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PERIODICITY FOR THE 3-STATE QUANTUM WALK ON CYCLES

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Dukes (2014) and Konno, Shimizu, and Takei (2017) studied the periodicity for 2-state quantum walks whose coin operator is the Hadamard matrix on cycle graph C_N with N vertices. The present paper treats the periodicity for 3-state quantum walks on C_N. Our results follow from a new method based on the cyclotomic field. This method gives a necessary condition for the coin operator for quantum walks to be periodic. Moreover, we reveal the period T_N of typical two kinds of quantum walks, the Grover and Fourier walks. We prove that both walks do not have any finite period except for $N = 3$, in which case $T_3 = 6$ (Grover), $= 12$ (Fourier).

Keywords: Quantum walks, Periodicity, Cycles

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1 Introduction

The discrete time quantum walk is defined as a quantum counterpart of the classical random walk [1, 2]. It is known that some theories of quantum walks are useful for developing new quantum algorithms. In other applications, for instance, quantum walks have been studied from viewpoints of topological insulator and quantum information science. Some reviews and books for the quantum walk are [15, 16, 17, 18]. In the present paper, we treat 3-state quantum walks on the cycle graph C_N with N vertices and focus on the property of periodicity. The periodicity of the quantum walk is widely studied with some applications [3, 4, 5]. For example, one of our motivations of the study on the periodicity of quantum walks is to characterise graphs [9, 11]. For some typical graphs, e.g., complete graphs, the generalized Bethe trees, and cycles, the periodicity is clarified [6, 7, 8]. Moreover, [10, 11] treat several famous graph classes (distance regular graph, Hamming graph, and Johnson graph). We should remark that the 3-state quantum walk on cycles is regarded as the quantum walk on cycles with self-loops.

From now on, we present the definition of 3-state quantum walks on C_N introduced by Sadowski et al.[12]. This model is given by adding a periodic boundary condition to 3-state quantum walks on Z, which is a typical walk causing localization [13, 14], where Z means

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the set of integers. We consider a quantum walk on the Hilbert space $\mathcal{H} = V(C_N) \otimes C^3$, where $V(C_N) = \{0, 1, \cdots, N-1\}$ is the vertex set of C_N ($N \geq 2$). Remark that we naturally identify $V(C_N)$ as Z/NZ, and use this identification throughout this paper without notice. The quantum walker has three kinds of chirality states, $\leftarrow$, $\bullet$, and $\rightarrow$, which means the direction of the motion of the walker. If the walker has $\leftarrow$ (resp. $\rightarrow$) chirality, it moves one step to the left (resp. right), and if it has the $\bullet$ chirality, it stays at the original position. Here, we correspond each state to the vectors as follows:

$$|\leftarrow\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad |\bullet\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad |\rightarrow\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Weight matrices P, R, and Q represent the walker hops to adjacent vertices or stays at the same vertex. They are defined by a division of the coin operator $C \in \mathrm{U}(3)$, respectively:

$$P = |\leftarrow\rangle\langle\leftarrow|C, \quad R = |\bullet\rangle\langle\bullet|C, \quad Q = |\rightarrow\rangle\langle\rightarrow|C,$$

where $\mathrm{U}(n)$ means the set of $n \times n$ unitary matrices. We note that above definitions of P, R, and Q are that of the moving shift type. The moving shift, which does not change the direction of the walker, is used for some fixed graphs, e.g., n-dimensional lattice, tree graphs. On the other hand, the flip-flop shift, which reverses the direction of the walker, is used for general graphs. The proof of our results in this paper are written only for the moving shift type. Note that we obtain similar results for the flip-flop shift version and assign it in Section 3 with its definition.

For the initial state $\Psi_0 \in \mathcal{H}$, the time evolution is $\Psi_t = U_N^t \Psi_0$ with the time evolution operator U_N defined as follows:

$$U_N = \begin{bmatrix} R & P & O & \cdots & O & Q \\ Q & R & P & \ddots & O & O \\ O & Q & R & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & O & \ddots & \ddots & R & P \\ P & O & \cdots & \cdots & Q & R \end{bmatrix} \in \mathrm{U}(3N) \quad (N \neq 2). \tag{1.1}$$

