|c_0|^2|c_2|^2 + |c_0|^2|c_3|^2 + 2|c_1|^4 + 2|c_1|^2|c_3|^2 + |c_2|^2|c_1|^2 + 2|c_2|^4$$
$$-2c_0^*c_2^*c_1^2 - c_0^*c_1c_2c_3^* - 2(c_1^*)^2c_0c_2 - 2c_1^*c_2^2c_3^* - c_0c_1^*c_2^*c_3 - 2c_1c_3(c_2^*)^2.$$

By replacing the expressions of X_2, X_3, X_4 in (24) in the right hand side of (30) one verifies that it coincides with the above expression of $\det(\rho_A)$. □

From (31) we obtain

$$\frac{\tau_{AB}}{2} \geq 2|X_2|^2 + 2|X_4|^2 - 4|X_2X_4| = 2\left(|X_2| - |X_4|\right)^2 \geq 0.$$

To have equality, that is, the pairwise entanglement equal to zero, both inequalities have to hold with the equal sign. We must have

$$|X_2| = |X_4|, \quad |X_3^2 - 4X_2X_4| = |X_3|^2 + 4|X_2||X_4| = |X_3|^2 + 4|X_2|^2 = |X_3|^2 + 4|X_4|^2.$$

The following also follows from Proposition 4.1.

Proposition 4.4 The only states in the symmetric sector that have both distributed and pairwise entanglements equal to zero are the separable states.

Pairwise entanglement τ_{AB} and distributed entanglement τ are local invariants, that is, they are functions invariant under local unitary transformations, which, in the symmetric case, are taken symmetric, i.e., of the form $X \otimes X \otimes X$, with $X \in U(2)$. A complete set of local invariants for general three qubits states and general local unitary transformations, is known. For symmetric qubit states a complete set of invariants can be obtained using the Majorana polynomial representation of symmetric states [17]. We briefly review this.[f] Given a general (not necessarily symmetric) product state $\psi_1 \otimes \psi_2 \otimes \psi_3$, with $\psi_j := \alpha_j|0\rangle + \beta_j|1\rangle$, $j = 1, 2, 3$, one can obtain a symmetric state of the form (4) as $A\Pi\psi_1 \otimes \psi_2 \otimes \psi_3$, where

$$\Pi := \frac{1}{3!} \sum_{P \in S_3} P \tag{33}$$

is the total symmetrizer, and A is a normalization factor. In particular, direct calculation shows, with the definitions (3),

$$\Pi(\psi_1 \otimes \psi_2 \otimes \psi_3) = \alpha_1\alpha_2\alpha_3\phi_0 + \left(\frac{\alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3}{3}\right)\phi_1 \tag{34}$$

$$+ \left(\frac{\alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\beta_3}{3}\right)\phi_2 + \beta_1\beta_2\beta_3\phi_3.$$

Viceversa, given a symmetric state (4), one considers the associated Majorana polynomial,

$$p_M(x) = c_0x^3 + 3c_1x^2 + 3c_2x + c_3,$$

which, by calculating the zeros, and up to a common factor, can be written as

$$p_M(x) = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)(\alpha_3 x + \beta_3).$$

By choosing $\psi_j := \alpha_j|0\rangle + \beta_j|1\rangle$ and using (34), we see that the resulting symmetric state $\Pi\psi_1 \otimes \psi_2 \otimes \psi_3$ is given by (4). Therefore every symmetric state is in correspondence with a not ordered triple of one qubit states ψ_1, ψ_2, ψ_3. Since each qubit state is in correspondence with a point on the Bloch sphere (see, e.g., [15]) a symmetric state is in correspondence with three not ordered vectors from the origin to the Bloch sphere in R^3. Furthermore, since for $X \in U(2)$, we have

$$X \otimes X \otimes X\Pi(\psi_1 \otimes \psi_2 \otimes \psi_3) = \Pi(X\psi_1 \otimes X\psi_2 \otimes X\psi_3),$$

applying a symmetric local unitary operation corresponds to a *simultaneous* rotation of the three Bloch vectors of the three one qubit states ψ_1, ψ_2 and ψ_3. Therefore the *angles* between the Bloch vectors give a complete set of invariants under local symmetric unitary operations.

We remark here that, using this representation of symmetric states, it is possible to assume that the states (4) can be written, after local symmetric unitary operations, in special forms. In particular, after a common rotation, it is possible to assume that one of the Bloch vectors corresponding to $\{\psi_1, \psi_2, \psi_3\}$ is in a special position, for example along the z-axis, while the remaining two can be rotated arbitrarily around the the first one. One possible special form

[f] We only discuss the Majorana polynomial representation in the three qubits case. For a general treatment, we refer to [14] and references therein.

to write the state (4) after a local unitary transformation is the one with $c_3 = 0$ and c_0 and c_2 real (or having the same phase, recall that states are defined up to a phase factor). In order to achieve this, take $\psi_1 \otimes \psi_2 \otimes \psi_3$ and choose $X \in U(2)$ so that $X\psi_1 = |0\rangle$ (up to a phase factor). Therefore we have

$$X \otimes X \otimes X\psi_1 \otimes \psi_2 \otimes \psi_3 = |0\rangle \otimes \left(\cos(\theta_1)|0\rangle + \sin(\theta_1)e^{i\chi_1}|1\rangle\right) \otimes \left(\cos(\theta_2)|0\rangle + \sin(\theta_2)e^{i\chi_2}|1\rangle\right),$$

for real parameters θ_1, θ_2, χ_1, χ_2. Now we can apply $Y \otimes Y \otimes Y$, with $Y = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix}$. We obtain

$$\begin{aligned}
&(Y \otimes Y \otimes Y)X \otimes X \otimes X\psi_1 \otimes \psi_2 \otimes \psi_3 \\
&= e^{i\chi}|0\rangle \otimes \left(\cos(\theta_1)e^{i\chi}|0\rangle + \sin(\theta_1)e^{i(\chi_1-\chi)}|1\rangle\right) \otimes \left(\cos(\theta_2)e^{i\chi}|0\rangle + \sin(\theta_2)e^{i(\chi_2-\chi)}|1\rangle\right) \\
&= |0\rangle \otimes \left(\cos(\theta_1)|0\rangle + \sin(\theta_1)e^{i(\chi_1-2\chi)}|1\rangle\right) \otimes \left(\cos(\theta_2)|0\rangle + \sin(\theta_2)e^{i(\chi_2-2\chi)}|1\rangle\right).
\end{aligned}$$

The choice $\chi := \frac{\chi_1+\chi_2}{4}$, gives, with $\eta = \frac{\chi_1-\chi_2}{2}$, the form

$$\psi_{prodcan} := |0\rangle \otimes \left(\cos(\theta_1)|0\rangle + \sin(\theta_1)e^{i\eta}|1\rangle\right) \otimes \left(\cos(\theta_2)|0\rangle + \sin(\theta_2)e^{-i\eta}|1\rangle\right). \tag{35}$$

Applying the total symmetrizer Π in (33) to $\psi_{prodcan}$ in (35), one obtains a symmetric state (4) with $c_3 = 0$ and c_0 and c_2 real. We remark that 3 is the minimum number of parameters necessary to identify equivalence classes of (unitary) locally equivalent states since states can be normalized and identified up to a common phase factor and therefore (in the symmetric sector) by 6 parameters and $SU(2)$ has dimension 3.

