Quantum Information and Computation, Vol. 21, No. 15&16 (2021) 1320–1352

UNIVERSAL CONTROL OF QUANTUM PROCESSES USING SECTOR-PRESERVING CHANNELS

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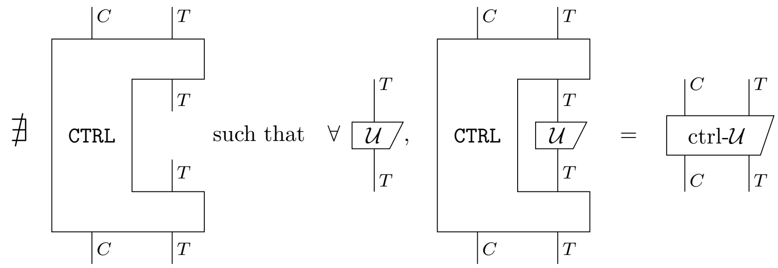
Received July 5, 2021
Revised October 6, 2021

No quantum circuit can turn a completely unknown unitary gate into its coherently controlled version. Yet, coherent control of unknown gates has been realised in experiments, making use of a different type of initial resources. Here, we formalise the task achieved by these experiments, extending it to the control of arbitrary noisy channels, and to more general types of control involving higher dimensional control systems. For the standard notion of coherent control, we identify the information-theoretic resource for controlling an arbitrary quantum channel on a d-dimensional system: specifically, the resource is an extended quantum channel acting as the original channel on a d-dimensional sector of a $(d+1)$-dimensional system. Using this resource, arbitrary controlled channels can be built with a universal circuit architecture. We then extend the standard notion of control to more general notions, including control of multiple channels with possibly different input and output systems. Finally, we develop a theoretical framework, called supermaps on routed channels, which provides a compact representation of coherent control as an operation performed on the extended channels, and highlights the way the operation acts on different sectors.

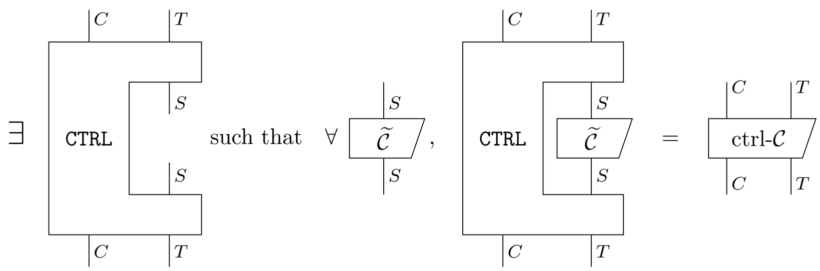
Keywords: coherent control, superposition of quantum channels, routed quantum circuits, sector-preserving channels
Communicated by: R Jozsa & J Eisert

1 Introduction

A number of quantum algorithms, such as Kitaev's phase estimation algorithm [1] and the DQC1 trace estimation algorithm [2], are based on the use of controlled unitary gates. Controlled gates represent a quantum version of the *if-then* clause, in which a subroutine is executed depending on the value of a control variable. In the controlled gate $\texttt{ctrl-}U$, the quantum state of a control system determines whether or not a target system is subject to a given unitary gate U. When the control system is in a superposition state, the target sys-



(a) **No-go theorem on coherent control with black box channels** [7, 8, 9, 10, 11, 12]. No supermap can convert an arbitrary unitary channel $\mathcal{U}$, acting on a target system T, into its controlled version `ctrl`-$\mathcal{U}$, acting on a control system C and on the target T.



(b) **Universal coherent control with sector-preserving channels.** There exists a supermap `CTRL` that transforms arbitrary sector-preserving channels $\widetilde{\mathcal{C}}$ acting on an extended input system S (with Hilbert space $\mathcal{H}_S = \mathbb{C} \oplus \mathcal{H}_T$) into arbitrary controlled channels `ctrl`-$\mathcal{C}$. In particular, the supermap `CTRL` maps arbitrary sector-preserving unitary channels $\widetilde{\mathcal{U}}$ into the corresponding controlled unitary channels `ctrl`-$\mathcal{U}$.

Fig. 1. Comparison between the standard no-go theorem and our universal controllisation circuit.

tem experiences a coherent superposition of quantum evolutions [3]. Quantum programming languages that exploit coherent control of quantum gates have been proposed in Refs. [4, 5, 6].

The standard way to construct quantum controlled gates is via universal gate sets. To build the controlled gate `ctrl`-U, one first decomposes the gate U into elementary gates, and then adds control to each of these gates [13]. This construction, however, requires a decomposition of the gate U into elementary gates. In many applications, such as quantum factoring [14], the decomposition is known, because the gate U is the quantum realisation of a classical function, for which a classical program is given. In other applications, however, the gate U may be completely unknown: in a cloud computing scenario, for example, the gate U may be implemented remotely by a server, and the program that generated U may be unknown to the client. In these situations, it would be convenient to have a way to generate the controlled gate `ctrl`-U from the access to an unknown, uncontrolled gate U. The ability to generate controlled gates would also benefit the implementation of standard quantum algorithms, providing them with an appealing modularity feature [11]. Besides quantum computation, the ability to control an unknown quantum process would be beneficial to other information-processing tasks, such as quantum communication [15, 16, 17], quantum metrology [18, 19], and quantum machine learning [20, 21].

The problem of the coherent control of an unknown channel can be phrased in the following way: 'Is there a universal protocol which, from the use of a black-box channel $\mathcal{C}$, implements its coherently controlled version?'. It has been proven several times, in ever stronger ways [7, 8, 9, 10, 11, 12], that the answer to this question is a resounding 'No': no quantum circuit can 'controllise' arbitrary operations. For general non-unitary channels, such a controllisation is not even unambiguously defined in the first place, as observed in Ref. [22].

Yet, as has been noted at the same time, coherent control is actually easily implementable in various contexts, such as optical systems [23, 24, 25, 26], trapped ions [25, 26], and superconducting qubits [27]. These realisations are not in contradiction with the no-go theorems because the resources they use are not black boxes: in the computer science terminology, they are grey boxes, whose action is partially known [8, 11] (see also Section 3.2 of this paper for a further elaboration of this point).

This mismatch between theory and experiments suggests that it may be necessary to revisit the terms of the problem. A suitable formulation of the problem would help understanding in which situations, from which resources, and with which protocols, one can implement a coherently controlled quantum channel. This understanding would allow to go beyond the existing examples of implementations of coherent control, and to compare their respective advantages. Another upshot of a better theoretical understanding is that it allows to neatly distinguish the informational, implementation-independent aspects of coherent control from the specific, system-dependent features of experimental implementations. In particular, it would help shift the focus away from optical implementations and towards a more implementation-neutral perspective. Finally, identifying the operational ingredients of coherent control helps elucidate some aspects of the existing no-go theorems, as studying protocols that can perform a certain task usually helps understanding why other protocols cannot.

In this paper we analyse the key features of the experimental implementations, and put forward a new formulation of the problem of coherent control based on these features. Our starting point is the observation that the crucial feature of the existing implementations is that they use *sector-preserving channels*; i.e., channels whose input systems can be partitioned into sectors (orthogonal subspaces), with the property that a state in a given sector always remains in this sector after the channel has acted. In this work, we focus on the case where some sectors are one-dimensional and others are d-dimensional. A sector-preserving channel acting on a system with a 1-dimensional sector and a d-dimensional sector will be called a *sector-preserving channel of type* $(1, d)$. More generally, a sector-preserving channel acting on a system with m 1-dimensional sectors and n d-dimensional sectors will be called a *sector-preserving channel of type* $(\underbrace{1, ..., 1}_{m \text{ times}}, \underbrace{d, ..., d}_{n \text{ times}})$.

The idea of regarding sector-preserving channels as resources originates from Ref. [15], and was further explored in Refs. [16, 17].[a] In these works, the focus was put on the use of sector-preserving channels for communication.[b] In contrast, the relevance of sector-preserving channels to the task of coherent control has not been explored before, and will be the focus of this paper.

Our main results are summarised in the following. For the standard notion of coherent control, we establish a perfect, one-to-one correspondence between sector-preserving channels

[a]In the past, a similar approach had independently been explored in Refs. [28, 29]. A different approach, based on the unitary extension of quantum channels, was developed in Refs. [30, 31, 32].

[b]This was part of a wider discussion about the communication advantages of coherent control of causal order [17, 33, 30, 34, 35].

of type $(1, d)$ and coherently controlled channels with target systems of dimension d. We then show that this one-to-one correspondence can be implemented physically, by inserting sector-preserving channels into a fixed, universal quantum circuit that generates the corresponding controlled channels. Mathematically, this universal circuit can be represented as a *quantum supermap* [36, 37, 38], that is, a transformation of quantum channels. We call this particular supermap the `CTRL` supermap, and show that it is invertible. Its inverse $\texttt{CTRL}^{-1}$ is also a supermap, corresponding to a universal circuit that transforms controlled channels on d-dimensional systems into sector-preserving channels of type $(1, d)$.

Summarising, coherently controlled channels on d-dimensional systems and sector-preserving channels of type $(1, d)$ are fully equivalent resources, and the interconversion of these resources is implemented by the `CTRL` supermap and by its inverse. It is worth contrasting this result with the existing no-go theorems on coherent control: while control cannot be achieved from general channels on d-dimensional systems, it can be achieved from sector-preserving channels of type $(1, d)$. The comparison is illustrated in Figure 1.

After establishing the above results, we extend them to more general versions of coherent control. For example, we show a one-to-one correspondence between sector-preserving channels of type $(1, 1, d)$ and compositely-controlled channels with two branches leading to application of the identity, and we build universal circuits that implement this correspondence in both ways. We then extend this result to compositely-controlled channels with any number of branches leading to application of the identity.

We also extend our results to the coherent control of N isometric channels, whose input and output spaces can be of different dimensions. As the initial resource, we take N sector-preserving isometric channels of type $(1 \to 1, d_{\text{in}} \to d_{\text{out}})$, meaning that *(i)* the input (output) is partitioned into a 1-dimensional sector and a d_{in}-dimensional (d_{out}-dimensional) sector, and *(ii)* states in the 1-dimensional input sector are mapped into states of in the 1-dimensional output sector, while states in the d_{in}-dimensional input sector are mapped into states of in the d_{out}-dimensional output sector. We then show that this resource can be used to construct a channel with coherent control between corresponding isometries. We study explicitly the $N = 2$ case, which readily generalises to arbitrary N. Mathematically, we show that there exists an invertible supermap `2-CTRL` that transforms every pair of sector-preserving isometric channels into the corresponding controlled channel.

In the non-isometric case, however, we find that sector-preserving channels of type $(1 \to 1, d_{\text{in}} \to d_{\text{out}})$ are generally not sufficient to achieve all possible controlled channels. Such channels can instead be realised using sector-preserving channels of type $(1 \to 1, d_{\text{in}} \to d_{\text{out}} d')$, where d' is the dimension of an auxiliary system, used to extend the original channels (from a d_{in}-dimensional system to a d_{out}-dimensional system) to isometries. Using this extra resource, we provide a universal protocol for the implementation of coherent control from N sector-preserving channels.

We conclude the paper by building a general framework for the manipulation of sector-preserving channels, and, more generally, of channels that maps input sectors into output sectors according to a prescribed rule, called the *route* [39]. The key ingredient of our framework is the notion of 'supermaps on routed channels,' a new kind of supermaps whose input is restricted to channels with a prescribed route. Examples of supermaps on routed channels are the `CTRL` and `2-CTRL` supermaps constructed earlier in the paper (or, more precisely, the restrictions of such maps to sector-preserving channels).

Our results open the way to several applications. First, by identifying the resources for the task of coherent control, we lay the basis for a resource-theoretic analysis of existing protocols and experiments. Second, the supermaps defined in this work can be easily extended to multiple channels, and to more elaborate architectures involving multiple instances of coherent control at different moments of time. This flexibility can help the design of complex protocols and algorithms, offering a built-in modularity feature. Finally, the new notions of composite control introduced in this paper have the potential to stimulate new theoretical protocols and experimental implementations with higher dimensional control systems.

The structure of the paper is as follows. In Section 2, we review the existing definitions of controlled unitaries and channels, and we address their extension to multiple channels, defining a new notion of compositely-controlled channels. In Section 3, we analyze the structure of the existing implementations of coherent control, and use it to motivate a study of sector-preserving channels of type $(1, d)$. We then show that these channels are in one-to one correspondence with controlled channels on a d-dimensional system. In Section 4, we show that the correspondence between sector-preserving channels of type $(1, d)$ and controlled channels can be physically implemented by a universal protocol, formalised by the `CTRL` supermap. In Section 5, we generalise this correspondence to the coherent control between N isometries, showing that it can also be realised via a universal protocol, and we discuss the case of the coherent control between N general channels, showing that it requires more involved resources. In Section 6, we extend the results of the previous sections to *compositely-controlled* channels. Finally, in Section 7 we define supermaps on routed channels, providing a general framework for the manipulation of sector-preserving channels and more general channels that transform sectors in a prescribed way.

2 Coherently controlled quantum channels

In this section, we review the existing definitions of coherently controlled unitaries and channels. Then, we provide a one-to-one parametrisation of the possible controlled versions of a channel in terms of a *'pinned Kraus operator'*. Finally, we discuss more general types of controlled quantum channels, and we provide one-to-one parametrisations for these in terms of pinned Kraus operators.

2.1 Controlled channels and pinned Kraus operators

Let us start with the most basic definition of controlled operation: controlled unitary gates. Given a unitary operator U acting on a d-dimensional Hilbert space $\mathcal{H}_T$, there is a standard notion of a 'controlled-U' channel: it is the channel corresponding to the unitary operator

$$\texttt{ctrl-}U := |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U\,, \qquad (1)$$

acting on a composite system, made of a two-dimensional *control* system C and of a d-dimensional *target* system T.

More generally, one may want to control the evolution of an open system. The general evolution of an open system T is described by a quantum channel $\mathcal{C}$, that is, a completely positive, trace-preserving map mapping density matrices on $\mathcal{H}_T$ into density matrices on $\mathcal{H}_T$. The action of the channel $\mathcal{C}$ on a generic density matrix ρ can be conveniently described in the Kraus representation, as $\mathcal{C}(\rho) = \sum_{i=1}^{n} C_i \rho C_i^\dagger$, where the operators $(C_i)_{i=1}^n$, called Kraus operators, satisfy the normalisation condition

$$\sum_{i=1}^{n} C_i^\dagger C_i = I\,, \tag{2}$$

I being the identity operator on $\mathcal{H}_T$.

Crucially, the Kraus representation of a channel is not unique: if V is a $l \times n$ isometry with matrix elements V_{ji}, the operators $(C_j')_{j=1}^l$ defined by $C_j' := \sum_i V_{ji}\, C_i'$ also form a Kraus representation of channel $\mathcal{C}$. The non-uniqueness of the Kraus representation will play an important role in this paper.