For $N = 2$, we define

$$U_2 = \begin{bmatrix} R & P+Q \\ P+Q & R \end{bmatrix} \in \mathrm{U}(6).$$

Let

$$\Psi_t = {}^t\left[\Psi_t(0),\ \Psi_t(1),\ \cdots,\ \Psi_t(N-2),\ \Psi_t(N-1)\right] \in \mathcal{H},$$

where t means the transpose operation. Then, (1.1) equals to

$$\Psi_{t+1}(x) = P\Psi_t(x+1) + R\Psi_t(x) + Q\Psi_t(x-1) \quad (x \in V(C_N)). \tag{1.2}$$

We introduce the Fourier transform of the quantum state as $\hat{\Psi}_t(k) = \sum_{x=0}^{N-1} e^{-\frac{2\pi k}{N}i}\Psi_t(x) \in C^3$ $(k = 0, 1, \ldots, N-1)$. Then, $\hat{U}(k) = \mathrm{diag}(e^{\frac{2\pi k}{N}i},\ 1,\ e^{-\frac{2\pi k}{N}i})C \in \mathrm{U}(3)$ gives the time evolution for the quantum state on the Fourier space, i.e., $\hat{\Psi}_t(k) = \left(\hat{U}(k)\right)^t \hat{\Psi}_0(k)$ $(k = 0, 1, \ldots, N-1)$. It is well known that $\mathrm{Spec}(U_N) = \bigcup_{k=0}^{N-1} \mathrm{Spec}(\hat{U}(k))$, where $\mathrm{Spec}(\cdot)$ denotes the set of eigenvalues. This equality follows from (1.2) and the unitarity of the Fourier transform. We put

$$\mathcal{N} = \{n \in N \ : \ (U_N)^n = I_{3N}\},$$

where N is the set of natural numbers and I_n is the $n \times n$ identity matrix. If $\mathcal{N} \neq \emptyset$, the period of the quantum walk $T_N = \min \mathcal{N}$. If $\mathcal{N} = \emptyset$, then $T_N = \infty$ and we say the quantum walk is not periodic. The following lemma is a useful method to distinguish whether a quantum walk is periodic or not.

Lemma 1.1 *For $T \in N$, the following (1) to (3) are equivalent.*

(1) $U_N^T = I_{3N}$.
(2) ${}^\forall \lambda \in \mathrm{Spec}(U_N),\ \lambda^T = 1$.
(3) ${}^\forall \lambda(k) \in \mathrm{Spec}(\hat{U}(k)),\ \lambda(k)^T = 1 \quad (k = 0, 1, \ldots, N-1)$.

We here introduce $Z[\zeta_N] = A \cap Q[\zeta_N]$, where $Z[\zeta_N]$ is the ring of integers of the n-th cyclotomic field, A is the set of algebraic integers, and $Q[\zeta_N]$ is the n-th cyclotomic field. By definitions of A and $Q[\zeta_N]$, we obtain the following lemma.
Lemma 1.2 *For $T \in N$, if $U_N^T = I_{3N}$, then the following relation holds.*

$$\lambda_1(k) + \lambda_2(k) + \lambda_3(k) \in A \cap Q[\zeta_T] = Z[\zeta_T] \qquad (k = 0, 1, \ldots, N-1),$$

where $\lambda_1(k), \lambda_2(k), \lambda_3(k) \in \mathrm{Spec}(\hat{U}(k))$ and ζ_n is a primitive n-th root of unity, i.e., $\zeta_n = e^{\frac{2\pi}{n}i}$.

2 Results

In this section, we present some results of the periodicity. We assign Section 2.1 as a necessary condition for the quantum walk, which has no assumptions on the coin, to be periodic. Section 2.2 clarifies the period of typical two kinds of walks, the Grover and Fourier walks.

2.1 Necessary condition for the coin operator

Theorem 2.1 *If $c_{11} \notin \frac{1}{N}Z[\zeta_{\mathrm{lcm}(N,T)}]$ or $c_{22} \notin \frac{1}{N}Z[\zeta_T]$ or $c_{33} \notin \frac{1}{N}Z[\zeta_{\mathrm{lcm}(N,T)}]$ is satisfied, then $U_N^T \neq I_{3N}$ for any $T \in N$, where the coin operator $C = (c_{ij})_{i,j=1,2,3}$ and $\mathrm{lcm}(\cdot,\cdot)$ is the least common multiple.*
Proof: Assume that $U_N^T = I_{3N}$. From Lemma **??**, we see that

$$\lambda_1(k) + \lambda_2(k) + \lambda_3(k) \in Z[\zeta_T]. \tag{2.3}$$

By definition of $\hat{U}(k)$, we have

$$\mathrm{tr}(\hat{U}(k)) = e^{\frac{2\pi k}{N}i}c_{11} + c_{22} + e^{-\frac{2\pi k}{N}i}c_{33}. \tag{2.4}$$