5 States in the Two Dimensional Invariant Subspaces and their Entanglement

We now consider the invariant subspaces of dimension two: V_1 and V_2 described in section 2. Since the orthogonal basis of V_2, given in equation (16), can be obtained from the orthogonal basis of V_1, given in equation (15), exchanging $a \leftrightarrow -a$, $b \leftrightarrow -b$, $c \leftrightarrow -c$, and (a, b, c) are free parameters (not all zero) we can consider without loss of generality only the subspace V_1. We shall calculate the pairwise entanglements and the distributed entanglement. We remark that since these states are in general not invariant under permutation (as opposed to states in the symmetric sector treated in the previous section) there is no a priori reason why τ_{AB} should be equal to τ_{AC}.

In what follows we assume that we have normalized the basis vectors $|v_i\rangle$, and $|w_i\rangle$ in equations (15) and (16). We denote by $\hat{x}_i$ and $\hat{y}_i$ the normalized coordinates corresponding to x_j and y_j, and, we still denote by $|v_i\rangle$, and $|w_i\rangle$ the basis vectors. Notice that in the special case, when $a = 0$, $c = -2b$ and $b < 0$, the two vectors $|v_1\rangle$, and $|w_1\rangle$ are already normalized (as well as the vectors $|v_2\rangle$, and $|w_2\rangle$, when $a = 0$, $c = -2b$ and $b > 0$). In the other cases, using the definitions of x_i and y_i, in terms of the constants a, b, c, and λ, given in Theorem 1, and the definition of λ given in equation (8), we have:

$$\begin{aligned}
x_2x_2^* + x_3x_3^* + x_5x_5^* &= (-\lambda + 6b + 6c)^2 + 36a^2 + (\lambda + 6b)^2 + 36a^2 + (2\lambda - 6c)^2 \\
&= 6\lambda^2 + 72b^2 + 72a^2 + 72c^2 + 72bc - 36\lambda c = 12\lambda^2 - 36\lambda c = 12\lambda(\lambda - 3c)
\end{aligned}$$

Thus, we have:

$$\hat{x}_i = \frac{x_i}{\sqrt{12\lambda(\lambda - 3c)}}. \tag{36}$$

Analogously, it holds:

$$\hat{y}_i = \frac{y_i}{\sqrt{12\lambda(\lambda + 3c)}}. \tag{37}$$

In order to simplify the calculation of the entanglement measures, it is convenient to anticipate a result on dynamics (treatment of dynamics will be done in the next section). We recall that we call *local (special) unitary symmetric operations* operations of the type $X \otimes X \otimes X$ with $X \in U(2)$ ($X \in SU(2)$).

Proposition 5.1 Given two states $|\psi_1\rangle$, $|\psi_2\rangle$ in the subspace V_1 (same for V_2) it is always possible to go from $|\psi_1\rangle$ to $|\psi_2\rangle$, using local operations.

This result is a direct consequence of Schur-Weyl duality and the Lie theoretic controllability criteria for quantum systems, in particular it is a consequence of the fact that $su^{S_3}(8)$ acts as $u(m)$ on each invariant subspace of dimension m (cf. [6]). We give a direct proof.

Proof. The Lie algebra corresponding to the Lie group of local symmetric special unitary matrices is spanned by the matrices $iH_{x,y,z}$ defined in (5). This Lie algebra leaves V_1 invariant. It is in fact the standard representation of $su(2)$. This can be verified directly. Explicit computation using (15) shows that $H_z|v_1\rangle = |v_1\rangle$, $H_z|w_1\rangle = -|w_1\rangle$.

Calculating $iH_x|v_1\rangle$ we get

$$iH_x|v_1\rangle = -i\hat{x}_3|101\rangle - i\hat{x}_2|110\rangle - i\hat{x}_5|011\rangle = -i|w_1\rangle$$

Similarly, we get that $iH_x|w_1\rangle = -i|v_1\rangle$. Thus on the orthonormal basis $\{|v_1\rangle, |w_1\rangle\}$ H_z and $-H_x$ act as σ_z and σ_x on the basis $\{|0\rangle, |1\rangle\}$ and therefore iH_z and $-iH_x$ generate the Lie algebra $su(2)$. Since the corresponding Lie group, $SU(2)$, is transitive on the complex sphere, the symmetric local transformations $X \otimes X \otimes X$, $X \in SU(2)$, are able to transfer any state to any other state (cf. [12]). □

We will now compute the entanglement measures for the states in the invariant subspaces of dimension 2. Since these quantities remain unchanged by using local operations, and all the states in V_1 (or V_2) can be reached using local operations starting from an arbitrary state, as proved in the previous proposition, it is enough to compute the measures for a particular state.

First we will see that the distributed entanglement τ is always zero.

Proposition 5.2 Let $|\psi\rangle \in V_1$ (or $|\psi\rangle \in V_2$) then $\tau = 0$

Proof. Direct calculation using (22) shows that $\tau = 0$ for the first basis vector $|v_1\rangle = \hat{x}_2|001\rangle + \hat{x}_3|010\rangle + \hat{x}_5|100\rangle$ in the definition (15). Therefore the result follows using Proposition 5.1.□

To calculate $\tau_{A(BC)}$, τ_{AB} and τ_{AC}, we recall that since $\tau = 0$ from (20)

$$\tau_{A(BC)} = \tau_{AB} + \tau_{AC}. \tag{38}$$

Since these quantities are constant on V_1, let us calculate them at $|v_1\rangle$ in (15). We only need to compute $\tau_{A(BC)}$ and τ_{AB} since τ_{AC} will follow from (38). Computation of ρ_A and ρ_{AB} for the state $|v_1\rangle$ gives

$$\rho_A = \begin{pmatrix} |\hat{x}_2|^2 + |\hat{x}_3|^2 & 0 \\ 0 & |\hat{x}_5|^2 \end{pmatrix}, \tag{39}$$

$$\rho_{AB} = \begin{pmatrix} |\hat{x}_2|^2 & 0 & 0 & 0 \\ 0 & |\hat{x}_3|^2 & \hat{x}_3\hat{x}_5^* & 0 \\ 0 & \hat{x}_5\hat{x}_3^* & |\hat{x}_5|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Using (21), we obtain

$$\tau_{A(BC)} = 4\left(|\hat{x}_2|^2 + |\hat{x}_3|^2\right)|\hat{x}_5|^2 = 4(|\hat{x}_2|^2 + |\hat{x}_3|^2)|\hat{x}_2 + \hat{x}_3|^2. \tag{41}$$

To compute τ_{AB} one has to calculate the eigenvalues of $\rho_{AB}\sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ which using formula (40) can be seen to be: zero with multiplicity two and the eigenvalues of the 2×2 matrix

$$\begin{pmatrix} 2|\hat{x}_3|^2|\hat{x}_5|^2 & 2|\hat{x}_3|^2 Re(\hat{x}_5\hat{x}_3^*) \\ 2|\hat{x}_5|^2 Re(\hat{x}_5\hat{x}_3^*) & 2|\hat{x}_5|^2|\hat{x}_3|^2 \end{pmatrix},$$

which are $\lambda_1^2 = 2|\mu|(|\mu| + |Re(\mu)|)$, $\lambda_2^2 = 2|\mu|(|\mu| - |Re(\mu)|)$, with $\mu := \hat{x}_5\hat{x}_3^*$. Using formula (19) we have since $\lambda_3 = \lambda_4 = 0$,

$$\tau_{AB} = (\lambda_1 - \lambda_2)^2 = 2|\mu|\left(\sqrt{|\mu| + |Re(\mu)|} - \sqrt{|\mu| - |Re(\mu)|}\right)^2 = 4|\mu|\left(|\mu| - |Im(\mu)|\right). \tag{42}$$

We also have $\tau_{AC} = \tau_{A(BC)} - \tau_{AB}$.