For a general quantum channel $\mathcal{C}$, the definition of coherent control is not straightforward. The naive generalisation of Eq. (1) would be to pick a Kraus representation (C_i) and define the controlled operators $\texttt{ctrl}\text{-}C_i = |0\rangle\langle 0| \otimes C_i + |1\rangle\langle 1| \otimes I$. This definition, however, would fail to give a quantum channel, because the above operators fail to satisfy the normalisation condition (2). A suitable generalisation of Eq. (1) was put forward in Ref. [22]: a controlled version of channel $\mathcal{C}$ is the channel with Kraus operators

$$\texttt{ctrl}_{\alpha_i}\text{-}C_i := \alpha_i\, |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes C_i\,, \tag{3}$$

where $(\alpha_i)_{i=1}^n$ are complex amplitudes satisfying the normalisation condition $\sum_{i=1}^n |\alpha_i|^2 = 1$.

This definition is a special case of the definition of coherent control of two general channels considered in Refs. [40, 28, 15, 30], in the special case where one of the two channels is the identity channel.

It is important to observe that the definition of the controlled channel does not depend only on the channel $\mathcal{C}$. In general, it can depend both on the set of Kraus operators $\mathbf{C} := (C_i)_{i=1}^n$ and on the set of amplitudes $\boldsymbol{\alpha} := (\alpha_i)_{i=1}^n$ used in Eq. (3). To emphasise the dependence on the Kraus operators $\mathbf{C}$ and on the amplitudes $\boldsymbol{\alpha}$, we will denote the controlled channel by $\texttt{ctrl}_{\boldsymbol{\alpha}}^{\mathbf{C}}\text{-}\mathcal{C}$.

Different choices of Kraus operators and amplitudes generally give rise to different versions of controlled channels, with none of these versions being straightforwardly more natural than the other (although some may be more or less coherent [22]). Given that the definition of controlled channels is non-unique, an important question is how to parametrise the possible controlled channels in a compact way. As it turns out, the parametrisation $\texttt{ctrl}_{\boldsymbol{\alpha}}^{\mathbf{C}}\text{-}\mathcal{C}$ is quite redundant: in fact, many choices of $\mathbf{C}$ and of $\boldsymbol{\alpha}$ give rise to the same controlled channel $\texttt{ctrl}_{\boldsymbol{\alpha}}^{\mathbf{C}}\text{-}\mathcal{C}$.

In the following, we provide a simple one-to-one parametrisation of the possible controlled channels corresponding to a given uncontrolled channel $\mathcal{C}$: the controlled channels are in one-to-one correspondence with pairs of the form $(\mathcal{C}, C_1)$, where C_1 is a fixed Kraus operator of $\mathcal{C}$. We call the pair $(\mathcal{C}, C_1)$ a *channel with a pinned Kraus operator*.

First, we prove that any controlled version of $\mathcal{C}$ has a Kraus representation in which only the first Kraus operator is coherent with the identity:

Lemma 1. *For every controlled channel $\texttt{ctrl}_{\boldsymbol{\alpha}}^{\mathbf{C}}\text{-}\mathcal{C}$, one can find a Kraus representation in which one Kraus operator is of the form $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes C_1'$ and all the others are of the form $|1\rangle\langle 1| \otimes C_j'$, where $\mathbf{C}' := (C_j')_{j=1}^n$ is a suitable Kraus representation of channel $\mathcal{C}$. In* *other words, one has*

$$\texttt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha}}\text{-}\mathcal{C} = \texttt{ctrl}^{\mathbf{C}'}_{\mathbf{u_n}}\text{-}\mathcal{C}\,, \tag{4}$$

where $\mathbf{u}_n$ *is the* n*-dimensional column vector with a* 1 *in the first entry, and 0 in the remaining* $n-1$ *entries.*

Proof. As $\boldsymbol{\alpha}$ is a normalised vector in $\mathbb{C}^n$, one can find a unitary matrix V sending it to the basis vector $\mathbf{u}_n$, i.e. $V\boldsymbol{\alpha} = \mathbf{u}_n$. Then, the Kraus operators $(C'_j)_{j=1}^n$ defined by $C'_j := \sum_j V_{ji} C_i$ form an alternative Kraus representation of $\mathcal{C}$, and the Kraus operators $(K_j)_{j=1}^n$ defined by $K_j := \sum_j V_{ji}(\texttt{ctrl}_{\alpha_i}\text{-}C_i)$ form an alternative Kraus representation of $\texttt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha}}\text{-}\mathcal{C}$. It is straightforward to see that $K_1 = \texttt{ctrl}_1\text{-}C'_1$ and $K_j = \texttt{ctrl}_0\text{-}C - j'$ for every $j > 1$. Hence, $\texttt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha}}\text{-}\mathcal{C}$ can be characterised as in (4). □

This result removes the freedom in the choice of the amplitudes $(\alpha_i)_{i=1}^n$: one can simply set the first amplitude to 1, and all the other amplitudes to zero. All the variability of the controlled channels is then included in the choice of Kraus representation for channel $\mathcal{C}$.

We now show a further simplification: the definition of the controlled channel depends only on the choice of the *first* Kraus operator in a Kraus representation of $\mathcal{C}$. In other words, the choice of the other Kraus operators does not affect the type of control one obtains.

Lemma 2. *Let* $\mathbf{C} := (C_i)_{i=1}^m$ *and* $\mathbf{C}' := (C'_j)_{j=1}^n$ *be two Kraus representations for channel* $\mathcal{C}$*. Then, the controlled channels* $\texttt{ctrl}^{\mathbf{C}}_{\mathbf{u}_m}\text{-}\mathcal{C}$ *and* $\texttt{ctrl}^{\mathbf{C}'}_{\mathbf{u_n}}\text{-}\mathcal{C}$ *coincide if and only if the operators* C_1 *and* C'_1 *coincide. In formula,*

$$\texttt{ctrl}^{\mathbf{C}}_{\mathbf{u}_m}\text{-}\mathcal{C} = \texttt{ctrl}^{\mathbf{C}'}_{\mathbf{u}_n}\text{-}\mathcal{C} \qquad \Longleftrightarrow \qquad C_1 = C'_1\,. \tag{5}$$

Proof. We start with the direct implication. Without loss of generality, we take $m = n$, as one can always include zero Kraus operators and match the cardinality of the Kraus representations of $\texttt{ctrl}^{\mathbf{C}}_{\mathbf{u}_m}\text{-}\mathcal{C}$ and $\texttt{ctrl}^{\mathbf{C}'}_{\mathbf{u}_n}\text{-}\mathcal{C}$. If the two controlled channels coincide, then there exists a unitary matrix W that connects their Kraus representations. In particular, one must have

$$C_1 \otimes |1\rangle\langle 1| + I \otimes |0\rangle\langle 0| = W_{11}\ (\, C'_1 \otimes |1\rangle\langle 1| + I \otimes |0\rangle\langle 0|\,) + \sum_{j>1} W_{1j}\, C'_j \otimes |1\rangle\langle 1|\,. \tag{6}$$

Taking the expectation value on the vector $|0\rangle$ on both sides of the equation, we then obtain the relation $I = W_{11}\, I$, which implies $W_{11} = 1$, and, since W is a unitary matrix, $W_{1j} = 0$ for every $j > 1$. Inserting this condition in Eq. (6) we obtain $C_1 = C'_1$.

For the converse implication, suppose that $C_1 = C'_1$. Then, for an arbitrary product state

$\rho_C \otimes \rho_T$ of the control and the target, we have

$$\begin{aligned}
\mathtt{ctrl}^{\mathbf{C}}_{\mathbf{u}_m}\text{-}\mathcal{C}(\rho_C \otimes \rho_T) &= \mathtt{ctrl}_1\text{-}C_1\ (\rho_C \otimes \rho_T)\ (\mathtt{ctrl}_1\text{-}C_1)^\dagger + \sum_{i>1} \mathtt{ctrl}_0\text{-}C_i\ (\rho_C \otimes \rho_T)\ (\mathtt{ctrl}_0\text{-}C_i)^\dagger \\
&= \mathtt{ctrl}_1\text{-}C_1\ (\rho_C \otimes \rho_T)\ (\mathtt{ctrl}_1\text{-}C_1)^\dagger + |0\rangle\langle 0|\rho_C|0\rangle\langle 0| \otimes \Big(\mathcal{C}(\rho_T) - C_1\rho_T C_1^\dagger\Big) \\
&= \mathtt{ctrl}_1\text{-}C'_1\ (\rho_C \otimes \rho_T)\ (\mathtt{ctrl}_1\text{-}C'_1)^\dagger + |0\rangle\langle 0| \otimes \rho_C|0\rangle\langle 0| \Big(\mathcal{C}(\rho_T) - C'_1\rho_T {C'_1}^\dagger\Big) \\
&= \mathtt{ctrl}_1\text{-}C'_1\ (\rho_C \otimes \rho_T)\ (\mathtt{ctrl}_1\text{-}C'_1)^\dagger + \sum_{j>1} \mathtt{ctrl}_0\text{-}C'_j\ (\rho_C \otimes \rho_T)\ \big(\mathtt{ctrl}_0\text{-}C'_j\big)^\dagger \\
&= \mathtt{ctrl}^{\mathbf{C}'}_{\mathbf{u}_n}\text{-}\mathcal{C}(\rho_C \otimes \rho_T)\,. \qquad (7)
\end{aligned}$$

Since ρ_C and ρ_T are arbitrary, we conclude $\mathtt{ctrl}^{\mathbf{C}}_{\mathbf{u}_m}\text{-}\mathcal{C} = \mathtt{ctrl}^{\mathbf{C}'}_{\mathbf{u}_n}\text{-}\mathcal{C}$. ☐

Combining Lemmas 1 and 2, we obtain a non-redundant parametrisation of the possible controlled versions of a given channel:

Theorem 1. *The controlled versions of channel $\mathcal{C}$, as defined by Eq. (3), are in one-to-one correspondence with the possible choices of a single Kraus operator for channel $\mathcal{C}$.*

By 'a Kraus operator for channel $\mathcal{C}$', we mean a Kraus operator appearing in at least one Kraus representation for $\mathcal{C}$. Equivalently, the possible Kraus operators for a given channel can be characterised as follows:

Lemma 3. *An operator C_1 is a Kraus operator for channel $\mathcal{C}$ if and only if the map $\mathcal{C}_- : \rho \mapsto \mathcal{C}(\rho) - C_1\rho C_1^\dagger$ is completely positive.*

Proof. The 'only if' part is immediate. For the 'if' part, a Kraus representation for $\mathcal{C}$ containing the operator C_1 can be built by picking an arbitrary Kraus representation for the map $\mathcal{C}_-$, say $(C_i)_{i=2}^n$. For any such choice, the operators $(C_i)_{i=1}^n$ form a Kraus representation for channel $\mathcal{C}$. ☐

Hereafter, we will call the single Kraus operator picked in Theorem 1 a *pinned Kraus operator*. A channel with a pinned Kraus operator will be represented by the pair $(\mathcal{C}, C_1)$. Given a pinned Kraus operator C_1, and an arbitrary completion of it into a Kraus representation $(C_i)_i$, the corresponding controlled version of $\mathcal{C}$ is given by the Kraus operators

$$\begin{cases} \widehat{C}_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes C_1 \\ \widehat{C}_i = |1\rangle\langle 1| \otimes C_i \qquad \forall i \geq 2\,. \end{cases} \qquad (8)$$

From now on, we will use the notation $\mathtt{ctrl}_{C_1}\text{-}\mathcal{C}$ to denote the controlled channel with the above Kraus operators. The action of the controlled channel $\mathtt{ctrl}_{C_1}\text{-}\mathcal{C}$ on a generic product state of the target system and of the control is

$$\begin{aligned}
\mathtt{ctrl}_{C_1}\text{-}\mathcal{C}(\rho_C \otimes \rho_T) &= \sum_i \widehat{C}_i\,(\rho_C \otimes \rho_T)\,\widehat{C}_i^\dagger \\
&= \langle 0|\rho_C|0\rangle\ \ |0\rangle\langle 0|_C \otimes\ \rho_T \\
&\quad + \langle 1|\rho_C|1\rangle\ \ |1\rangle\langle 1|_C \otimes\ \mathcal{C}(\rho_T) \\
&\quad + \langle 1|\rho_C|0\rangle\ \ |1\rangle\langle 0|_C \otimes\ C_1\,\rho_T + \text{h.c.}\,, \qquad (9)
\end{aligned}$$

where h.c. denotes the Hermitian conjugate. In the above formula, the first two terms in the sum represent the classical control on the channel, while the second two terms represent the 'coherent part' of the controlled operation.

This pinned Kraus operator C_1 coincides with the 'transformation matrix' of Ref. [30], the 'vacuum interference operator' of Ref. [15], and the 'K operator' of Ref. [22]. Ref. [30] derived the 'transformation matrix' from a Stinespring dilation of the channel $\mathcal{C}$, and interpreted it as the additional information that has to be provided about the physical implementation of channel $\mathcal{C}$ in order to build a controlled channel. In contrast, Ref. [15] derived the 'vacuum interference operator' from an extension of channel $\mathcal{C}$ to a larger channel that can act also on the vacuum. In this paper, we will make connection with the latter approach, showing that the controlled channel $\mathtt{ctrl}_{C_1}\text{-}\mathcal{C}$ is in one-to-one correspondence, both mathematically and physically, with a particular extension of the original channel $\mathcal{C}$, corresponding to the vacuum extension of Ref. [15].

Compared to Refs. [30, 15, 22], our presentation makes it evident that the operator characterising a controlled version of channel $\mathcal{C}$ can be simply understood as a Kraus operator of this channel, a fact that has not been pointed out before.[c] In addition, the explicit relation between control and pinned Kraus operators suggests further extensions of the notion of quantum control, as discussed in the next subsection.

2.2 Control between multiple noisy channels

We now consider a generalisation of the notion of coherent control: the case in which each of the two values of the control is associated to the execution of a different channel on the target system. In other words, we now consider the coherent control between the execution of two channels $\mathcal{A}$ and $\mathcal{B}$, rather than between one channel and the identity channel. We will now take the input and output target systems, T_{in} and T_{out}, to be of possibly different dimensions.

Before entering into the technical details, it may be helpful to note that different authors have used different names for what is essentially the same notion: Refs. [3], [40], [28], [30], [15] use the expressions 'superposition of time evolutions', 'interference of CP maps', 'gluing of CP maps', 'coherent control of quantum channels', and 'superposition of quantum channels', respectively. We review the existing terminologies in Appendix 1.