Remark 5.3 From formula (41) it follows that $\tau_{A(BC)}$ is zero if and only if $\hat{x}_5 = 0$ ($\hat{x}_2$ and $\hat{x}_3$ cannot be simultaneously zero because this would imply the vector $|v_1\rangle$ to be zero). In this case, $|v_1\rangle$ would be a product state (of the form $|0\rangle \otimes \tilde{\psi}_{BC}$ for a state $\tilde{\psi}_{BC}$ on the subsystem (BC)). Since the local symmetric unitary group is transitive on the subspace V_1, every state in this subspace is a product state as expected when the entanglement is zero. The condition on the entanglement τ_{AB} is less intuitive. It can be stated by saying that $\mu := \hat{x}_5\hat{x}_3^*$ is purely imaginary.

We finish this section by observing (see next proposition) that, if we take a state restricted to the direct sum of V_1 and V_2, then the distributed entanglement τ can be strictly positive, but it is always less than or equal to 1/3.

Proposition 5.4 Let $|\psi\rangle \in V_1 \bigoplus V_2$ then $\tau \leq \frac{1}{3}$, with the bound being sharp.

Proof. A general state $|\psi\rangle \in V_1 \bigoplus V_2$, can be written as

$$|\psi\rangle = \alpha|v_1\rangle + \beta|w_1\rangle + \gamma|v_2\rangle + \delta|w_2\rangle,$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$, and where the vectors $|v_i\rangle$, and $|w_i\rangle$ are the basis vectors defined in equations (15) and (16), after normalization. Writing explicitly $|\psi\rangle$, we have:

$$|\psi\rangle = (\alpha\hat{x}_2 + \gamma\hat{y}_2)|001\rangle + (\alpha\hat{x}_3 + \gamma\hat{y}_3)|010\rangle + (\alpha\hat{x}_5 + \gamma\hat{y}_5)|100\rangle$$

$$+ (\beta\hat{x}_2 + \delta\hat{y}_2)|110\rangle + (\beta\hat{x}_3 + \delta\hat{y}_3)|101\rangle + (\beta\hat{x}_5 + \delta\hat{y}_5)|011\rangle.$$

Now we compute the distributed entanglement.

Denoting by $A_i = \alpha\hat{x}_i + \gamma\hat{y}_i$ and $B_i = \beta\hat{x}_i + \delta\hat{y}_i$, $i = 2, 3, 5$, we have, using equation (22),

$$\tau = 4\left|A_2^2B_2^2 + A_3^2B_3^2 + A_5^2B_5^2 - 2(A_2B_2A_3B_3 + A_2B_2A_5B_5 + A_3B_3A_5B_5)\right|$$

Using the fact that $A_5 = -A_2 - A_3$ and $B_5 = -B_2 - B_3$ from (17) (18), we have:

$$\tau = 4\left|\left(A_2^2B_2^2 - A_3^2B_3^2\right)^2 + (A_2 + A_3)^2(B_2 + B_3)^2 - 2(A_2 + A_3)(B_2 + B_3)(A_2B_2 + A_3B_3)\right|$$

$$= 4\left|(A_2^2B_2^2 - A_3^2B_3^2)^2 + (A_2 + A_3)(B_2 + B_3)(A_3B_2 + A_2B_3 - A_2B_2 - A_3B_3)\right|$$

$$= 4\left|\left(A_2^2B_2^2 - A_3^2B_3^2\right)^2 + \left(A_2^2B_3^2 + A_3^2B_2^2\right)^2 - \left(A_2^2B_2^2 + A_3^2B_3^2\right)^2\right|$$

$$= 4\left|\left(A_2^2B_3^2 - A_3^2B_2^2\right)^2\right|.$$

Using the definitions of A_i and B_i, we get:

$$\tau = 4\left|(\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2)^2(\alpha\delta - \beta\gamma)^2\right|. \tag{43}$$

Now by using the definitions of x_i and y_i, in terms of the constants a, b, c, and λ, given in Theorem 1, and equations (36) and (37), we have:[g]

$$(\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2) = \frac{(-\lambda + 6b + 6c + 6ia)(-\lambda + 6b + 6ia) - (-\lambda - 6b - 6ia)(-\lambda - 6b - 6c - 6ia)}{12\lambda\sqrt{(\lambda + 3c)(\lambda - 3c)}}$$

$$= \frac{-24\lambda b - 24i\lambda a - 12\lambda c}{12\lambda\sqrt{(\lambda + 3c)(\lambda - 3c)}} = -\frac{c + 2b + 2ia}{\sqrt{\lambda^2 - 9c^2}}$$

which gives:

$$|(\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2)^2| = \frac{(c + 2b)^2 + 4a^2}{\lambda^2 - 9c^2} = \frac{1}{3} \tag{44}$$

Thus using this equality in (43), we have:

$$\tau = \frac{4}{3}\left|(\alpha\delta - \beta\gamma)^2\right| \le \frac{1}{3}. \tag{45}$$

In the previous equation, the last inequality, is obtained, by observing that:

$$|\alpha\delta - \beta\gamma| \le |\alpha\delta| + |\beta\gamma| \le \frac{1}{2}(|\alpha|^2 + |\delta|^2) + \frac{1}{2}(|\beta|^2 + |\gamma|^2) = \frac{1}{2}.$$

Equality in (45) can be obtained by choosing $\alpha = \beta = \delta = \frac{1}{2}$, $\gamma = -\frac{1}{2}$. □

Notice that Proposition 5.2 is obtained as a corollary in the case $\alpha = \beta = 0$ or $\gamma = \delta = 0$.

[g] In the definition of the basis vectors $|v_1\rangle$, and $|w_1\rangle$ there is also the special case, when $a = 0$, $c = -2b$ and $b < 0$. However in this case we have $\hat{x}_2 = \frac{1}{\sqrt{2}} = -\hat{x}_3$ and $\hat{y}_2 = \hat{y}_3 = \frac{1}{\sqrt{6}}$, thus equation (44) below, which is our target, still holds. Similarly for the case $a = 0$, $c = -2b$ and $b > 0$.

6 Symmetric Dynamics on the Invariant Subspaces

We now study how the Lie group $U^{S_3}(8)$ of symmetric dynamics acts on its invariant subspaces. On the two dimensional invariant subspaces V_1 and V_2 this group acts as $U(2)$ and its induced dynamics is not more rich than the one of the group of symmetric local transformations $X \otimes X \otimes X$. These transformations do not modify the entanglement measures and in particular the distributed entanglement which is zero for each of the two subspaces, as we have seen in the previous section.

More interesting is the dynamics of $U^{S_3}(8)$ on the four dimensional symmetric sector which can be proven by Schur-Weyl duality (see, e.g., [1]) to be given by all possible unitary 4×4 matrices. The local symmetric unitary transformations are a Lie subgroup of $U^{S_3}(8)$ whose Lie algebra is spanned by $iH_{x,y,z}$ in (5) and multiples of the identity. If we consider the orthonormal basis obtained from (3) $\{\phi_0, \frac{\phi_1}{\sqrt{3}}, \frac{\phi_2}{\sqrt{3}}, \phi_3\}$, the matrices $\frac{i}{2}H_{x,y,z}$ give the four dimensional irreducible representation of $su(2)$. The corresponding matrices can be computed using the Clebsch-Gordan coefficients [9] (or directly by computing their action on the given basis). They are

$$S_x := \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & 2i & 0 \\ 0 & 2i & 0 & \sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}, \tag{46}$$

$$S_y := \frac{1}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \tag{47}$$

$$S_z := \frac{1}{2}\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}, \tag{48}$$

and satisfy the standard commutation relations for $su(2)$,

$$[S_x, S_y] = S_z, \qquad [S_y, S_z] = S_x, \qquad [S_z, S_x] = S_y. \tag{49}$$

The Lie subgroup of $SU(4)$ corresponding to the Lie algebra spanned by $S_{x,y,z}$ (corresponding to local symmetric transformations) leaves the measures of entanglement unchanged. In fact, it is a *maximal* Lie subgroup having this property as shown in the following two propositions whose proofs are postponed to the Appendix.

Proposition 6.1 The *local Lie group* corresponding to the Lie algebra spanned by ($i\times$) the 4×4 identity and $S_{x,y,z}$ is maximal among the Lie groups leaving the distributed tangle τ unchanged on the 4-dimensional symmetric sector. That is, there is no Lie group leaving such measure invariant which is larger than the local Lie group.

Proposition 6.2 The local Lie group corresponding to the Lie algebra spanned by ($i\times$) the 4×4 identity and $S_{x,y,z}$ is a maximal Lie group leaving the pairwise tangle $\tau_{AB} = \tau_{AC} = \tau_{BC}$ unchanged on the 4-dimensional symmetric sector.

The symmetric Lie group $U^{S_3}(8)$ acts on the symmetric sector spanned by the vectors $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ as the unitary group $U(4)$ and the associated Lie algebra acts like $u(4)$ (see, e.g., [1]). The Lie algebra spanned by $S_{x,y,z}$, in (46), (47), (48), is a subalgebra of $u(4)$

corresponding to local symmetric operations and isomorphic to $su(2)$, the symmetric sector giving an irreducible representation of $su(2)$. Since $U(4)$ is transitive on the complex unit sphere, it is possible, using elements of the Lie group $U^{S_3}(8)$, to transfer the state from a product state $\phi \otimes \phi \otimes \phi$ to any arbitrary state in the symmetric sector, independently of the entanglement value of the target state. In order to analyze how the elements of $U^{S_3}(8)$ modify the entanglement on the symmetric sector, we analyze the structure of the Lie algebra $u(4)$ starting with describing how the Lie algebra spanned by $\{S_x, S_y, S_z\}$, which leaves such measures unchanged, 'sits' in $u(4)$. Our goal is to arrive at a factorization of elements of $U(4)$ which separates factors which modify the entanglement measures from the symmetric local transformations that do not, trying to use as many as possible of the latter ones.

The Lie algebra $u(4)$ admits, up to conjugacy, a Cartan decomposition (see, e.g., [10])

$$u(4) = sp(2) \oplus sp^{\perp}(2), \tag{50}$$

where $sp(2)$ is the symplectic Lie algebra and $sp^{\perp}(2)$ is its orthogonal complement.[h] They satisfy the basic (Cartan-like) commutation relations

$$[sp(2), sp(2)] \subseteq sp(2), \qquad \left[sp^{\perp}(2), sp(2)\right] \subseteq sp^{\perp}(2), \qquad \left[sp^{\perp}(2), sp^{\perp}(2)\right] \subseteq sp(2). \tag{51}$$

As it is customary, we denote by $Sp(n)$ the connected Lie group associated with $sp(n)$. According to Cartan decomposition theorem, every unitary 4×4 matrix U can be written as

$$U := K_1 A K_2, \tag{52}$$

where $K_{1,2}$ are in $Sp(2)$ and A is the exponential of an element in a maximal Abelian subalgebra in $sp^{\perp}(2)$ which, in this case, has dimension 2, since we are including multiples of the identity as well in $sp^{\perp}(2)$.[i]

The symplectic Lie algebra $sp(n)$ (which has dimension $n(2n+1)$) and its associated Lie group $Sp(n)$ have several important properties that are of interest for the study of quantum dynamics. In particular, $sp(n)$ is a *maximal Lie subalgebra* of $su(2n)$ which means that $sp(n)$ along with any nonzero element $X \notin sp(n)$ of $su(2n)$ generates all of $su(2n)$. Every Lie subalgebra of $su(2n)$ which is isomorphic to $sp(n)$ is actually *conjugate* to $sp(n)$. Furthermore, $Sp(n)$ is *transitive* on the complex unit sphere S^{2n-1} representing quantum states. This means that, for any two (normalized) quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, there exists a matrix in $Sp(n)$ such that $|\psi_2\rangle = X|\psi_1\rangle$. This means, in particular, that any possible value of the entanglement in the symmetric sector can be achieved by only using the transformations K_1 and K_2 in (52).

For our purposes we consider a Lie subalgebra $\mathcal{S}$ *conjugate* to $sp(2)$ in $u(4)$. We consider the Lie algebra of 4×4 matrices of the form

$$F := \begin{pmatrix} ir & \alpha & \beta & \gamma \\ -\alpha^* & is & \delta & -\beta \\ -\beta^* & -\delta^* & -is & \alpha \\ -\gamma^* & \beta^* & -\alpha^* & -ir \end{pmatrix},$$

[h] The inner product considered is the inner product $\langle A, B\rangle = kTr(AB^{\dagger})$, for an appropriate positive constant k.