If we start with the basic case of two isometric gates, represented by two isometries $U, V : \mathcal{H}_{T_{\text{in}}} \to \mathcal{H}_{T_{\text{out}}}$, the standard notion of a 'controlled-(U, V)' channel is given by the isometry

$$\mathtt{ctrl}\text{-}(U, V) := |0\rangle\langle 0| \otimes U + |1\rangle\langle 1| \otimes V \,. \tag{10}$$

Extending this definition to the case of the control between two noisy evolutions, represented by CPTP maps $\mathcal{A}, \mathcal{B} : \mathcal{L}(\mathcal{H}_{T_{\text{in}}}) \to \mathcal{L}(\mathcal{H}_{T_{\text{out}}})$, requires more work. Once again, there are a variety of ways of defining the controlled version of $\mathcal{A}$ and $\mathcal{B}$. These different versions can be obtained by picking Kraus representations of same length[d] $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ for $\mathcal{A}$ and $\mathcal{B}$ and defining the Kraus operators:

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[c] A proof in Ref. [22] mentioned that any possible 'K operator' is a Kraus operator of $\mathcal{C}$, without however discussing the reverse implication.

[d] Note that any two Kraus representations can be taken to be of the same length by adjoining 0's to the shortest one.

$$\texttt{ctrl-}(A_i, B_i) := |0\rangle\langle 0| \otimes A_i + |1\rangle\langle 1| \otimes B_i\,. \tag{11}$$

A one-to-one parametrisation of the possible choices is provided in the following theorem, proven in Appendix 2:

Theorem 2. *Given a Kraus representation $(A_i)_{i=1}^n$ of minimal length of $\mathcal{A}$, the choice of a control between $\mathcal{A}$ and $\mathcal{B}$ is in one-to-one correspondence with the choice of n Kraus operators of $\mathcal{B}$.*

By 'n Kraus operators of $\mathcal{B}$', we mean n operators that appear together in at least one Kraus representation of $\mathcal{B}$. Calling these operators B_i's, and arbitrarily completing them into a Kraus representation $(B_i)_{i=1}^{n'}$ of $\mathcal{B}$, Kraus operators for the corresponding controlled channel are given by the concatenation of the $(\texttt{ctrl-}(A_i, B_i))_{i=1}^n$ and the $(\texttt{ctrl-}(0, B_i))_{n<i\leq n'}$. Note that in this parametrisation, only the Kraus operators of $\mathcal{B}$ vary; those of $\mathcal{A}$ are fixed from the start.

The previous considerations can be extended to the case of a control system of dimension N, controlling between the execution of N channels $\mathcal{C}^1, \ldots, \mathcal{C}^N$. A strategy would be to proceed via recursion, first picking a control between $\mathcal{C}^1$ and $\mathcal{C}^2$, then picking a control between this controlled channel and $\mathcal{C}^3$, etc.

3 A new resource for coherent control: sector-preserving channels

Here we discuss the physical resources needed to implement coherent control of general quantum channels.

3.1 A no-go theorem for coherent control of unitary gates, and a way to evade it

It has been proven in various ways that it is impossible to construct a controlled unitary gate starting from a black box that implements the corresponding uncontrolled unitary gate [8, 7, 11, 9, 10, 12]. Mathematically, the no-go theorem is that it is impossible to find a quantum supermap that transforms a generic unitary channel $\mathcal{U} : \rho \mapsto U\rho U^\dagger$ into the controlled unitary channel $\texttt{ctrl-}\mathcal{U} : \rho \mapsto \texttt{ctrl-}U\,\rho\,\texttt{ctrl-}U^\dagger$ with the operator $\texttt{ctrl-}U$ defined in Eq. (1).

The origin of the impossibility is that the uncontrolled unitary channel $\mathcal{U}$ is provided as a *black box*, without any further information on its action except for the fact that $\mathcal{U}$ is known to be unitary. One way to evade the no-go theorem is to start from a device that is not a complete black box, but rather a *grey box*, whose action is partially known. For example, one could be given a device that implements a unitary gate $\widetilde{U} = |\phi_0\rangle\langle\phi_0| \oplus U$, where $\widetilde{U}$ acts on $\mathcal{H}$ and U is an unknown unitary gate acting on a d-dimensional sector (i.e. orthogonal subspace) $\mathcal{H}^1 \subseteq \mathcal{H}$, and $|\phi_0\rangle$ is another state, orthogonal to all the states in $\mathcal{H}^1$. In this case, the action of the device in the sector $\mathcal{H}^1$ is unknown, while the action of the device on the vector $|\phi_0\rangle$ is known. In this setting, the controlled gate $\texttt{ctrl-}U$ can be built from the gate $\widetilde{U}$ using a simple quantum circuit [8, 25, 11].

The use of grey boxes that act in a known way on some input states is central to all existing proposals for experimental implementations of coherent controls of unitary gates.

For example, photonic implementations [23, 24] achieve coherent control of certain optical devices, such as polarisation rotators, by exploiting the fact that such devices are passive, and therefore transform the vacuum state into itself. In these examples, the sector $\mathcal{H}^1$ is spanned by single-photon polarisation states, and the state $|\phi_0\rangle$ is the zero-photon Fock state.

In trapped-ions implementations [25, 26], the input device uses a laser pulse to implement a unitary gate by stimulating the transition between the two electronic levels. The pulse is far off resonance with the transition between the other electronic levels of the ion, and therefore the device acts trivially on such levels. In this case, the state $|\phi_0\rangle$ can be any of the levels that are unaffected by the pulse. A similar situation arises in superconducting-qubits implementations [27].

In summary, all the existing proposals of experimental implementations use grey box unitary gates $\widetilde{U}$ that act

1. as unknown gates U on a sector $\mathcal{H}^1 \simeq \mathcal{H}_T$, and
2. as the identity gate I on another sector $\mathcal{H}^0$, orthogonal to $\mathcal{H}^1$.

In the following we will extend this scheme from unitary gates to arbitrary noisy channels, and to the case of gates acting as the identity on several sectors, showing that access to a suitable grey box channel allows one to build a controlled channel that is in one-to-one correspondence with it.

We will restrict ourselves to the case in which the sectors on which the identity is applied are one-dimensional; however, all our arguments could be extended to the case in which they are multi-dimensional and the grey boxes act as the identity on each of them. Note that when the extension sectors have the same dimension as $\mathcal{H}^0$, the above requirements lead to the usual definition of controlled channels.

3.2 Modelling noisy grey boxes: sector-preserving channels

We now consider how the grey box approach of the previous section can be extended from unitary gates to arbitrary noisy channels.

To this purpose, we consider a noisy quantum channel $\widetilde{\mathcal{C}}$ that acts on a system S with a Hilbert space $\mathcal{H}_S$ partitioned into two sectors, $\mathcal{H}_S = \mathcal{H}_S^0 \oplus \mathcal{H}_S^1$, with $\mathcal{H}_S^0$ one-dimensional and $\mathcal{H}_S^1 \simeq \mathcal{H}_T$. The channel $\widetilde{\mathcal{C}}$ will act

1. as a completely unknown channel $\mathcal{C} : \mathcal{L}(\mathcal{H}_S^1) \to \mathcal{L}(\mathcal{H}_S^1)$ on the input states in $\mathcal{L}(\mathcal{H}_S^1)$, and
2. as the identity channel I on the unique input state in $\mathcal{L}(\mathcal{H}_S^0)$.

Such grey boxes have a simple characterisation: they are the channels that preserve the sectors $\mathcal{H}_S^m$, thus called sector-preserving channels.[e]

Definition 1. *Let $\mathcal{H}_S = \bigoplus_{k=0}^m \mathcal{H}_S^k$ be a Hilbert space with a preferred partition into sectors. A channel $\widetilde{\mathcal{C}} : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_S)$ is* sector-preserving *if it preserves the set of states with support in the subspace $\mathcal{H}_S^k$, for every $k \in \{0, \ldots, m\}$. In formula,*

$$\forall k,\, \forall \rho \in \mathcal{L}(\mathcal{H}_S^k), \quad \mathcal{C}(\rho) \in \mathcal{L}(\mathcal{H}_S^k)\,, \tag{12}$$

[e] We note that the notion of sector-preservingness has been independently introduced in the past, under the name 'subspace-preservingness'; see Ref. [28].

Note that $\rho \in \mathcal{L}(\mathcal{H}_S^k)$ equivalently means that $\mathsf{Supp}(\rho) \subseteq \mathcal{H}_S^k$, where $\mathsf{Supp}(\rho)$ denotes the support of ρ.

Sector-preserving channels can be seen as a special case of the notion of channels 'following a route' (i.e., satisfying given sectorial constraints), introduced in Ref. [39]: namely, they are the channels that follow the identity route $\delta \times \delta$. The condition (12) was called the 'no-leakage condition' in Ref. [15].

When some of the sectors $\mathcal{H}_S^k$ are one-dimensional, the condition of sector preservation (12) implies that the channel $\widetilde{\mathcal{C}}$ acts as the identity channel on each of them. In the following, we will denote the sector preserving channels with $\dim(\mathcal{H}_S^k) = 1\,\forall k < m$ and $\dim(\mathcal{H}_S^m) = d$ as *sector-preserving channels of type* $(\underbrace{1, \dots, 1}_{m \text{ times}}, d)$. In particular, the channels we asked for in this Section are the sector-preserving channels of type $(1, d)$.

The approach of considering an extended channel that acts as $\mathcal{C}$ on a given sector was introduced in Ref. [15]. There, there was only one one-dimensional sector, which was called the 'vacuum sector', and the channel $\widetilde{\mathcal{C}}$ was called a 'vacuum extension', with this terminology motivated by the photonic implementations. Here, however, we prefer to use the expressions ' extension sectors' and 'extended channel', which are neutral with respect to the choice of experimental implementations.

The key point of our paper is that the grey box channel $\widetilde{\mathcal{C}}$, and not the black box channel $\mathcal{C}$, should be regarded as the initial resource for the implementation of coherent control. In other words, we argue that one should shift the terms of the problem away from the question 'what can one do with an unknown channel $\mathcal{C}$?'. Instead, one should ask the question 'what can one do with a channel $\widetilde{\mathcal{C}}$ that acts as an unknown channel on a given sector?'.

A similar shift of perspective was proposed in Refs. [15, 17, 16] for the purpose of defining quantum communication protocols where messages can travel in a coherent superposition of multiple trajectories. In this context, extended channels were used to describe communication devices that can take as input either one particle (corresponding, in our notations, to the sector $\mathcal{H}_S^1$) or the vacuum (corresponding to the sector $\mathcal{H}_S^0$). This modelling was essential to define resource theories of quantum communication [17], where the initial resources are communication devices that can be connected in a coherent superposition of multiple configurations. Our paper can be viewed as an application of the same approach to the task of the coherent control of quantum channels: the extended channel represents the initial resource, and the question is which types of controlled channel can be constructed from such resource.

3.3 The case of one extension sector

The case where there is only one extension sector $\mathcal{H}_S^1$ (i.e., of sector-preserving channels of type $(1, d)$) is particularly relevant in this paper, because, as we will show later, it provides the fundamental resource for the realisation of the controlled channels defined in Eq. (3).

In terms of Kraus representation, the sector-preserving channels of type $(1, d)$ can be characterised as the channels with Kraus operators of the form

$$\widetilde{C}_i = \alpha_i \oplus C_i, \tag{13}$$

where $(C_i)_i$ is a Kraus representation of some channel acting on sector $\mathcal{H}_S^1 \simeq \mathcal{H}_T$, and the α_i's are amplitudes satisfying the normalisation condition $\sum_i |\alpha_i|^2 = 1$. For a proof of the above equation, see Lemma 1 in Ref. [15] (this can also be seen as a consequence of the more general Theorem 6 in Ref. [39]).

A one-to-one parametrisation of the sector-preserving channels of type $(1, d)$ can be obtained with the same approach as in Section 2.

Lemma 4. *Every sector-preserving channel of type $(1, d)$ has a Kraus representation of the form*

$$\begin{cases} \widetilde{C}_1 = 1 \oplus C_1 \\ \widetilde{C}_i = 0 \oplus C_i \qquad \forall i \geq 2\,, \end{cases} \tag{14}$$

where $(C_i)_i$ is a Kraus representation of some channel on the d-dimensional sector.

Proof. As in the proof of Lemma 1, this alternative Kraus representation can be found by using a unitary matrix $(V_{ji})_{ji}$ that sends the normalised vector $(\alpha_i)_i$ to $(1, 0, \ldots, 0)$. □

Using the same arguments as in Section 2, it is easy to see that the sector-preserving channels $\widetilde{\mathcal{C}}$ are in one-to-one correspondence with pairs $(\mathcal{C}, C_1)$, consisting of a channel acting on sector $\mathcal{L}(\mathcal{H}_S^1)$, and of a Kraus operator for $\mathcal{C}$. In short, we have the following.

Theorem 3. *The sector-preserving channels of type $(1, d)$ are in one-to-one correspondence with channels with a pinned Kraus operator on their d-dimensional sector.*

The sector-preserving channel of type $(1, d)$ that corresponds to the channel $\mathcal{C}$ with the pinned Kraus operator C_1 on its d-dimensional sector shall be called $\widetilde{\mathcal{C}}[C_1]$. In the case of unitary channels, the characterisation is particularly simple.

Corollary 1. *Sector-preserving unitary channels of type $(1, d)$ are in one-to-one correspondence with unitary operators in dimension d. Explicitly, the correspondence between sector-preserving unitary channels $\widetilde{\mathcal{U}}$ and unitary operators U is given by the relation*

$$\widetilde{\mathcal{U}}(\rho) = (1 \oplus U)\ \rho\ (1 \oplus U)^\dagger \qquad \forall \rho \in \mathcal{L}(\mathcal{H}_S)\,. \tag{15}$$

This is in contrast with the general situation for unitary channels, which correspond to unitary operators only up to an arbitrary global phase. The crucial fact here is that the one-dimensional extension sector can be used to fix this phase gauge in the d-dimensional sector.