[i] In general the maximal Abelian subalgebra for the Cartan decomposition $su(n) = sp(\frac{n}{2}) \oplus sp^{\perp}(\frac{n}{2})$ has dimension $\frac{n}{2} - 1$ which would give 1 in this case. However, we have included multiples of the identity since we looked at the decomposition for $u(4)$. A full treatment of the decompositions for $u(n)$ can be found in [10] and a summary with applications for quantum systems can be found in [5].

with r and s arbitrary real numbers and α, β, γ, δ arbitrary complex numbers. Matrices in the Lie algebra $\mathcal{S}$ satisfy,

$$FJ + JF^T = 0,$$

where J is the matrix

$$J := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{53}$$

and formulas (50)-(52) hold with $sp(2)$ replaced by $\mathcal{S}$. The reason for this choice is that the matrices $S_{x,y,z}$, (46), (47), (48), giving the 4-dimensional irreducible representation of $su(2)$, belong to $\mathcal{S}$ (see, e.g., [7] for a treatment of how irreducible representations of $su(2)$ fit in the corresponding unitary Lie algebra). The decomposition (52) therefore holds with K_1 and K_2 belonging to the connected Lie group $e^{\mathcal{S}}$ conjugate to $Sp(2)$ and associated with the Lie algebra $\mathcal{S}$.[j] On the other hand, for the matrix A in (52), we can take the product of the exponentials of two elements in a Cartan subalgebra in $\mathcal{S}^{\perp}$. For such a Cartan subalgebra, we take $\texttt{span}\{i\mathbf{1}_4, iH_{zz}\}$ where on $(C^2)^{\otimes 3}$, H_{zz} is defined as H_{zz} in (6). In the symmetric sector, in the basis $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ it is given by the matrix

$$iH_{zz} = \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 3i \end{pmatrix}. \tag{54}$$

With this choice, formula (52) can be written as

$$U = K_1 e^{i\mathbf{1}_4 z} e^{iH_{zz} w} K_2, \tag{55}$$

for real parameters z and w, with K_1 and K_2 in $e^{\mathcal{S}}$ (isomorphic to $Sp(2)$). We now turn to the factorization of K_1 and K_2 in (55).

The Lie algebra $sp(2)$ has a Cartan decomposition $sp(2) = \mathcal{L} \oplus \mathcal{L}^{\perp}$ that can be chosen (up to conjugacy) between two possibilities denoted by **CI** and **CII** (cf. Chapter X in [10]). Given such a decomposition, the matrices K_1 and K_2 in (52) can be written as K in the following formula

$$K = L_1 \hat{A} L_2, \tag{56}$$

where $\hat{A}$ is the exponential of a matrix in a maximal Abelian subalgebra inside $\mathcal{L}^{\perp}$. We choose the decomposition **CI** because this allows us to separate S_x, S_y and S_z in $\mathcal{L}$ and $\mathcal{L}^{\perp}$. In particular, we have $\mathcal{L} = \mathcal{S} \cap so(4)$ which is given by the matrices of the form

$$\hat{L} := \begin{pmatrix} 0 & a & k & r \\ -a & 0 & f & -k \\ -k & -f & 0 & a \\ -r & k & -a & 0 \end{pmatrix}, \tag{57}$$

[j] Here and in the following we use the convention of denoting by $e^{\mathcal{S}}$ the connected Lie group associated with a Lie algebra $\mathcal{S}$.

for four real parameter a, k, r, f.[k] The matrix S_y in (47) belongs to the Lie subalgebra $\mathcal{L}$ (choose $a = -\frac{\sqrt{3}}{2}$, $f = -1$ and the other parameters equal to zero) while S_x and S_z belong to $\mathcal{L}^{\perp}$. The dimension of the Cartan subalgebra associated to this decomposition (the *rank* of the decomposition) is 2. We choose as Cartan subalgebra the one spanned by S_z and H, where $H := \texttt{diag}(0, \frac{i}{2}, -\frac{i}{2}, 0)$. Therefore $\hat{A}$ in formula (56) can be written as

$$\hat{A} = e^{S_z x} e^{H y}, \tag{58}$$

for real values x and y. The Lie algebra $\mathcal{L}$ is isomorphic to $u(2)$ with the isomorphism given by

$$i\mathbf{1}_2 \leftrightarrow J, \tag{59}$$

with J in (53),

$$\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{61}$$

$$\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leftrightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{62}$$

Consider the matrix in $\mathcal{L}$

$$R := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -\sqrt{3} & 0 \\ 0 & \sqrt{3} & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{63}$$

which is orthogonal to S_y. Performing an Euler-like decomposition on $e^{\mathcal{L}}$, we can write any element L in $e^{\mathcal{L}}$, such as L_1 and L_2 in (56), as $L := e^{S_y t_1} e^{R t_2} e^{S_y t_3} e^{J t_4}$. Combining this with $\hat{A}$ in (58) we obtain that every element K in $e^{\mathcal{S}}$ can be written as

$$K := e^{S_y t_1} e^{R t_2} e^{S_y t_3} e^{J t_4} e^{S_z t_5} e^{H t_6} e^{J t_7} e^{S_y t_8} e^{R t_9} e^{S_y t_{10}}. \tag{64}$$

In this decomposition, the only elements that change the entanglement measures are factors of the type e^{Rt}, e^{Jt}, and e^{Ht}. In the resulting decomposition for $U \in U(4)$ (55), to these needs to be added the evolution $e^{iH_{zz}w}$ in (55).

The full decomposition of unitary transformations which combines (55) with (64) separates factors which do not change the entanglement on the symmetric sector from factors that do. In particular the factors of the type $e^{S_y t}$ and $e^{S_z t}$ correspond to *local* transformations which leave the measures of entanglement unchanged, while the other factors change them. In this context, the decomposition is similar in spirit to the one found in [22] for the two qubits case.

[k] The Lie algebra $\mathcal{S}$ is only conjugate to $sp(2)$, therefore, the Lie algebra characterizing the decomposition **CI** is different but conjugate with respect to the one for $sp(2)$. The matrix M which gives the conjugacy is $M := \begin{pmatrix} U_1 & 0 \\ 0 & U_1^T \end{pmatrix}$ with $U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. We have $M sp(2) M^T = \mathcal{S}$.

7 Control of a Three Spin Symmetric Network

In this section, we briefly consider a possible physical scenario where the analysis carried out in the previous sections applies. We consider in particular a network of three identical spin $\frac{1}{2}$ particles subject to a controlling electromagnetic field which simultaneously interacts with the three spins. The three spins interact with each other via Ising interaction. The time varying Hamiltonian for this system is

$$H_S := H_S(t) = H_{zz} + H_x u_x(t) + H_y u_y(t) + H_z u_z(t), \tag{65}$$

with H_{zz} defined in (6) and $H_{x,y,z}$ defined in (5). The functions $u_{x,y,z} = u_{x,y,z}(t)$ represent (spatially uniform) components of an electromagnetic field in the x, y, z direction. The dynamics of the state is given by the Schrödinger equation

$$\dot{\psi} = -iH_S(t)\psi, \qquad \psi(0) = \psi_0. \tag{66}$$

The controllability analysis of a quantum system (see, e.g., [5]) describes the set of states that can be reached starting from ψ_0. In this case, such a set is [1]

$$\mathcal{R}_{\psi_0} := \{X\psi_0 \,|\, X \in U^{S_3}(8)\}. \tag{67}$$

In particular, if ψ_0 belongs to one of the invariant subspaces of $U^{S_3}(8)$ (or $u^{S_3}(8)$), then, for every control function, the state will remain in that subspace. However, within each invariant subspace every state transfer is possible [1]. The invariant subspaces for the group $U^{S_3}(8)$ were described in section 2.