Going back to the case of general channels, Theorem 3 establishes a one-to-one correspondence between sector-preserving channels of type $(1, d)$ and controlled channels:

Corollary 2. *For any d, the following sets are in one-to-one correspondence:*

1. *controlled channels as defined in (3), with a d-dimensional target system;*
2. *sector-preserving channels of type $(1, d)$;*
3. *channels with a pinned Kraus operator in dimension d.*

Let us comment on the respective roles, for our purposes, of the three notions which Corollary 2 shows to be mathematically equivalent. The first (controlled channels) is essentially an informational notion, with practical use in quantum protocols: this is typically what one wants to eventually obtain. The second (sector-preserving channels of type $(1, d)$) can be

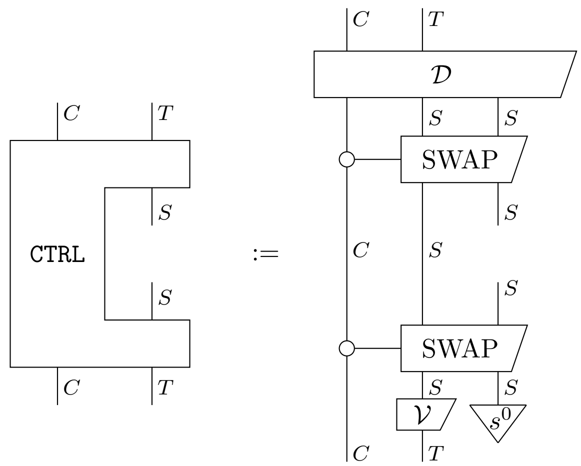

Fig. 2. Quantum circuit for `CTRL` supermap. The supermap transforms sector-preserving channels acting on a system S with Hilbert space $\mathcal{H}_S = \mathcal{H}_S^0 \oplus \mathcal{H}_S^1$ into controlled channels acting on the composite system $C \otimes T$, consisting of a control system C and of a target system T with Hilbert space $\mathcal{H}_T \simeq \mathcal{H}_S^1$. The sector-preserving channel in input is inserted between two controlled `SWAP` operations, which in turn are placed between two quantum channels $\mathcal{V}$ and $\mathcal{D}$, which serve as 'adaptors', between the systems T and S, and between the systems $C \otimes S \otimes S$ and $C \otimes T$, respectively.

understood as the physical resource (with the sector-preserving property often corresponding to physical features of an interaction, such as conservation laws) allowing to implement the first one. Finally, the third (channels with a pinned Kraus operator) is a purely mathematical notion, with no direct practical interpretation, which serves to provide a simple one-to-one mathematical parametrisation to the first two.

In fact, a more careful inspection also reveals that the one-to-one correspondence between the above sets can be implemented by linear maps. For the sets of controlled channels and sector-preserving channels, the correspondence can be implemented physically by quantum circuits that convert sector-preserving channels into controlled channels, and vice-versa. This physical correspondence is the object of the next section.

4 The control supermap and the equivalence between sector-preserving and controlled channels

4.1 The control supermap

In the previous section, we showed that the controlled channels on target systems of dimension d (the $\texttt{ctrl}_{C_1}$-$\mathcal{C}$) are in one-to-one correspondence with sector-preserving channels of type $(1, d)$ (the $\widetilde{\mathcal{C}}[C_1]$).

Our point is now to show that for any given d, there is a universal circuit architecture in which an agent who possesses the sector-preserving channel $\widetilde{\mathcal{C}}[C_1]$ can insert this channel in order to implement the controlled channel $\texttt{ctrl}_{C_1}$-$\mathcal{C}$.

We thus introduce the *control supermap*, a supermap which takes as input any sector-preserving channel $\widetilde{\mathcal{C}}[C_1]$ of type $(1, d)$, and yields the controlled channel $\texttt{ctrl}_{C_1}$-$\mathcal{C}$ acting on a target system of dimension d.

Theorem 4. *Let $\mathcal{H}_S = \mathcal{H}_S^0 \oplus \mathcal{H}_S^1$ be a Hilbert space, with $\dim(\mathcal{H}_S^0) = 1$ and $\dim(\mathcal{H}_S^1) = d$, let $\mathcal{H}_C$ be a control space of dimension 2, and $\mathcal{H}_T$ be a target space, with $\mathcal{H}_T \simeq \mathcal{H}_S^1$.*

There exists a supermap $\texttt{CTRL}$ *of type $(S \to S) \to (C \otimes T \to C \otimes T)$ such that for any sector-preserving channel $\widetilde{\mathcal{C}}[C_1]$,*

$$\texttt{CTRL}[\widetilde{\mathcal{C}}[C_1]] = \texttt{ctrl}_{C_1}\text{-}\mathcal{C}\,. \tag{16}$$

Furthermore, this supermap is unitary-preserving on the sector-preserving channels on S.

Proof. Let $V : \mathcal{H}_T \to \mathcal{H}_S$ be the isometry that maps $\mathcal{H}_T$ into the subspace $\mathcal{H}_S^1 \simeq \mathcal{H}_T$, let $|s^0\rangle$ be a unit vector in $\mathcal{H}_S^0$, let $W : \mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_S \to \mathcal{H}_C \otimes \mathcal{H}_T$ be the coisometry defined by $W := I \otimes V^\dagger \otimes \langle s^0|$, and let $\mathcal{D}$ be the quantum channel defined by $\mathcal{D}(\rho) := W\rho W^\dagger + \rho_0\ \mathrm{Tr}[P\,\rho]$, where ρ_0 is a fixed density matrix on $\mathcal{H}_C \otimes H_T$ and $P := I - W^\dagger W$[f]. We then define the supermap $\texttt{CTRL}$ through its action on a generic linear map $\mathcal{M} : \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_T) \to \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_T)$:

$$\texttt{CTRL}(\mathcal{M}) := \mathcal{D} \circ \texttt{ctrl-}\mathcal{SWAP} \circ (\mathcal{I}_C \otimes \mathcal{I}_S \otimes \mathcal{M}) \circ \texttt{ctrl-}\mathcal{SWAP} \circ (\mathcal{I}_C \otimes \mathcal{V} \otimes |s^0\rangle\langle s^0|) \tag{17}$$

where $\mathcal{V}$ is the quantum channel corresponding to the isometry V, and $\texttt{ctrl-}\mathcal{SWAP}$ is the unitary channel corresponding to the controlled SWAP operator (see Figure 2 for an illustration).

With this definition, one can verify that the condition $\texttt{CTRL}[\widetilde{\mathcal{C}}[C_1]] = \texttt{ctrl}_{C_1}\text{-}\mathcal{C}$ holds. Let us prove it by showing that they act in the same way on pure states, using a Kraus representation for the channel $\widetilde{\mathcal{C}}[C_1]$ with Kraus operators $\widetilde{C}_i = \delta_{i1} \oplus C_i$. From there, it can then be deduced by linearity that the two channels act in the same way on any density matrix, and therefore that they are equal. We take a strict equality $T = S^1$ to avoid unnecessary clutter.

Taking an arbitrary state $|\psi\rangle_{CT}$, we obtain

$$\texttt{ctrl-}\mathrm{SWAP}(V \otimes |s^0\rangle)\,|\psi\rangle_{CT} = |0\rangle_C \otimes {}_C\langle 0|\psi\rangle_{CS} \otimes |s^0\rangle_S + |1\rangle_C \otimes |s^0\rangle_S \otimes {}_C\langle 1|\psi\rangle_{CS}$$

and thus

$$\begin{aligned}
|\psi_i\rangle &:= \texttt{ctrl-}\mathrm{SWAP}\,(I_C \otimes I_S \otimes \widetilde{C}_i)\,\texttt{ctrl-}\mathrm{SWAP}(V \otimes |s^0\rangle)\,|\psi\rangle_{CT} \\
&= \texttt{ctrl-}\mathrm{SWAP}\,\big(\delta_{i1}\ |0\rangle_C \otimes {}_C\langle 0|\psi\rangle_{CS} \otimes |s^0\rangle_S + |1\rangle_C \otimes |s^0\rangle_S \otimes C_i\ {}_C\langle 1|\psi\rangle_{CS}\big) \\
&= \big(\delta_{i1}\,|0\rangle_C \otimes \langle 0|\psi\rangle_{CS} + C_i\ \ |1\rangle_C \otimes \langle 1|\psi\rangle_{CS}\big) \otimes |s^0\rangle\ .
\end{aligned} \tag{18}$$

Now, one has $P\,|\psi_i\rangle = 0$, and $W\,|\psi_i\rangle = \delta_{i1}\ |0\rangle \otimes {}_C\langle 0|\psi\rangle_{CS} + |1\rangle_C \otimes C_i\ \ {}_C\langle 1|\psi\rangle_{CS} \equiv \texttt{ctrl}_{\delta_{i1}}\text{-}C_i\ |\psi\rangle_{CT}$. Summarising, if the control and target start off in the state $|\psi\rangle_{CT}$, and if the subprocess corresponding to the Kraus operator $\widetilde{C}_i$ takes place, then the final (subnormalized)

[f] Note that the only thing that matters is how $\mathcal{D}$ acts on the sector $\mathcal{H}_C \otimes \mathcal{H}_S^1 \otimes \mathcal{H}_S^0$ of its input; its action on other sectors is irrelevant and can be defined in an arbitrary way, as long as it gives a CPTP map.

state is $W|\psi_i\rangle = \texttt{ctrl}_{\delta_{i1}}\text{-}C_i\ |\psi\rangle_{CT}$. On average over all possible values of i, we obtain the evolution

$$\begin{aligned}\texttt{CTRL}(\widetilde{\mathcal{C}}[C_1])(|\psi\rangle\langle\psi|) &= \sum_i W|\psi_i\rangle\langle\psi_i|W^\dagger \\ &= \sum_i (\texttt{ctrl}_{\delta_{i1}}\text{-}C_i)\,|\psi\rangle\langle\psi|\,(\texttt{ctrl}_{\delta_{i1}}\text{-}C_i)^\dagger \\ &= \texttt{ctrl}_{C_1}\text{-}\mathcal{C}(|\psi\rangle\langle\psi|)\,.\end{aligned}$$

As for the preservation of unitarity on sector-preserving channels, it is sufficient to recall Corollary 1: unitary sector-preserving channels of type $(1,d)$ are of the form $\widetilde{\mathcal{U}} : \rho \mapsto (1 \oplus U)\rho(1 \oplus U)^\dagger$. By the previous calculation, one then has $\texttt{CTRL}(\widetilde{\mathcal{U}}) : \rho \mapsto (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U)\rho(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U)^\dagger$, which is a unitary channel.

□

The supermap CTRL constitutes a rigorous theoretical formalisation of the existing experimental schemes for the implementation of coherent control. It is the universal protocol through which sector-preserving channels of type $(1,d)$ can be turned into their corresponding controlled channel.

We note that even though we defined this supermap as accepting as input *any* possible channel $S \to S$, the only thing we are interested in is in fact its action on *sector-preserving* channels. An alternative way of defining it would be to formally restrict its inputs to be only sector-preserving channels (or extensions of those); this would make clearer the fact that this protocol is only useful when sector-preserving channels are used, and would also allow to get rid of superfluous information in the specification of the supermap – namely, information that only modifies the action of the supermap on non-sector-preserving channels. We will do this in Section 7, coining the notion of *supermaps on routed channels*.

Let us also comment on the specific case of unitary channels. Per Corollary 1, we know that sector-preserving unitary *channels* of type $(1,d)$ are in one-to-one correspondence with unitary *operators* on their d-dimensional sector. Noting as U the unitary operator corresponding to the unitary sector-preserving channel $\widetilde{\mathcal{U}}$, the control supermap will then precisely map any sector-preserving unitary channel $\widetilde{\mathcal{U}}$ to the gate applying the controlled-unitary $\texttt{ctrl}\text{-}U$ defined in equation (1):

$$\forall \widetilde{\mathcal{U}}\ \text{unitary},\ \texttt{CTRL}[\widetilde{\mathcal{U}}] = \texttt{ctrl}\text{-}U\,. \tag{19}$$

The control supermap thus also realises, in particular, the coherent control of unitary gates.

4.2 Sector-preserving and controlled channels are equivalent resources

The previous section showed that there is a universal circuit structure which turns sector-preserving channels of type $(1,d)$ into their corresponding controlled channel. As resources, sector-preserving channels of type $(1,d)$ thus allow one to obtain controlled channels. We now show the opposite: from a controlled channel, one can obtain its corresponding sector-preserving channel of type $(1,d)$, once again using a universal circuit structure.

Theorem 5. *Let $\mathcal{H}_T \simeq \mathcal{H}_S^1$ be a target space, and let $\mathcal{H}_C$ be a control space of dimension 2. Taking $\mathcal{H}_S^0 \cong \mathbb{C}$, $\mathcal{H}_S^1 \simeq \mathcal{H}_T$ and $\mathcal{H}_S := \mathcal{H}_S^0 \oplus \mathcal{H}_S^1$, there exists a supermap $\mathtt{CTRL}^{-1}$ of type $(C \otimes T \to C \otimes T) \to (S \to S)$ such that for any controlled channel $\mathtt{ctrl}_{C_1}$-$\mathcal{C}$,*

$$\mathtt{CTRL}^{-1}[\mathtt{ctrl}_{C_1}\text{-}\mathcal{C}] = \widetilde{\mathcal{C}}[C_1]\,. \tag{20}$$

Furthermore, this supermap is unitary-preserving on the controlled channels on $C \otimes T$.

Proof. One can define $\mathtt{CTRL}^{-1}$'s action on a given map $\mathcal{K}$ of type $C\otimes T \to C\otimes T$ as $\mathtt{CTRL}^{-1}[\mathcal{K}] = \mathcal{W} \circ \mathcal{K} \circ \mathcal{V}$, where $\mathcal{V}$ is the channel corresponding to the isometry $V : \mathcal{H}_S \to \mathcal{H}_C \otimes \mathcal{H}_T$ that acts as $V|\psi\rangle = |1\rangle \otimes |\psi\rangle$ for $|\psi\rangle \in \mathcal{H}_S^1$, and $V|\psi\rangle = |0\rangle \otimes |\phi_0\rangle$ for $|\psi\rangle \in \mathcal{H}_S^0$ where $|\phi^0\rangle$ is a fixed arbitrary state in $\mathcal{H}_T$, and channel $\mathcal{W}$ acts as $\mathcal{V}^\dagger$ on $\mathcal{V}$'s range and in an arbitrary way elsewhere.

From this definition, a simple computation shows that (20) holds. □

The existence of this inverse control supermap shows that sector-preserving channels of type $(1, d)$ and controlled channels are fully equivalent resources: one can go from a sector-preserving channel to its corresponding controlled channel and back again, using a universal circuit architecture in both cases. This concludes our demonstration of the main claim of this paper.

Note that $\mathtt{CTRL}^{-1} \circ \mathtt{CTRL}$ acts as the identity supermap only on input channels that are sector-preserving. A way of formally restricting the $\mathtt{CTRL}$ supermap to only act on sector-preserving channels will be described in Section 7. Once viewed in this way, the $\mathtt{CTRL}$ supermap can be said to be unitary-preserving and invertible.

5 Implementing coherent control of multiple channels

5.1 The case of isometric channels

We now show how the previous methods apply to the coherent control of $N \geq 2$ channels, as defined in Section 2.2. For simplicity, we restrict ourselves to the case of isometric channels, and to $N = 2$. The methods we present are readily extendable to the $N > 2$. Note that the coherent control of isometric gates includes that of unitary gates and of pure states, as both are specific examples of isometric gates.

If we define the task of coherent control between two isometric gates as that of implementing controlled-(U, V) (as defined in equation (10)) from uses of the isometric gates U and V, then it is a direct consequence of the aforementioned no-go theorems that such a task cannot be achieved via a universal circuit architecture.