Let us restrict ourselves to the symmetric sector where the dynamics was described in section 6 and let us pose the problem of reaching, starting from a symmetric product state, a state with maximum distributed entanglement $\tau = 1$, using an appropriate control function. The problem will be solved if we prove that there exists a symmetric local state $\hat{\psi}_0 := \phi \otimes \phi \otimes \phi$ and a time $\bar{t}$ such that $e^{-iH_{zz}\bar{t}}\hat{\psi}_0$ has maximum distributed entanglement. This property is referred to, in the case of (pairwise) entanglement of two qubits, as $e^{-iH_{zz}\bar{t}}$ being a *perfect entangler* [22] and we shall adopt this terminology here, mutatis mutandis. If there exists such a product state $\hat{\psi}_0$, we can use in (66) high amplitude short time pulses (so that we can neglect in the dynamics the effect of H_{zz}) which will produce a local symmetric transformation from ψ_0 to the state $\hat{\psi}_0$. Then we can set the controls equal to zero and allow the evolution go as $\hat{\psi}_0 \to e^{-iH_{zz}t}\hat{\psi}_0$ for time $\bar{t}$. We are left with proving that $e^{-iH_{zz}\bar{t}}$ is a perfect entangler.

Proposition 7.1 There exists a $\bar{t}$ such that $e^{-iH_{zz}\bar{t}}$ is a perfect entangler transferring the local symmetric state

$$\hat{\psi}_0 = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)^{\otimes 3}, \tag{68}$$

to a state of maximum distributed entanglement $\tau = 1$.

Proof. The state $\hat{\psi}_0$ in (68) can be written in the form (4) with $c_0 = c_1 = c_2 = c_3 = \frac{1}{2\sqrt{2}}$. Using the explicit expression of H_{zz} in (54) we have that $c_{0,1,2,3}$ vary with time as

$$c_0(t) = c_3(t) = \frac{e^{3it}}{2\sqrt{2}}, \qquad c_1(t) = c_2(t) = \frac{e^{-it}}{2\sqrt{2}}. \tag{69}$$

Using the formula for the distributed entanglement τ in (27) together with the expressions of X_2, X_3, X_4 in (24) we get $X_2 = X_4 = \frac{1}{8}(e^{2it} - e^{-2it})$, $X_3 = \frac{1}{8}(e^{6it} - e^{-2it})$, and

$$\tau(t) = 4\left|X_3^2 - 4X_2X_4\right| = \frac{1}{16}\left|(e^{6it} - e^{-2it})^2 - 4(e^{2it} - e^{-2it})^2\right| = \frac{1}{16}\left|(e^{6it} - e^{-2it})^2 + 16\sin^2(2t)\right|.$$

The function on the right hand side has a maximum equal to 1 at $t = \frac{\pi}{4}$. Therefore the proposition holds with $\bar{t} = \frac{\pi}{4}$. □

8 Conclusions

In this paper, we have given an analysis of the states of a three qubit quantum system under the action of the Lie group $U^{S_3}(8)$ of unitary matrices which commute with the symmetric group of three objects. This is motivated by the controlled dynamics of symmetric spin networks with three spin $\frac{1}{2}$ particles, as described in section 7. The Hilbert space of three qubits splits into subspaces of dimension 4, 2 and 2, which are invariant under the action of the Lie group $U^{S_3}(8)$. The subspace of dimension 4 is uniquely determined and corresponds to the so called symmetric sector W of states which are invariant under permutation (symmetric states). The subspaces of dimension 2 are not uniquely determined although they are orthogonal to W and orthogonal to each other. We have provided the following results:

We have parametrized all the possible decompositions of the state space in invariant subspaces under the Lie group $U^{S_3}(8)$ (Theorem 1).

For states in the symmetric sector W, we have introduced three quantities X_2, X_3, X_4 in (24) which are easily calculated from the expression of the state and have given a simple criterion of separability (Proposition 4.1).

We have calculated expressions of the distributed entanglement and the pairwise entanglement in terms of the quantities X_2, X_3, X_4 (Propositions 4.2, 4.3) and concluded that the only states on the symmetric sector which have both entanglements equal to zero are the separable states (Proposition 4.4).

For states in the two dimensional invariant subspaces, we have proven that the distributed entanglement is always equal to zero (Proposition 5.2) while the pairwise entanglement depends on the subspace considered. We provided a simple formula for it (formula (42)). States that are in the direct sum of the two, two dimensional, invariant subspaces, may have distributed entanglement different from zero but always bounded by $\frac{1}{3}$ (Proposition 5.4).

We have proven that there is no connected Lie subgroup of $U^{S_3}(8)$ which properly contains the Lie subgroup of local symmetric transformations and leaves unchanged the distributed entanglement (Proposition 6.1) or the pairwise entanglement (Proposition 6.2) on the symmetric sector.

We have given a decomposition of any evolution in $U^{S_3}(8)$ on the symmetric sector into (local) elements which do not modify the entanglement and factors which modify it (formulas (55) and (64)).

We have proven that the free evolution given by a pairwise Ising interaction is a perfect entangler for distributed entanglement on a symmetric network of three spin (Proposition 7.1) and used this to prescribe a control law to drive a separable symmetric state to a state of maximal distributed entanglement.

In future research, it will be of interest to extend the results presented here to states for more than three qubits. Such extensions depend on and will lead to a better understanding of multipartite entanglement beyond the three qubits case.

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Appendix A Proofs of Propositions 6.1 and 6.2

We consider a general vector in the symmetric sector

$$\psi = c_0\phi_0 + \hat{c}_1\frac{1}{\sqrt{3}}\phi_1 + \hat{c}_2\frac{1}{\sqrt{3}}\phi_2 + c_3\phi_3,$$

where, comparing with (4), we have $\hat{c}_1 = \sqrt{3}c_1$, $\hat{c}_2 = \sqrt{3}c_2$.

Given either a constant c or avector y or a matrix M, we follow the convention of denoting by c_R and c_I, y_R and y_I, M_R and M_I, their real and imaginary parts respectively. Moreover given a matrix M, we denote by $M_{i,j}$, its i,j element.

Proof of Proposition 6.1

Proof. Recall, see (27), that the distributed tangle τ is given by $\tau = 4|X_3^2 - 4X_2X_4|$. Consider the quantity $X_3^2 - 4X_2X_4$ written separating its real and imaginary parts as $X_3^2 - 4X_2X_4 := R + iI$

Then invariance of τ is equivalent to invariance of the function $f := \frac{1}{2}(R^2 + I^2)$. R and I are functions of the complex vector $\vec{v} := (c_0, \hat{c}_1, \hat{c}_2, c_3)^T := \vec{v}_R + i\vec{v}_I$, for real vectors $\vec{v}_R$ and $\vec{v}_I$, where $\vec{v}_R := (c_{0,R}, \hat{c}_{1,R}, \hat{c}_{2,R}, c_{3,R})^T$, $\vec{v}_I := (c_{0,I}, \hat{c}_{1,I}, \hat{c}_{2,I}, c_{3,I})^T$. If F is an element of the Lie algebra associated to a given Lie subgroup of $U(4)$, under the action of e^{Ft}, $\vec{v}$ changes as $\vec{v} \to e^{Ft}\vec{v}$ and therefore $f(t)$ varies as

$$f(t) = \frac{1}{2}\left(R^2(e^{Ft}\vec{v}) + I^2(e^{Ft}\vec{v})\right). \tag{A.1}$$

Since F is skew-Hermitian, it can be written as $F = A + iB$ with A skew symmetric and B symmetric (with both of them real). Invariance of $f = f(t)$ implies

$$0 = \frac{df}{dt}|_{t=0} = (R\nabla R + I\nabla I)\begin{pmatrix} A\vec{v}_R - B\vec{v}_I \\ A\vec{v}_I + B\vec{v}_R \end{pmatrix}. \tag{A.2}$$

The idea of the proof is to show that if the matrix $F := A + iB$ satisfies (A.2) for all possible vectors $\vec{v}$, then it must be in the Lie algebra spanned by $S_{x,y,z}$, plus the $i\times$ identity. To show this, we will first compute (A.2) for special vectors.