To circumvent this, we will instead keep our perspective of considering coherent control as a task performed on sector-preserving channels. Here, as in Section 2.2, we take the input and output target systems to be of possibly different dimensions. Accordingly, we will slightly extend the relevant definitions. For instance, Definition 1 can be extended in a straightforward way to encompass sector-preserving channels from $\mathcal{H}_{S_{\text{in}}} := \bigoplus_k \mathcal{H}_{S_{\text{in}}}^k$ to $\mathcal{H}_{S_{\text{out}}} := \bigoplus_k \mathcal{H}_{S_{\text{out}}}^k$. In the case in which the Hilbert spaces are both partitioned between a multi-dimensional sector and several one-dimensional ones, we will refer to these channels as being sector-preserving of type $(1 \to 1, \ldots, 1 \to 1, d \to d')$. Structural theorems about these channels can be seen to extend from those of Section 3.3 (Lemma 4, Theorem 3 and Corollaries 1 and 2) in a natural way.

In particular, Corollary 1 can be extended to a statement about isometric sector-preserving channels $\mathcal{C}$ of type $(1 \to 1, d \to d')$: they are in one-to-one correspondence with isometric operators $U_{\mathcal{C}}$ in dimension $d \to d'$. Our point is to implement this correspondence physically in order to create a control between two isometric gates. We single out a version of the control supermap that allows one to build the coherent control between two isometric gates from the two sector-preserving isometric channels of type $(1 \to 1, d \to d')$ corresponding to these isometries. This supermap was originally introduced in Ref. [15] (in the case $d = d'$), in a slightly different framework.

Theorem 6. *Let $\mathcal{H}_{S_{\text{in}}} = \mathcal{H}^0_{S_{\text{in}}} \oplus \mathcal{H}^1_{S_{\text{in}}}$ and $\mathcal{H}_{S_{\text{out}}} = \mathcal{H}^0_{S_{\text{out}}} \oplus \mathcal{H}^1_{S_{\text{out}}}$ be partitioned spaces, with $\mathcal{H}^0_{S_{\text{in}}}$ and $\mathcal{H}^0_{S_{\text{out}}}$ one-dimensional, let $\mathcal{H}_C$ be a control space of dimension 2, and let $\mathcal{H}_{T_{\text{in}}}$ and $\mathcal{H}_{T_{\text{out}}}$ be target spaces, with $\mathcal{H}_{T_{\text{in}}} \simeq \mathcal{H}^1_{S_{\text{in}}}$ and $\mathcal{H}_{T_{\text{out}}} \simeq \mathcal{H}^1_{S_{\text{out}}}$.*

There exists a supermap `2-CTRL` *of type $(S_{\text{in}} \to S_{\text{out}}) \otimes (S_{\text{in}} \to S_{\text{out}}) \to (C \otimes T_{\text{in}} \to C \otimes T_{\text{out}})$ such that for any pair of isometric sector-preserving channels $\mathcal{C}$ and $\mathcal{D}$,*

$$\texttt{2-CTRL}[\mathcal{C} \otimes \mathcal{D}] = \texttt{ctrl-}(U_{\mathcal{C}}, U_{\mathcal{D}})\,. \tag{21}$$

Proof. This can be easily computed from the formulation of the `2-CTRL` supermap shown in Figure 3, in full analogy to the computation in the proof of Theorem 4. □

Theorem 6 can serve as a formalisation of the existing experimental schemes for coherently controlling two unitaries, such as the superposition of paths [15]. It is easy to see that it could be readily generalised to the coherent control between N isometries by a control system of dimension N.

In particular, one can see in this formulation that the coherent control of two isometries can be implemented with a simple parallel combination of the two resource sector-preserving channels.

5.2 What about general channels?

A natural question to ask would be whether the previous result can be extended to the case of controls between two general noisy channels, as defined in equation (11) and classified in Theorem 2: i.e., whether a given version of a control between two channels $\mathcal{A}$ and $\mathcal{B}$ can be obtained from the application of the `2-CTRL` supermap on suitably chosen sector-preserving channels of type $(1 \to 1, d \to d')$. The answer to this question, however, is negative.

To see this, take $\mathcal{A} = \mathcal{B} = \mathcal{D}$, where $\mathcal{D}$ is the depolarising channel on a qubit, i.e. $\mathcal{D} : \rho \mapsto \frac{1}{2}(\rho + Z\rho Z)$. One natural version of a control between $\mathcal{A}$ and $\mathcal{B}$ is then given by the channel $\mathcal{I}_C \otimes \mathcal{D}_{S^1}$: i.e., $\mathcal{D}$ is always applied to S^1 and the control doesn't play any part. However, no use of the `2-CTRL` supermap on $\mathcal{A}$ and $\mathcal{B}$ can yield this channel. This is essentially because, in channels obtained from the use of the `2-CTRL` supermap, there can only be full coherence between one Kraus operator of $\mathcal{A}$ and one Kraus operator of $\mathcal{B}$.

Implementing the control between two noisy channels in general will therefore require the use of a more elaborate scheme, using more involved resources. In Appendix C, we propose such a scheme. Rather than sector-preserving channels of the form $\mathbb{C} \oplus \mathcal{H}^1_{S_{\text{in}}} \to \mathbb{C} \oplus \mathcal{H}^1_{S_{\text{out}}}$, this scheme will require the use of sector-preserving channels of the form $\mathbb{C} \oplus \mathcal{H}^1_{S_{\text{in}}} \to \mathbb{C} \oplus \mathcal{H}^1_{S_{\text{out}}} \otimes \mathcal{H}^1_E$, where $\mathcal{H}^1_E$ is an auxiliary Hilbert space, representing the environment. In such a scheme, the

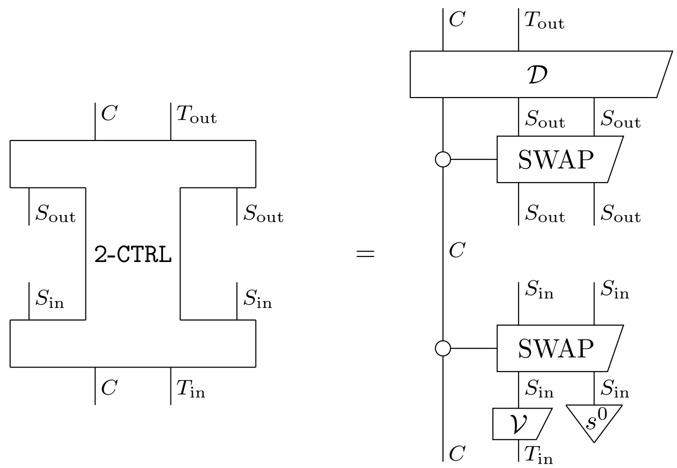

Fig. 3. Quantum circuit for the 2-CTRL supermap. The input of the supermap are two sector-preserving channels transforming a system S_{in} with Hilbert space $\mathcal{H}_{S_{\text{in}}} = \mathcal{H}^0_{S_{\text{in}}} \oplus \mathcal{H}^1_{S_{\text{in}}}$ into a system S_{out} with Hilbert space $\mathcal{H}_{S_{\text{out}}} = \mathcal{H}^0_{S_{\text{out}}} \oplus \mathcal{H}^1_{S_{\text{out}}}$. The output of the supermap is a controlled channel transforming the composite system $C \otimes T_{\text{in}}$ with $\mathcal{H}_{T_{\text{in}}} \simeq \mathcal{H}_{S^1_{\text{in}}}$ into the composite system $C \otimes T_{\text{out}}$ with $\mathcal{H}_{T_{\text{out}}} \simeq \mathcal{H}_{S^1_{\text{out}}}$. The channels $\mathcal{V}$ and $\mathcal{D}$ and the state $|s^0\rangle$ are defined as in Theorem 4. A very similar supermap was defined in Ref. [15] for the case $T_{\text{in}} = T_{\text{out}}$.

number of Kraus operators of $\mathcal{A}$ and $\mathcal{B}$ that can be coherent with each other in the produced controlled channel is capped by the dimension of $\mathcal{H}^1_E$.

6 Compositely-controlled channels

In this section, we consider another generalisation of the notion of controlled quantum channels, corresponding to higher-dimensional control systems, and we show how to implement it, via a universal architecture, using as resources sector-preserving channels of type $(d, 1, \dots, 1)$.

6.1 Compositely-controlled channels and multiple pinned operators

We introduce a generalisation that can be useful in the description of quantum programs, which may contain instructions of the form 'if $f(x) = 1$, then execute channel $\mathcal{C}$, otherwise do nothing', where f is a Boolean function taking as input a parameter x labelling the different branches of the computational process.

To get started, consider a three-dimensional control system C, with basis states $\{|0\rangle, |1\rangle, |2\rangle\}$. We associate state $|0\rangle$ to the execution of the given channel $\mathcal{C}$, and states $|1\rangle$ and $|2\rangle$ to the 'do nothing' option. This corresponds to choosing the Boolean function f to be $f(0) = 1$ and $f(1) = f(2) = 0$. A controlled channel can then be defined in terms of the Kraus operators

$$\texttt{ctrl}_{\alpha_i,\beta_i}\text{-}C_i := |0\rangle\langle 0| \otimes C_i + \alpha_i\, |1\rangle\langle 1| \otimes I + \beta_i\, |2\rangle\langle 2| \otimes I\,, \tag{22}$$

where $\mathbf{C} := (C_i)_{i=1}^n$ is a Kraus representation of channel $\mathcal{C}$, and $\boldsymbol{\alpha} := (\alpha_i)_{i=1}^n$ and $\boldsymbol{\beta} := (\beta_i)_{i=1}^n$ are complex amplitudes satisfying the normalisation conditions $\sum_{i=1}^n |\alpha_i|^2 = 1$ and $\sum_{i=1}^n |\beta_i|^2 = 1$, respectively. We shall call a controlled channel as defined in (22) a 2-*compositely-controlled* channel. In the following, this controlled channel will be denoted by

$\mathtt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$-$\mathcal{C}$.

As with standard controlled channels, different choices of Kraus representations and of amplitudes generally lead to different kinds of controlled channels, and again, one may ask for a one-to-one parametrisation. The generalisation of Theorem 1 is the following.

Theorem 7. *The 2-compositely-controlled versions of channel $\mathcal{C}$, as defined by Eq. (22) are in one-to-one[g] correspondence with triples of the form $(C'_1, C'_2, \gamma_{12})$, where C'_1 and C'_2 are two Kraus operators for channel $\mathcal{C}$, and of a complex amplitude $\gamma_{12} \in \mathbb{C}$ satisfying $|\gamma_{12}| \leq 1$. Explicitly, the Kraus operators for the controlled channel can be written as*

$$\begin{cases} \widehat{C}'_1 = |0\rangle\langle 0| \otimes C'_1 + |1\rangle\langle 1| \otimes I + \gamma_{12}\ |2\rangle\langle 2| \otimes I \\ \widehat{C}'_2 = |0\rangle\langle 0| \otimes C'_2 + \sqrt{1-|\gamma_{12}|^2}\ |2\rangle\langle 2| \otimes I \\ \widehat{C}'_i = |0\rangle\langle 0| \otimes C'_i \qquad \forall i \geq 3\,, \end{cases} \tag{23}$$

for some suitable Kraus representation $(C'_i)_i$ of channel $\mathcal{C}$.

Proof. The proof is a generalisation of the proof of Theorem 1. Starting from the Kraus operators in Eq. (22), one can generate a new Kraus representation of the controlled channel using a unitary matrix. To choose the appropriate unitary matrix, we apply the Gram-Schmidt construction to the column vectors $|\alpha\rangle = (\alpha_i)_{i=1}^n$ and $|\beta\rangle = (\beta_i)_{i=1}^n$. In other words, we construct an orthonormal basis $(|v_i\rangle)_{i=1}^n$ where the first vector is $|v_1\rangle = |\alpha\rangle$ and the second vector is $|v_2\rangle = |\beta\rangle - \langle\alpha|\beta\rangle\ |\alpha\rangle / \|\ |\beta\rangle - \langle\alpha|\beta\rangle\ |\alpha\rangle\ \|$. One can then define the unitary operator $U = \sum_{j=1}^n |j\rangle\langle v_j|$, and use its matrix elements to define a new Kraus representation $\widehat{C}_j = \sum_i U_{ji}\,\mathtt{ctrl}_{\alpha_i,\beta_i}\text{-}C_i$. Explicit calculation of the Kraus operators yields Eq. (23), with $\gamma_{12} = \langle\alpha|\beta\rangle$, $C'_1 = \sum_i \overline{\alpha}_i\, C_i$ and $C'_2 = \sum_i (\overline{\beta}_i - \langle\beta|\alpha\rangle\ \overline{\alpha}_i)/\sqrt{1-|\gamma_{12}|^2}\, C_i$.

For every given controlled channel $\mathtt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$-$\mathcal{C}$, the pinned Kraus operators C_1 and C_2, and the amplitude γ_{12} can be uniquely determined from the action of the channel on a generic product state of the target and the control. Explicitly, one has

$$\begin{aligned} &\mathtt{ctrl}^{\mathbf{C}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\text{-}\mathcal{C}(\rho_C \otimes \rho_T) \\ &\quad = \langle 0|\rho_C|0\rangle\ \ |0\rangle\langle 0|_C \otimes \mathcal{C}(\rho_T) \\ &\qquad + \langle 1|\rho_C|1\rangle\ \ |1\rangle\langle 1|_C \otimes \rho_T \\ &\qquad + \langle 2|\rho_C|2\rangle\ \ |2\rangle\langle 2|_C \otimes \rho_T \\ &\qquad + \langle 0|\rho_C|1\rangle\ C'_1\ |0\rangle\langle 1|_C \otimes \rho_T + \text{h.c.} \\ &\qquad + \langle 0|\rho_C|2\rangle\ \sqrt{1-|\gamma_{12}|^2}\ \ |0\rangle\langle 2|_C \otimes C'_2\,\rho_T + \text{h.c.} \\ &\qquad + \langle 1|\rho_C|2\rangle\ \gamma_{12}\ |1\rangle\langle 2|_C \otimes \rho_T + \text{h.c.}\,, \end{aligned} \tag{24}$$

from which the operators C_1 and C_2, and the amplitude γ_{12} can be extracted by taking the appropriate matrix elements of the output state.

In summary, every 2-compositely-controlled channel as defined by Eq. (22) can be parameterised by two pinned Kraus operators (C'_1, C'_2) and one amplitude γ_{12}, and the triple $(C'_1, C'_2, \gamma_{12})$ is uniquely determined by the channel. □

[g] Except in the case where $|\gamma_{12}| = 1$; the choice of C'_2 is then irrelevant. Given that this is a set of measure 0, we will neglect the existence of this case in the rest of this paper.

The above notion of controlled channel can be easily extended to higher dimensional systems, introducing controlled Kraus operators of the form

$$\texttt{ctrl-}C_i = |0\rangle\langle 0| \otimes C_i + \sum_{k=1}^{m} \alpha_i^k \; |k\rangle\langle k| \otimes I \,, \tag{25}$$

where, for each $k \in \{1, \dots, m\}$, the amplitudes $(\alpha_i^k)_i$ satisfy the normalisation condition $\sum_i \left|\alpha_i^k\right|^2 = 1$. Controlled channels of the form (25) will be called m-compositely controlled channels. In this case, the controlled channel is in one-to-one correspondence with m pinned Kraus operators, and $m-1$ complex amplitudes: using the same argument as in the proof of Theorem 7, one can show that the controlled channel has a Kraus representation of the form

$$\begin{cases} \widehat{C}_1 = |0\rangle\langle 0| \otimes C_1' + |1\rangle\langle 1| \otimes I + \sum_{k>1} \gamma_{1k} \; |k\rangle\langle k| \otimes I \\ \widehat{C}_2 = |0\rangle\langle 0| \otimes C_2' + \sqrt{1-|\gamma_{12}|^2} \; |2\rangle\langle 2| \otimes I + \sum_{k>2} \gamma_{2k} \; |k\rangle\langle k| \otimes I \\ \widehat{C}_3 = |0\rangle\langle 0| \otimes C_3' + \sqrt{1-|\gamma_{13}|^2 - |\gamma_{23}|^2} \; |3\rangle\langle 3| \otimes I + \sum_{k>3} \gamma_{3k} \; |k\rangle\langle k| \otimes I \\ \vdots \\ \widehat{C}_j = |0\rangle\langle 0| \otimes C_j' + \sqrt{1-\sum_{i<j} |\gamma_{ij}|^2} \; |j\rangle\langle j| \otimes I + \sum_{k>j} \gamma_{jk} \; |k\rangle\langle k| \otimes I \qquad \forall j \le m \\ \widehat{C}_j = |0\rangle\langle 0| \otimes C_j' \qquad \forall j > m \,, \end{cases} \tag{26}$$

where (C_i') is a Kraus representation of channel $\mathcal{C}$, and $(\gamma_{ij})_{i<k\le m}$ are suitable amplitudes.

In summary, controlled channels can represent if-then clauses in the execution of a quantum program, and every branch of the program corresponding to the 'do nothing' instruction corresponds to a pinned Kraus operator.

6.2 A resource: sector-preserving channels of type $(d, 1 \dots, 1)$

We now consider the types of channels that can be used as resources for the implementation of m-composite control. To do this, we extend the approach of section 3.2 to consider sector-preserving channels with one d-dimensional sector and m 1-dimensional sectors, i.e. those of type $(d, \underbrace{1, \dots, 1}_{m \text{ times}})$.

The Kraus operators of such channels have the form

$$\widetilde{C}_i = C_i \oplus \alpha_i^1 \; \oplus \alpha_i^2 \oplus \cdots \oplus \alpha_i^m \,, \tag{27}$$

where $(C_i)_i$ is a Kraus representation of a channel $\mathcal{C}$ in dimension d, and, for every $k \in \{1, \dots, m\}$, $(\alpha_i^k)_i$ are amplitudes satisfying the condition $\sum_i |\alpha_i^k|^2 = 1$. The existence of this form is a consequence of Theorem 6 in Ref. [39].

A one-to-one parametrisation can be obtained using the same argument as in the proof of Theorem 7, which allows us to show that every sector-preserving channel of this type is in one-to-one correspondence with m pinned Kraus operators of a channel $\mathcal{C}$ in dimension d, and with a set of complex amplitudes $(\gamma_{ij})_{1\le i<j\le m}$. To illustrate the situation, we consider the $m = 2$ case. In this case, it is possible to show that every sector-preserving channel $\widetilde{\mathcal{C}}$ admits

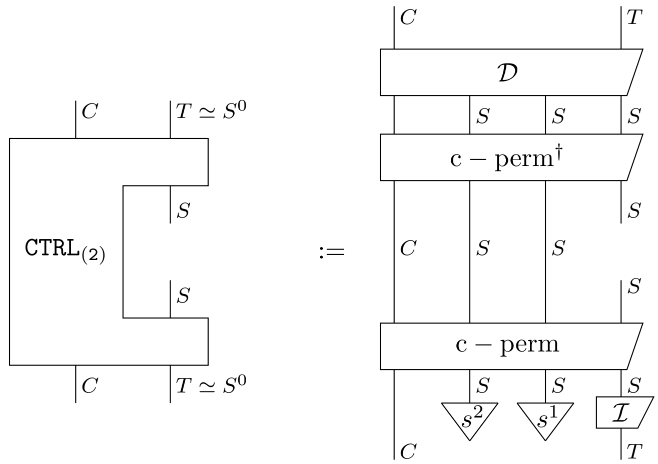

Fig. 4. The $\mathtt{CTRL}_{(2)}$ supermap. s^1 and s^2 are the only normalised states in $\mathcal{L}(\mathcal{H}_S^1)$ and $\mathcal{L}(\mathcal{H}_S^2)$, respectively. $\mathcal{I} : \mathcal{L}(\mathcal{H}_T) \to \mathcal{L}(\mathcal{H}_S)$ is an isometric channel such that $\mathcal{I}(\rho) = \rho$ (i.e. it just embeds $\mathcal{L}(\mathcal{H}_T)$ within $\mathcal{L}(\mathcal{H}_S)$). $\mathrm{c} - \mathrm{perm}$ is the unitary gate which, depending on the state of C, performs a cyclic permutation of the three S wires: it performs the identity if C is in state $|0\rangle$, connects each wire to its right neighbour if C is in the state $|1\rangle$, and connects each wire to its left neighbour if C is in the state $|2\rangle$. $\mathcal{D} : \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_S \otimes \mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_T)$ reduces to the identity on the sector $\mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_S^2 \otimes \mathcal{H}_S^1 \otimes \mathcal{H}_S^0) \simeq \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_S^0) \simeq \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_T)$ of its input space; its action on other sectors is irrelevant and can be arbitrarily defined, as long as it makes $\mathcal{D}$ into a CPTP map.

a Kraus representation of the form

$$\begin{cases} \widetilde{C}_1 = C_1 \oplus 1 \oplus \gamma_{12} \\ \widetilde{C}_2 = C_2 \oplus 0 \oplus \sqrt{1 - |\gamma_{12}|^2} \\ \widetilde{C}_i = C_i \oplus 0 \oplus 0 \qquad \forall i \geq 3 \,, \end{cases} \tag{28}$$

where $(C_i)_i$ are Kraus operators of a suitable quantum channel $\mathcal{C}$ in dimension d. Note that this expression is completely analogous to Eq. (23). In this case, it is possible to show that the quadruple $(\mathcal{C}, C_1, C_2, \gamma_{12})$ provides a one-to-one parametrisation:

Theorem 8. *The sector-preserving channels of type* $(d, 1, 1)$ *are in one-to-one correspondence with quadruples of the form* $(\mathcal{C}, C_1, C_2, \gamma_{12})$*, where* $\mathcal{C} : \mathcal{L}(\mathcal{H}_S^0) \to \mathcal{L}(\mathcal{H}_S^0)$ *is a channel in dimension* d*,* C_1 *and* C_2 *are two Kraus operators for* $\mathcal{C}$*, and* γ *is a complex amplitude satisfying* $|\gamma_{12}| \leq 1$*.*

Combining Theorem 8 with Theorem 7, we obtain the following.

Corollary 3. *The following sets are in one-to-one correspondence:*

1. *2-compositely-controlled channels, as defined in (22), with a* d*-dimensional target system;*
2. *quadruples of the form* $(\mathcal{C}, C_1, C_2, \gamma_{12})$*, where* $(\mathcal{C}, C_1, C_2)$ *is a channel with two pinned Kraus operators in dimension* d*, and* γ_{12} *is a complex amplitude satisfying* $|\gamma_{12}| \leq 1$*;*
3. *sector-preserving channels of type* $(d, 1, 1)$*.*

The case of arbitrary $m \geq 2$ can be treated similarly, and also in this case, one can show that there exists a one-to-one correspondence between the set of m-compositely-controlled quantum channels of type (25) and the set of sector-preserving channels of type $(d, \underbrace{1, \ldots, 1}_{m \text{ times}})$.

6.3 *Implementing compositely-controlled channels via a universal circuit architecture*

We now turn to the generalisation of the result of Section 4 to the implementation of m-compositely-controlled channels. For illustration, we once again focus on the case $m = 2$.

As stated in Corollary 3, for any given d, there is indeed a one-to-one correspondence between the 2-compositely-controlled channels on target systems of dimension d (which can be written as the $\mathrm{ctrl}_{\mathrm{C}_1,\mathrm{C}_2,\gamma_{12}}\text{-}\mathcal{C}$), and the sector-preserving channels of type $(d, 1, 1)$ (which can be written as the $\widetilde{\mathcal{C}}[C_1, C_2, \gamma_{12}]$). This correspondence can also be implemented via a universal circuit architecture.

Theorem 9. *Let $\mathcal{H}_S = \mathcal{H}_S^0 \oplus \mathcal{H}_S^1 \oplus \mathcal{H}_S^2$ be a Hilbert space, with $\dim(\mathcal{H}_S^0) = d$ and $\dim(\mathcal{H}_S^1) = \dim(\mathcal{H}_S^2) = 1$, and let $\mathcal{H}_C$ be a control space of dimension 3.*

There exists a supermap $\mathtt{CTRL}_{(2)}$ of type $(S \to S) \to (C \otimes S^0 \to C \otimes S^0)$ such that for any sector-preserving channel $\widetilde{\mathcal{C}}[C_1, C_2, \gamma_{12}]$,

$$\mathtt{CTRL}_{(2)}[\widetilde{\mathcal{C}}[C_1, C_2, \gamma_{12}]] = \mathrm{ctrl}_{\mathrm{C}_1,\mathrm{C}_2,\gamma_{12}}\text{-}\mathcal{C}\,. \tag{29}$$

Furthermore, this supermap is unitary-preserving on the sector-preserving channels on $\mathcal{H}_S$.

This Theorem can be proven in a straightforward way using the formulation of the $\mathtt{CTRL}_{(2)}$ supermap presented in Figure 4.

Similarly, an inverse $\mathtt{CTRL}_{(2)}^{-1}$ of this control map can easily be defined, showing that sector-preserving channels of type $(d, 1, 1)$ and 2-compositely-controlled channels are fully equivalent resources.

These results can be generalised in a straightforward way to the case of general m: for any given m, d, there exists a universal circuit architecture (represented by a supermap $\mathrm{CTRL}_{(\mathrm{m})}$) turning a sector-preserving channel of type $(d, \underbrace{1, \ldots, 1}_{m \text{ times}})$ into its corresponding m-compositely-controlled channel, and a universal circuit architecture realising the converse task.

7 Supermaps on routed channels

We now turn to a formal construction allowing to describe the `CTRL` and `2-CTRL` supermaps as acting solely on sector-preserving channels. We achieve this through the introduction of the notion of supermaps on routed channels.

Supermaps, first introduced in [36], can be conceptually defined as 'operations on operations': they are linear transformations taking quantum channels as input and mapping them to output quantum channels. Their main use is to model the different ways of using and connecting together 'black-box' operations [36], for example in a quantum comb [37] or in more exotic setups, such as the quantum switch [38]; they provide a rigorous framework for studying the features and relative advantages of these manipulations of the black boxes.

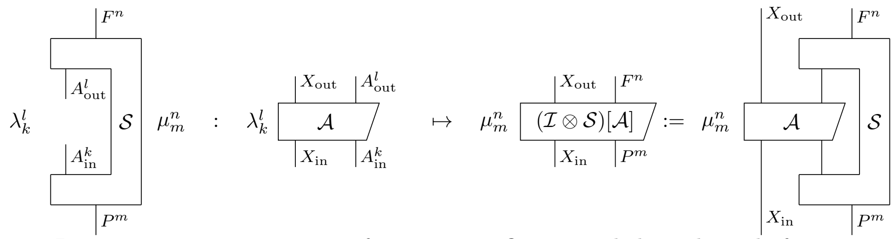

Fig. 5. Diagrammatic representation of a supermap $\mathcal{S}$ on routed channels, and of its action on a routed channel $(\lambda, \mathcal{A})$ (also acting on an auxiliary system), yielding a routed channel $(\mu, (\mathcal{I}\otimes\mathcal{S})[\mathcal{A}])$.

Here, we define (deterministic) 'supermaps on routed channels' as supermaps which only accept a subset of all channels as input; namely, in the language of Ref. [39], those that follow a certain route – i.e. satisfy a given set of sectorial constraints. These restrictions will make the possible supermaps more diverse, as they are no longer required to be well-defined on all possible input channels. Fortunately, a good deal of the formal work necessary in order to define such supermaps on routed channels has been undertaken already: in [38], deterministic supermaps on a restricted subset of quantum channels were defined in general. We will recall the main parts of this definition, then apply it to the definition of supermaps on routed channels.

We denote a system X as corresponding to a finite-dimensional Hilbert space $\mathcal{H}_X$. For two systems A_{in} and A_{out}, we denote $\text{Herm}(A_{\text{in}} \to A_{\text{out}})$ to be the real vector space of Hermitian-preserving linear maps from $\mathcal{L}(\mathcal{H}_{A_{\text{in}}})$ to $\mathcal{L}(\mathcal{H}_{A_{\text{out}}})$, and $\text{QChan}(A_{\text{in}} \to A_{\text{out}})$ to be its subset containing quantum channels of type $A_{\text{in}} \to A_{\text{out}}$. We also note $\text{St}(X) \subseteq \text{Herm}(\mathcal{H}_X)$ to be the set of states for system X. The first notion we need is that of an extension of a set of channels, which allows us to take into consideration channels which also act on an auxiliary system. Given a subset of channels $S \subseteq \text{QChan}(A_{\text{in}} \to A_{\text{out}})$ and two systems X_{in}, X_{out}, the extension of S in $\text{QChan}(A_{\text{in}}X_{\text{in}} \to A_{\text{out}}X_{\text{out}})$ is the set $\text{Ext}_{X_{\text{in}}\to X_{\text{out}}}(S) := \{\mathcal{C} \in \text{QChan}(A_{\text{in}}X_{\text{in}} \to A_{\text{out}}X_{\text{out}}) \,|\, \forall \sigma \in \text{St}(X_{\text{in}}), \text{Tr}_{X_{\text{out}}}(\mathcal{C} \circ (\mathbb{1}_{A_{\text{in}}} \otimes \sigma_{X_{\text{in}}})) \in S\}$. With this notion, one can define deterministic supermaps on a restricted subset of channels [38].