Let us consider the case of vectors for which $\vec{v}_I = 0$. A direct calculation using the definitions (24) shows that $I = 0$ so that (A.2) simplifies as

$$\dot{f}(0) = R\nabla R\begin{pmatrix} A\vec{v}_R \\ B\vec{v}_R \end{pmatrix} = 0. \tag{A.3}$$

Using the definitions (24) we get

$$R = X_{3R}^2 - X_{3I}^2 - 4\left(X_{2R}X_{4R} - X_{2I}X_{4I}\right), \tag{A.4}$$

and an explicit calculation, using the constraint that $\vec{v}_I = 0$, gives $X_{2I} = X_{3I} = X_{4I} = 0$, so that we have

$$\nabla R(\vec{v}_R, 0) = 2X_{3R}\nabla X_{3R} - 4X_{4R}\nabla X_{2R} - 4X_{2R}\nabla X_{4R}. \tag{A.5}$$

Now we specialize further the vector $\vec{v}$ in (A.3).

1. Set $\hat{c}_2 = c_2 = \hat{c}_1 = c_1 = 0$. We have $X_2 = X_4 = 0$ and $X_{3R} = c_{0R}c_{3R}$. A direct calculation gives

$$\nabla X_{3R} = \nabla\left(c_{0R}c_{3R} - c_{0I}c_{3I} + \frac{\hat{c}_{1I}\hat{c}_{2I}}{3} - \frac{\hat{c}_{1R}\hat{c}_{2R}}{3}\right)$$

$$= \left(c_{3R},\, -\frac{\hat{c}_{2R}}{3},\, -\frac{\hat{c}_{1R}}{3},\, c_{0R}\, - c_{3I},\, \frac{\hat{c}_{2I}}{3},\, \frac{\hat{c}_{1I}}{3},\, -c_{0I}\right),$$

which, using $\hat{c}_1 = \hat{c}_2 = 0$ along with $c_{0I} = c_{3I} = 0$, gives

$$\nabla X_{3R} = (c_{3R},\, 0,\, 0,\, c_{0R},\, 0,\, 0,\, 0,\, 0). \tag{A.6}$$

Placing this and $X_{3R} = c_{0R}c_{3R}$ in (A.3), we have

$$\dot{f}(0) = c_{0R}c_{3R}\begin{pmatrix} c_{3R} & 0 & 0 & c_{0R}\end{pmatrix} A \begin{pmatrix} c_{0R} \\ 0 \\ 0 \\ c_{3R}\end{pmatrix} = 0. \tag{A.7}$$

Assume $c_{0R}c_{3R} \neq 0$. Using the fact that A is skew-symmetric, and using $c_{0R}^2 \neq c_{3R}^2$, this relation implies $A_{1,4} = A_{4,1} = 0$.

2. Set $c_0 = c_2 = 0$. A direct calculation using (24) gives $X_3 = 0$ and (A.4) gives $R = 4c_{1R}^3c_{3R}$. We have, using (A.5) and $X_3 = 0$,

$$\nabla R = -4X_{4R}\nabla X_{2R} - 4X_{2R}\nabla X_{4R} = -4c_{1R}c_{3R}\nabla X_{2R} + 4c_{1R}^2\nabla X_{4R}. \tag{A.8}$$

Now use

$$\nabla X_{2R} = \nabla\left(c_{0R}\frac{\hat{c}_{2R}}{\sqrt{3}} - c_{0I}\frac{\hat{c}_{2I}}{\sqrt{3}} - \frac{\hat{c}_{1R}^2}{3} + \frac{\hat{c}_{1I}}{3}\right) \tag{A.9}$$

$$= \left(\frac{\hat{c}_{2R}}{\sqrt{3}},\, -\frac{2}{3}\hat{c}_{1R},\, \frac{c_{0R}}{\sqrt{3}},\, 0,\, -\frac{\hat{c}_{2I}}{\sqrt{3}},\, \frac{2\hat{c}_{1I}}{3},\, -\frac{c_{0I}}{\sqrt{3}},\, 0\right),$$

and

$$\nabla X_{4R} = \nabla\left(\frac{\hat{c}_{1R}}{\sqrt{3}}c_{3R} - \frac{\hat{c}_{1I}}{\sqrt{3}}c_{3I} - \frac{\hat{c}_{2R}^2}{3} + \frac{\hat{c}_{2I}^2}{3}\right) \tag{A.10}$$

$$= \left(0,\, \frac{c_{3R}}{\sqrt{3}},\, -\frac{2}{3}\hat{c}_{2R},\, \frac{\hat{c}_{1R}}{\sqrt{3}},\, 0,\, -\frac{c_{3I}}{\sqrt{3}},\, \frac{2}{3}\hat{c}_{2I},\, -\frac{\hat{c}_{1I}}{\sqrt{3}}\right),$$

with $c_0 = \hat{c}_2 = 0$ and $\hat{c}_{1I} = c_{3I} = 0$, in (A.8). We get

$$\nabla R = \frac{4}{\sqrt{3}} \hat{c}_{1R}^2 \left(0,\, c_{3R},\, 0,\, \frac{\hat{c}_{1R}}{3},\, 0,\, 0,\, 0,\, 0 \right).$$

Placing this in (A.3) , we get

$$\dot{f}(0) = \frac{16}{9} \hat{c}_{1R}^5 c_{3R} \begin{pmatrix} 0 & c_{3R} & 0 & \frac{\hat{c}_{1R}}{3} \end{pmatrix} A \begin{pmatrix} 0 \\ \hat{c}_{1R} \\ 0 \\ c_{3R} \end{pmatrix} = 0. \tag{A.11}$$

This gives using $A_{1,4} = A_{4,1} = 0$, $(c_{3R}^2 - \frac{\hat{c}_{1R}^2}{3}) A_{2,4} = 0$ which implies $A_{2,4} = 0$, if we choose $\hat{c}_{1R}$ and c_{3R} different from zero.