Definition 2 (Deterministic supermaps on a restricted subset of quantum channels)**.** *Let* $S \subseteq \text{QChan}(A_{\text{in}} \to A_{\text{out}})$ *and* $T \subseteq \text{QChan}(P \to F)$ *be subsets of channels. A deterministic supermap of type* $S \to T$ *is a linear map* $\mathcal{S}$ *from* $\text{Herm}(A_{\text{in}} \to A_{\text{out}})$ *to* $\text{Herm}(P \to F)$ *such that, for any auxiliary systems* X_{in}*,* X_{out} *and for any channel* $\mathcal{C} \in \text{Ext}_{X_{\text{in}}\to X_{\text{out}}}(S)$*, one has*

$$(\mathcal{S} \otimes \mathcal{I}_{X_{\text{in}}\to X_{\text{out}}})[\mathcal{C}] \in \text{Ext}_{X_{\text{in}}\to X_{\text{out}}}(T)\,, \tag{30}$$

where $\mathcal{I}_{X_{\text{in}}\to X_{\text{out}}}$ *is the identity supermap on* $\text{Herm}(X_{\text{in}} \to X_{\text{out}})$*.*

We can first apply this notion to the definition of supermaps acting on a single routed channel. First, we briefly recall the basic notions introduced in Ref. [39]. Here, we will restrict ourselves to routes with *full coherence*, i.e., only encoding sectorial constraints and not coherence constraints.[h] A partitioned Hilbert space X^k is a Hilbert space with a preferred

[h]This leads us to adopting notations that are somewhat different from those of Ref. [39]. There, routes for general channels were taken to be completely positive relations $\Lambda^{ll'}_{kk'}$. As fully coherent routes are those Λ's which can be written as $\Lambda^{ll'}_{kk'} = \lambda^l_k \lambda^{l'}_{k'}$, we simplify our notations in the present article by just referring to

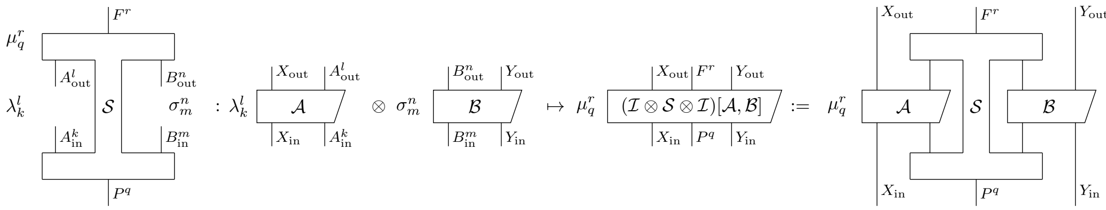

Fig. 6. Diagrammatic representation of a supermap $\mathcal{S}$ acting on a pair of routed channels $(\lambda, \mathcal{A})$ and $(\sigma, \mathcal{B})$ (also acting on auxiliary systems), yielding a routed channel $(\mu, (\mathcal{I}\otimes\mathcal{S}\otimes\mathcal{I})[\mathcal{A},\mathcal{B}])$.

orthogonal partition, labelled by a finite set $\mathcal{Z}_X$; i.e., $\mathcal{H}_X := \oplus_{k\in\mathcal{Z}_X}\mathcal{H}^k_X$. Given two such partitioned spaces A^k_{in} and A^l_{out}, and a relation $\lambda : \mathcal{Z}_{A_{\text{in}}} \to \mathcal{Z}_{A_{\text{out}}}$ (or, in other terms, a Boolean matrix $(\lambda^l_k)^{l\in\mathcal{Z}_{A_{\text{out}}}}_{k\in\mathcal{Z}_{A_{\text{in}}}}$), we say that a channel $\mathcal{A} \in \text{QChan}(A_{\text{in}} \to A_{\text{out}})$ *follows the route* λ if

$$\forall k,\ \forall\rho \in \mathcal{L}(\mathcal{H}^k_{A_{\text{in}}}),\ \mathcal{A}(\rho) \in \mathcal{L}\left(\bigoplus{}_{l|\lambda^l_k=1}\mathcal{H}^l_{A_{\text{out}}}\right). \tag{31}$$

Equivalently (see Theorem 6 in Ref. [39]), given any Kraus representation $(K_i)_i$ of $\mathcal{A}$, $\mathcal{A}$ follows λ if and only if

$$\forall i, \forall k,\ \forall|\psi\rangle \in \mathcal{H}^k_{A_{\text{in}}},\ K_i\,|\psi\rangle \in \bigoplus{}_{l|\lambda^l_k=1}\mathcal{H}^l_{A_{\text{out}}}. \tag{32}$$

We denote the set of channels of type $A_{\text{in}} \to A_{\text{out}}$ that follow the route λ as $\text{QChan}^\lambda(A^k_{\text{in}} \to A^l_{\text{out}}) \subseteq \text{QChan}(A_{\text{in}} \to A_{\text{out}})$. We will also say that these channels have type $A^k_{\text{in}} \xrightarrow{\lambda} A^l_{\text{out}}$; it is this type of channels on which we want to define supermaps. It is easy to prove that the condition defining the extension of $A^k_{\text{in}} \xrightarrow{\lambda} A^l_{\text{out}}$ to auxiliary systems can be simplified.

Lemma 5. *For a type $A^k_{\text{in}} \xrightarrow{\lambda} A^l_{\text{out}}$ and auxiliary systems X_{in}, X_{out}, one has:*

$$\begin{aligned}&\text{Ext}_{X_{\text{in}}\to X_{\text{out}}}(\text{QChan}^\lambda(A^k_{\text{in}} \to A^l_{\text{out}}))\\ &= \text{QChan}^\lambda(A^k_{\text{in}}\otimes X_{\text{in}} \to A^l_{\text{out}}\otimes X_{\text{out}}).\end{aligned} \tag{33}$$

In other terms, the extension of the set of channels $A_{\text{in}} \to A_{\text{out}}$ following a route λ to a type $X_{\text{in}} \to X_{\text{out}}$ is simply the set of channels $A_{\text{in}}\otimes X_{\text{in}} \to A_{\text{out}}\otimes X_{\text{out}}$ following λ. The definition of supermaps on routed channels then derives naturally from Definition 2.

Definition 3 (Supermap on routed channels)**.** *Let A^k_{in}, A^l_{out}, P^m and F^n be partitioned Hilbert spaces, and let $\lambda : \mathcal{Z}_{A_{\text{in}}} \to \mathcal{Z}_{A_{\text{out}}}$ and $\mu : \mathcal{Z}_P \to \mathcal{Z}_F$ be two relations. A deterministic supermap of type $(A^k_{\text{in}} \xrightarrow{\lambda} A^l_{\text{out}}) \to (P^m \xrightarrow{\mu} F^n)$ is a linear map $\mathcal{S}$ from* Herm$(A_{\text{in}} \to A_{\text{out}})$ *to* Herm$(P \to F)$ *such that, for any auxiliary systems X_{in}, X_{out} and for any channel $\mathcal{C} \in$* $\text{QChan}^\lambda(A^k_{\text{in}}\otimes X_{\text{in}} \to A^l_{\text{out}}\otimes X_{\text{out}})$*, one has*

$$(\mathcal{S}\otimes\mathcal{I}_{X_{\text{in}}\to X_{\text{out}}})[\mathcal{C}] \in \text{QChan}^\mu(P^m\otimes X_{\text{in}} \to F^n\otimes X_{\text{out}}). \tag{34}$$

---

them as λ^l_k.

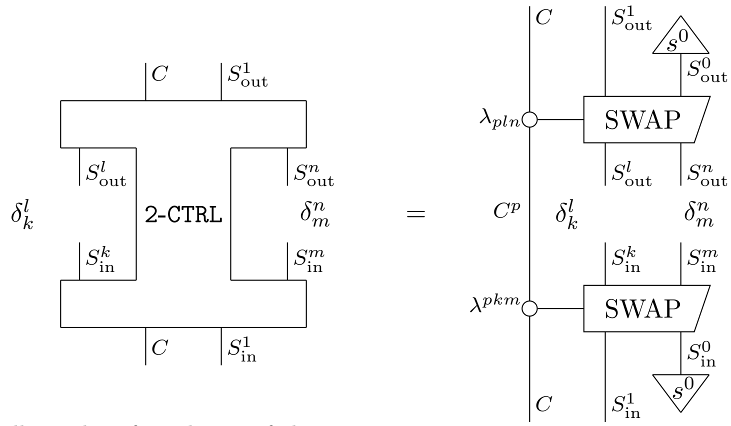

Fig. 7. Fully explicit formulation of the 2-CTRL supermap as a supermap on sector-preserving channels, in the framework of routed quantum circuits [39]. s^0 is the only state on the one-dimensional sector $\mathcal{L}(\mathcal{H}^0_{S\text{in}})$. The Boolean vector $(\lambda^{pkm})_{p,k,m\in\{0,1\}}$ has coefficients 1 for indices 001 and 110, and 0 elsewhere. $(\lambda_{pln})_{p,l,n\in\{0,1\}}$ is its transpose. An advantage of the routed formulation is to allow one to get rid of the superfluous embedding operations that were present in the standard formulation (Figure 3).

We show how supermaps on routed channels can be represented graphically in Figure 5. The CTRL supermap described in Theorem 4 can be characterised as a supermap on routed channels, with type $(A^k \xrightarrow{\delta} A^l) \to (C \otimes S^1 \to C \otimes S^1)$.

Let us now turn to supermaps acting on multiple routed channels. To avoid clutter, we will present the construction for supermaps acting on a pair of channels, the generalisation to $N \geq 2$ being immediate. Formally, these have to be defined as supermaps whose input channels should be product channels, with each channel in this product following a given route. For some partitioned spaces A^k_{in}, A^l_{out}, B^m_{in} and B^n_{out}, and for two relations $\lambda : \mathcal{Z}_{A_{\text{in}}} \to \mathcal{Z}_{A_{\text{out}}}$ and $\sigma : \mathcal{Z}_{B_{\text{in}}} \to \mathcal{Z}_{B_{\text{out}}}$, we thus define $\text{ProdChan}^{\lambda\times\sigma}(A^k_{\text{in}} \otimes B^m_{\text{in}} \to A^l_{\text{out}} \otimes B^n_{\text{out}})$ to be the intersection of the set of product channels $\text{ProdChan}(A_{\text{in}} \otimes B_{\text{in}} \to A_{\text{out}} \otimes B_{\text{out}})$ with $\text{QChan}^{\lambda\times\sigma}(A^k_{\text{in}} \otimes B^m_{\text{in}} \to A^l_{\text{out}} \otimes B^n_{\text{out}})$. One can then define supermaps acting on such a set, once again following Definition 2.

Definition 4 (Supermaps on pairs of routed channels). *Let A^k_{in}, A^l_{out}, B^m_{in}, B^n_{out}, P^q and F^r be partitioned Hilbert spaces, and let $\lambda : \mathcal{Z}_{A_{\text{in}}} \to \mathcal{Z}_{A_{\text{out}}}$, $\sigma : \mathcal{Z}_{B_{\text{in}}} \to \mathcal{Z}_{B_{\text{out}}}$ and $\mu : \mathcal{Z}_P \to \mathcal{Z}_F$ be relations. A deterministic supermap of type $(A^k_{\text{in}} \xrightarrow{\lambda} A^l_{\text{out}}) \otimes (B^m_{\text{in}} \xrightarrow{\sigma} B^n_{\text{out}}) \to (P^q \xrightarrow{\mu} F^r)$ is a linear map $\mathcal{S}$ from $\text{Herm}(A_{\text{in}} \otimes B_{\text{in}} \to A_{\text{out}} \otimes B_{\text{out}})$ to $\text{Herm}(P \to F)$ such that, for any auxiliary systems X_{in}, X_{out}, Y_{in}, Y_{out} and for any pair of channels $\mathcal{A} \in \text{QChan}^{\lambda}(A^k_{\text{in}} \otimes X_{\text{in}} \to A^l_{\text{out}} \otimes X_{\text{out}})$, $\mathcal{B} \in \text{QChan}^{\sigma}(B^m_{\text{in}} \otimes Y_{\text{in}} \to B^n_{\text{out}} \otimes Y_{\text{out}})$, one has*

$$\begin{aligned}&(\mathcal{I}_{X_{\text{in}}\to X_{\text{out}}} \otimes \mathcal{S} \otimes \mathcal{I}_{Y_{\text{in}}\to Y_{\text{out}}})[\mathcal{A} \otimes \mathcal{B}] \\ &\in \text{QChan}^{\mu}(X_{\text{in}} \otimes P^q \otimes Y_{\text{in}} \to X_{\text{out}} \otimes F^r \otimes Y_{\text{out}})\,.\end{aligned} \tag{35}$$

We show how supermaps on pairs of routed channels can be represented graphically in Figure 6. The 2-CTRL supermap described in Theorem 6 can be characterised as a supermap on routed channels, with type $(A^k \xrightarrow{\delta} A^l) \otimes (A^k \xrightarrow{\delta} A^l) \to (C \otimes S^1 \to C \otimes S^1)$; we show in

Figure 7 how it can then be written in a fully explicit way in the language of routed quantum circuits. Figure 7 can thus be seen as a more compact rewriting of Figure 3 , which contained the additional operations $\mathcal{V}$ and $\mathcal{D}$. The role of these operations was simply to embed the target systems into suitable sectors. While in some specific realisations these embeddings may correspond to non-trivial physical operations, from the information-theoretic point of view they are irrelevant, and they can be completely absorbed into the graphical language of routed circuits.

8 Conclusion

In this work, we showed that sector-preserving channels of type $(1, d)$ are the necessary resource for implementing controlled channels on a d-dimensional system. We demonstrated that sector-preserving channels and controlled channels are into one-to-one correspondence, and can be faithfully parametrised by channels with a pinned Kraus operator. In addition, we showed that this mathematical one-to-one correspondence can be implemented physically: for any given d, there exist two universal circuit architectures that convert sector-preserving channels into controlled channels, and vice-versa.

In addition to characterising the resources for the standard notion of control, we defined a generalised type of controlled channels, called compositely-controlled, in which several states of the control are associated with the 'do-nothing' option. Also in this case, we showed that the controlled channels are in (both mathematical and physical) correspondence with sector-preserving channels, in this case of type $(d, 1, \ldots, 1)$. We also generalised these results to the implementation of coherent control between N channels, and showed that, when these channels are not isometries, such an implementation requires the use of sector-preserving channels of type $(1, d)$ which reduce, on their d-dimensional sectors, to isometric extensions of the channels to be controlled.

The framework of sector-preserving channels provides an information-theoretic underpinning to the existing experimental schemes for the implementation of universal coherent control [23, 24, 25, 26, 27] , as well as a pathway to the generalisation of such schemes to more complex architectures. Furthermore, it lays down the conceptual and mathematical framework required to analyse and compare the performance of implementations of coherent control as well as the advantages that they yield, e.g. in computation or in communication. As a byproduct, it also motivates new experiments on the realisation of composite control and theoretical investigations of its uses.