This shows that every matrix $F = A + iB$ in the Lie algebra corresponding to the Lie group which leaves τ unchanged has to be such that $A_{2,4} = A_{1,4} = A_{4,2} = A_{4,1} = 0$. Since we assume that S_y in (47) also belongs to such Lie algebra and $[S_y, A]$ is real (and skew-symmetric) while $[S_y, B]$ is purely imaginary (and symmetric), imposing this condition on $[S_y, A]$, shows that we must have also $A_{1,3} = A_{3,1} = 0$ and $A_{3,4} = -A_{4,3} = \frac{\sqrt{3}}{2} A_{2,3}$. Furthermore, imposing the condition that the $(1, 3)$ component is zero to $[S_y, A]$, we also get $A_{1,2} = A_{3,4}$. This shows that the real part of $A + iB$ must be a multiple of S_y in (47).

Now consider the restrictions on the symmetric matrix B. Since, with S_z in (48), $[S_z, iB]$ is real, it must be proportional to S_y. From this restriction, it follows that B must be the sum of a diagonal matrix and a matrix proportional to S_x in (46). Then, considering the Lie bracket $[S_x, iB]$ which must also be proportional to S_y it follows that the diagonal part of B must be a linear combination of the identity and S_z. This concludes the proof.
□

Proof of Proposition 6.2

Proof. We use the notations of the proof of Proposition 6.1. Let us first consider the function $\det(\rho_A)$ in (30) as we act on the vector $\vec{v}$ as defined in the previous proof of Proposition 6.1 with e^{Ft}. That is, similarly to (A.1), we define a function

$$g(t) = \det(\rho_A) \left(e^{Ft} \vec{v} \right) = \left| X_3(e^{Ft} \vec{v}) \right|^2 + 2 \left| X_2(e^{Ft} \vec{v}) \right|^2 + 2 \left| X_4(e^{Ft} \vec{v}) \right|^2, \tag{A.12}$$

and we calculate $\frac{d}{dt}|_{t=0} g(t)$. Notice that this is not set to zero yet. Similarly to what was done in (A.2), we have

$$\frac{d}{dt}|_{t=0} g(t) = (\nabla |X_3|^2 + 2\nabla |X_2|^2 + 2\nabla |X_4|^2) \begin{pmatrix} A\vec{v}_R - B\vec{v}_I \\ A\vec{v}_I + B\vec{v}_R \end{pmatrix}. \tag{A.13}$$

Choose now an initial point so that $\vec{v}_I = 0$ which implies that the imaginary parts of X_2, X_3 and X_4 are also zero. Therefore (A.13) gives

$$\frac{d}{dt}|_{t=0} g(t) = (2X_{3R} \nabla X_{3R} + 4X_{2R} \nabla X_{2R} + 4X_{4R} \nabla X_{4R}) \begin{pmatrix} A\vec{v}_R \\ B\vec{v}_R \end{pmatrix}. \tag{A.14}$$

We now proceed considering subcases as in Proposition 6.1, in fact, the same subcases.

1. Set $\hat{c}_1 = \hat{c}_2 = 0$. In this case $X_2 = X_4 = 0$ and ∇X_{3R} was already computed in (A.6). Using this expression and the expression of X_{3R}, which in this case is $X_{3R} = c_{0R} c_{3R}$, we obtain
$$\frac{d}{dt}|_{t=0} g(t) = 2c_{0R}c_{3R}K, \tag{A.15}$$
where (cf. (A.7))
$$K := \begin{pmatrix} c_{3R} & 0 & 0 & c_{0R} \end{pmatrix} A \begin{pmatrix} c_{0R} \\ 0 \\ 0 \\ c_{3R} \end{pmatrix}.$$
With the given definitions, the expression of the pairwise entanglement (31) as a function of t is
$$\tau_{AB}(t) = 2\left(g(t) - \sqrt{2f(t)}\right).$$
The condition $\frac{d\tau_{AB}}{dt}|_{t=0} = 0$ gives
$$\dot{g}(0) - \frac{\dot{f}(0)}{\sqrt{2f(0)}} = 0, \tag{A.16}$$
which, using (A.15), (A.2), (A.7) and the expression of $f(0) = \frac{1}{2}c_{0R}^2 c_{3R}^2$, gives
$$2c_{0R}c_{3R}|c_{0R}c_{3R}|K - c_{0R}c_{3R}K = 0.$$
The quantity $c_{0R}c_{3R}$ can be chosen so that this implies $K = 0$ and, as in the proof of Proposition 6.1, this gives $A_{14} = A_{4,1} = 0$.

2. Set $c_0 = c_2 = 0$.

This gives $X_3 = 0$. We also have $X_{2R} = -c_{1R}^2 = -\frac{\hat{c}_{1R}^2}{3}$, $X_{4R} = c_{1R}c_{3R} = \frac{\hat{c}_{1R}}{\sqrt{3}}c_{3R}$. The quantities ∇X_{2R} and ∇X_{4R} in this case were calculated in (A.9) (A.10). These formulas give, with $c_0 = c_2 = 0$,
$$\nabla X_{2R} = \left(0, -\frac{2}{3}\hat{c}_{1R}, 0, 0, 0, 0, 0, 0\right), \tag{A.17}$$
$$\nabla X_{4R} = \left(0, \frac{c_{3R}}{\sqrt{3}}, 0, \frac{\hat{c}_{1R}}{\sqrt{3}}, 0, 0, 0, 0\right). \tag{A.18}$$
Using these formulas in (A.14), we obtain
$$\frac{d}{dt}|_{t=0} g(t) = \frac{4}{3}\hat{c}_{1R}\left(0, c_{3R}^2 + \frac{2}{3}\hat{c}_{1R}^2, 0, \hat{c}_{1R}c_{3R}, 0, 0, 0, 0\right)\begin{pmatrix} A\vec{v}_R \\ B\vec{v}_R \end{pmatrix}. \tag{A.19}$$
In this case, $X_3 - 4X_2X_4 = 4c_{1R}^3 c_{3R} = \frac{4}{3}\frac{\hat{c}_{1R}^3}{\sqrt{3}}c_{3R}$, and therefore $f(0) = \frac{1}{2}|X_3 - 4X_2X_4|^2 = \frac{8}{27}\hat{c}_{1R}^6 c_{3R}^2$. Replacing this in (A.16), together with the expression of $\dot{f}(0)$ calculated in (A.11), we get
$$\frac{4}{3}\hat{c}_{1R}\left(0, c_{3R}^2 + \frac{2}{3}\hat{c}_{1R}^2, 0, \hat{c}_{1R}c_{3R}\right) A \begin{pmatrix} 0 \\ \hat{c}_{1R} \\ 0 \\ c_{3R} \end{pmatrix} - \frac{4}{3}\sqrt{3}\frac{\hat{c}_{1R}^5 c_{3R}}{|c_{1R}^3 c_{3R}|}(0, c_{3R}, 0, \frac{\hat{c}_{1R}}{3}) A \begin{pmatrix} 0 \\ \hat{c}_{1R} \\ 0 \\ c_{3R} \end{pmatrix} = 0. \tag{A.20}$$

Now choose $c_{3R} = \hat{c}_{1R}$. Using this in (A.20), we obtain, after simplifications,

$$\begin{pmatrix} 0 & 5-3\sqrt{3} & 0 & 3-\sqrt{3} \end{pmatrix} A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

This, using $A_{1,4} = A_{4,1} = 0$, implies $A_{2,4} = A_{4,2} = 0$.

The rest of the proof proceeds as the proof of Proposition 6.1.

□