Finally, in order to properly characterise the supermaps we defined, we introduced the notion of supermaps on routed channels, and gave them a rigorous mathematical definition, building on the framework of Ref. [39]. However, the supermaps presented in the present work can always be extended to act on general channels (even though this will make them lose their unitary-preserving property). An interesting open question is whether there exist supermaps on routed channels which cannot be extended to act on general channels. This might in particular prove relevant to the study of Indefinite Causal Order: it has been shown [41] that some indefinite causal structures could be investigated using index-matching circuits, a specific type of routed circuits.

9 Acknowledgments

It is a pleasure to thank Hlér Kristjánsson and Matt Wilson for helpful discussions, advice and comments. AV is supported by the EPSRC Centre for Doctoral Training in Controlled Quantum Dynamics. This work is supported by the Hong Research Grant Council through grant 17300918 and through the Senior Research Fellowship "Quantum Causal Discovery and the Foundations of Quantum Artificial Intelligence," by the Croucher Foundation, by the HKU Seed Funding for Basic Research, and by the John Templeton Foundation through grant 61466, The Quantum Information Structure of Spacetime (qiss.fr). Research at the Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

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Appendix A A review of the terminology on coherent control in previous literature

The notion of 'coherent control' has been studied under several different names in the literature, which might lead to some confusion. In this appendix, we provide a review of the different terms previously used, arguing that they all essentially refer to the same notion. We will then motivate the choice of the term 'coherent control' employed in this paper.

Coherent control was first considered for unitary gates in the work of Aharonov and coauthors [3]. In this work, controlled unitary gates were used to build what was called a 'superposition of time evolutions'. More precisely, the authors discussed the possibility of implementing evolutions of the form $\sum_j c_j\, U_j$, where the U_j's are unitary operators, and the c_j's are complex coefficients. It was proven that such an evolution could be realised, for arbitrary c_j's, using auxiliary systems and postselection. The protocol described in Ref. [3] consists in realising the controlled unitary gate $\sum_j\, |j\rangle\langle j|\otimes U_j$, initialising the control system in a superposition state, measuring the control system in a suitable basis, and then postselecting on a specific measurement outcome.

Another early instance was in the work of Åberg [28, 29], in which some of what would later come to be seen as the crucial features of coherent control were pointed out and analysed under different names. Indeed, Ref. [28] introduces the concept of so-called subspace-preserving channels, asking how their mathematical form can be obtained from that of their restrictions to each subspace, a procedure called *gluing of completely positive maps*, which is noted to be non-unique. This procedure is a mathematical avatar of the task of coherent control; and, even though the question of physical implementation is not discussed in detail, the comment on the non-uniqueness can be regarded as an early observation of the ill-definedness of the control between two quantum channels. This ill-definedness is noted to be due to the incompleteness of the description of the channels one wants to glue. An application of these methods to single-particle interferometry is described in Ref. [29].

Around the same time, Oi [40] studied the *interference of CP maps*, proposing that the combination of quantum channels in an interferometic setup could reveal additional properties of their physical implementation that are not included in the mathematical expression of quantum channels. In the light of our results, the ability to probe additional properties of the implementation is due to the fact that the channels inserted in the interferometric setup are not the original channels, but rather sector-preserving channels of type $(1, d)$ which coincide with them on their d-dimensional sector. It is the properties of these sector-preserving channels, not of the original ones, that become visible through interferometry.

Finally, Chiribella and Kristjánsson [15] considered *superpositions of quantum channels*, in the context of a communication model where the information carriers move on a superpositions of trajectories. Even though this paper focused on applications to communication, its framework also yields an implementation of the task of coherent control, as shown by the present paper. In this perspective, superpositions of trajectories represent one of the possible physical implementations of coherent control.

The term we adopted here, 'coherent control' (or sometimes 'quantum control', or simply 'control'), is commonly found in both experimental [23, 24, 25, 26, 27] and theoretical [8, 9, 12, 22, 30, 31, 32] works. Consistency with this relatively large body of works is one of the benefits of choosing the term 'control'. Moreover, this choice has the advantage of referring to a clearly defined operational task, rather than to analogies with properties of quantum states (such as 'superpositions of quantum evolutions' or 'superpositions of quantum channels'), to mathematical procedures ('gluing of CP maps'), to possible phenomena ('interference of CP maps'), or to specific types of physical implementations ('superpositions of trajectories').

Appendix B Parametrising the coherent control between two channels

In this Appendix, we prove Theorem 2. We fix a Kraus representation $(A_i)_{i=1}^n$ of minimal length of $\mathcal{A}$. We first prove that any version of a controlled channel between $\mathcal{A}$ and $\mathcal{B}$ admits a Kraus decomposition $(K_j)_{j=1}^m$, where $m \geq n$, $K_j = \texttt{ctrl-}(A_j, B_j)$ for $j \leq n$ and $K_j = \texttt{ctrl-}(0, B_j)$ for $j > n$. Let us take such a channel, given by Kraus operators $(\texttt{ctrl-}(A'_i, B'_i))_{i=1}^m$. The A'_i form a Kraus representation of $\mathcal{A}$; therefore, $m \geq n$ and there exists an unitary matrix $(V_{ji})_{i,j=1}^m$ such that $\sum_i V_{ji} A'_i = A_j$ for $j \leq n$ and 0 for $j > n$. Then, $(\sum_j V_{ji} K_i)_{j=1}^m$ is a Kraus representation of the right form for the controlled channel.

We now prove that, given two choices $(B_i)_{i=1}^m$ and $(B'_j)_{j=1}^{m'}$ of Kraus representations for $\mathcal{B}$, the controlled channels that they define are equal if and only if $\forall i \leq n, B_i = B'_i$. First, suppose that the latter equation holds. Then, taking an isometry matrix $(V_{ji})_{n<i\leq m}^{n<j\leq m'}$ relating the Kraus decompositions $(B_i)_{i=n+1}^m$ and $(B_j)_{j=n+1}^{m'}$, we can complete it into a unitary matrix $(V_{ji})_{1\leq i\leq m}^{1\leq j\leq m'}$ by taking $\forall i, j \leq n, V_{ji} = \delta_{ji}$; one then has $\forall i, j, \sum_i V_{ji} \texttt{ctrl-}(A_i, B_i) = \texttt{ctrl-}(A_j, B'_j)$. Reciprocally, suppose that the controlled channels defined by the choices $(B_i)_{i=1}^m$ and $(B'_j)_{j=1}^{m'}$ are equal. Taking then $(V_{ji})_{1\leq i\leq m}^{1\leq j\leq m'}$ to be an isometry matrix relating the associated Kraus decompositions, one has in particular $\forall i, j \leq n, \sum_i V_{ji} A_i = A_j$. Yet, that $(A_i)_{i=1}^n$ is a Kraus representation of minimal length implies in particular that the A_i's are linearly independent; therefore $\forall i, j \leq n, V_{ji} = \delta_{ji}$, which implies $\forall i \leq n, B'_i = B_i$.

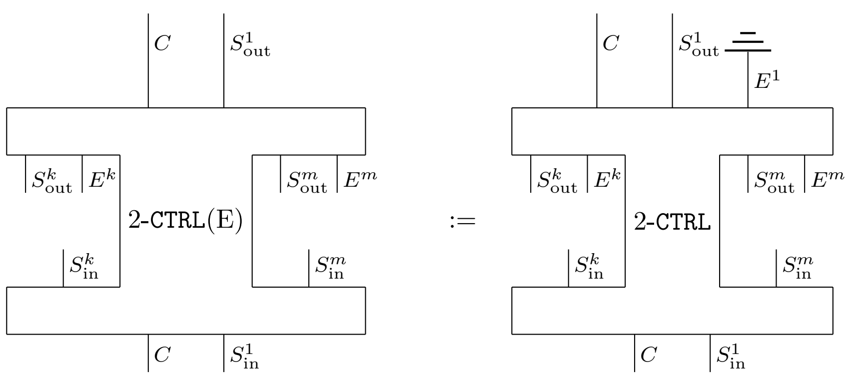
Appendix C Control of two noisy channels

In this Appendix, we propose a universal circuit implementation for all possible versions of the control between two noisy channels $\mathcal{A}$ and $\mathcal{B}$ from $\mathcal{L}(\mathcal{H}_{T_{\text{in}}})$ to $\mathcal{L}(\mathcal{H}_{T_{\text{out}}})$. To avoid clutter, we will take the isomorphisms $T_{\text{in}} \simeq S^1_{\text{in}}$ and $T_{\text{out}} \simeq S^1_{\text{out}}$ to be strict, that is, as will assume $T_{\text{in}} = S^1_{\text{in}}$ and $T_{\text{out}} = S^1_{\text{out}}$.

Recall that, as proven in Section 5, in the case where $\mathcal{A}$ and $\mathcal{B}$ are isometric channels the controlled version could be implemented using as resources sector-preserving channels from $\mathcal{L}(\mathcal{H}^0_{S_{\text{in}}} \oplus \mathcal{H}^1_{S_{\text{in}}})$ to $\mathcal{L}(\mathcal{H}^0_{S_{\text{out}}} \oplus \mathcal{H}^1_{S_{\text{out}}})$, where $\mathcal{H}^1_{S_{\text{out}}} := \mathcal{H}_{S_{\text{out}}}$, $\mathcal{H}^1_{S_{\text{in}}} := \mathcal{H}_{S_{\text{in}}}$, and $\mathcal{H}^0_{S_{\text{in}}} \cong \mathcal{H}^0_{S_{\text{out}}} \cong \mathbb{C}$, with these channels restricting respectively to $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{L}(\mathcal{H}^1_{S_{\text{in}}})$. However, the controlled channels yielded by this method can feature full coherence only between at most one Kraus operator of $\mathcal{A}$ and one operator of $\mathcal{B}$.

Here, we shall therefore make use of more complex resources. These resources will be sector-preserving channels whose multi-dimensional output sector will not be $\mathcal{H}^1_{S_{\text{out}}}$, but $\mathcal{H}^1_{S_{\text{out}}} \otimes \mathcal{H}^1_E$, where $\mathcal{H}^1_E$ is an auxiliary Hilbert space. The restrictions of these channels to this sector will have to yield $\mathcal{A}$ and $\mathcal{B}$ when E^1 is traced out. In other words, to get the full scope of controls between $\mathcal{A}$ and $\mathcal{B}$ we need to use sector-preserving channels that restrict to (possibly partial) purifications of $\mathcal{A}$ and $\mathcal{B}$ on their multi-dimensional sectors. Using such resources, the number of Kraus operators of $\mathcal{A}$ and $\mathcal{B}$ between which there can be full coherence in the controlled channel is capped by the dimension of $\mathcal{H}^1_E$. In particular a sufficiently large $\mathcal{H}^1_E$ will ensure that all possible controlled channels can be generated.

More formally, we define the supermap `2-CTRL(E)` from the supermap `2-CTRL` in the following way:[i]



(C.1)

Let us now prove that, for a given choice of E, `2-CTRL(E)` can produce all controlled channels in which the number of coherent pairs of Kraus operators is less than the dimension of E.

Theorem C.1. *We fix an environment E with dimension D, and use the one-to-one parametrisation of the control between two channels provided by Theorem 2: i.e., given a Kraus representation $(A_i)_{i=1}^n$ of $\mathcal{A}$ of minimal length, the parametrisation is given by the choice of n Kraus operators B_i of $\mathcal{B}$.*

Then any choice of a control in which only the D first operators B_i are non-zero can be obtained from the use of the `2-CTRL(E)` *supermap.*

[i] Here, we defined this supermap as a routed one (also using the convention of contracting Kronecker deltas) for clarity, but this could also be arbitrarily expanded into a supermap acting on all channels from $\mathcal{L}(\mathcal{H}_{S_{\text{in}}})$ to $\mathcal{L}(\oplus_{k\in\{0,1\}} \mathcal{H}^k_{S_{\text{out}}} \otimes \mathcal{H}^k_E)$. Note that when writing such a non-routed supermap, one would have to write the combination of S_{out} and E as a single wire, as the way in which they combine to form $S^k_{\text{out}} E^k$ is not a tensor product and cannot be expressed using standard quantum circuits.

Proof. In the case $D = 1$ (i.e. that of the 2-CTRL supermap), it can easily be computed, from the formula of Fig. 3, that any controlled version in which there is coherence between A_1 and B_1 can be obtained by plugging the channels $\widetilde{\mathcal{A}}^{A_1}$ and $\widetilde{\mathcal{B}}^{B_1}$ in 2-CTRL.

Considering now the case $D > 1$, let us take a version $\mathcal{C}$ of a control between $\mathcal{A}$ and $\mathcal{B}$ for which a Kraus representation is $\Big(|0\rangle\langle 0|_C \otimes A_1 + |1\rangle\langle 1|_C \otimes B_1, \ldots, |0\rangle\langle 0|_C \otimes A_D + |1\rangle\langle 1|_C \otimes B_D, |0\rangle\langle 0|_C \otimes A_{D+1}, \ldots, |0\rangle\langle 0|_C \otimes A_n, |1\rangle\langle 1|_C \otimes B_{D+1}, \ldots, |1\rangle\langle 1|_C \otimes B_m \Big)$. Then a (possibly partial) purification of $\mathcal{C}$ is given by the channel of type $CS^1_{\text{in}} \to CS^1_{\text{out}}E$ for which a Kraus representation is $\Big(\sum_{i=1}^{D} \big(|0\rangle\langle 0|_C \otimes A_i + |1\rangle\langle 1|_C \otimes B_i \big) \otimes |i\rangle_E \,, |0\rangle\langle 0|_C \otimes A_{D+1} \otimes |1\rangle_E \,, \ldots, |0\rangle\langle 0|_C \otimes A_n \otimes |1\rangle_E \,, |1\rangle\langle 1|_C \otimes B_{D+1} \otimes |1\rangle_E \,, \ldots, |1\rangle\langle 1|_C \otimes B_m \otimes |1\rangle_E \Big)$. This latter channel can be seen as being a version of a control between two channels $S^1_{\text{in}} \to S^1_{\text{out}}E^1$ with coherence between one pair of Kraus operators. By the first part of the proof, it can thus be obtained by applying the 2-CTRL supermap to suitable sector-preserving channels of type $S^k_{\text{in}} \to S^k_{\text{out}}E^k$. Discarding E^1 then yields $\mathcal{C}$. The 2-CTRL(E) as defined in (C.1) thus yields $\mathcal{C}$ when applied to the same sector-preserving channels. □

In particular, as any channel $S^1_{\text{in}} \to S^1_{\text{out}}$ admits a Kraus representation of length less than the product of the dimensions of S^1_{in} and S^1_{out}, all versions of controlled channels can be obtained from the use of the supermap 2-CTRL(E) when E is of that dimension.