Combining (2.3) with (2.4) gives

$$e^{\frac{2\pi k}{N}i}c_{11} + c_{22} + e^{-\frac{2\pi k}{N}i}c_{33} \in Z[\zeta_T]. \tag{2.5}$$

It follows from (2.5) and $e^{\pm\frac{2\pi k}{N}i}Z[\zeta_T] \subset Z[\zeta_{\mathrm{lcm}(N,T)}]$ that

$$\sum_{k=0}^{N-1}\left(e^{\frac{2\pi k}{N}i}c_{11} + c_{22} + e^{-\frac{2\pi k}{N}i}c_{33}\right) = Nc_{22} \in Z[\zeta_T],$$
$$\sum_{k=0}^{N-1} e^{-\frac{2\pi k}{N}i}\left(e^{\frac{2\pi k}{N}i}c_{11} + c_{22} + e^{-\frac{2\pi k}{N}i}c_{33}\right) = Nc_{11} \in Z[\zeta_{\mathrm{lcm}(N,T)}],$$
$$\sum_{k=0}^{N-1} e^{\frac{2\pi k}{N}i}\left(e^{\frac{2\pi k}{N}i}c_{11} + c_{22} + e^{-\frac{2\pi k}{N}i}c_{33}\right) = Nc_{33} \in Z[\zeta_{\mathrm{lcm}(N,T)}].$$

Thus, if $U_N^T = I_{3N}$, then $c_{11} \in \frac{1}{N}Z[\zeta_{\mathrm{lcm}(N,T)}]$, $c_{22} \in \frac{1}{N}Z[\zeta_T]$, and $c_{33} \in \frac{1}{N}Z[\zeta_{\mathrm{lcm}(N,T)}]$. The desired conclusion is given.

2.2 Periodicity for the typical quantum walks

2.2.1 The Grover walk

The Grover walk is determined by the coin operator as the Grover matrix, i.e.,

$$C = G(3) = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Here, $G(n) = (g_{i,j})_{i,j=1,2,\dots,n}$ is the Grover matrix with size n, which is defined by

$$g_{i,j} = \begin{cases} \frac{2}{n} - 1 & (i = j) \\ \frac{2}{n} & (i \neq j) \end{cases}.$$

Then, the characteristic polynomial of $\hat{U}(k)$ is

$$\det\left(\lambda(k)I_3 - \hat{U}(k)\right) = (\lambda(k) - 1)\left(\lambda(k)^2 + \frac{4 + e^{\frac{2\pi k}{N}i} + e^{-\frac{2\pi k}{N}i}}{3}\lambda(k) + 1\right). \tag{2.6}$$

The above equation gives $\mathrm{Spec}(\hat{U}(k)) = \{\lambda_1(k), \lambda_2(k), \lambda_3(k)\}$ with $\lambda_2(k) = \overline{\lambda_1(k)}$ and $\lambda_3(k) = 1$.

Theorem 2.2 *For any $N \geq 2$, the period of the Grover walk is as follows.*

$$T_N = \begin{cases} 6 & (N = 3) \\ \infty & (N \neq 3) \end{cases}.$$

Proof: Firstly, we prove $N = 3$ case. From (2.6), we get the eigenvalues of $\hat{U}(k)$ as follows.

$$\lambda_1(k) = \overline{\lambda_2(k)} = \begin{cases} -1 & (k = 0) \\ e^{\frac{2\pi k}{3}i} & (k = 1, 2) \end{cases}.$$

Hence, Lemma 1.1 gives $T_3 = \mathrm{lcm}(2,3) = 6$. Next, we prove $N \neq 3$ case. Assume that $U_N^T = I_{3N}$. By Lemma 1.2, we see that

$$\lambda_1(k) + \lambda_2(k) + \lambda_3(k) \in A \cap Q[\zeta_T].$$

Since $\lambda_3(k) = 1$, we have

$$\lambda_1(k) + \lambda_2(k) \in A \cap Q[\zeta_T]. \tag{2.7}$$

On the other hand, definition of $Q[\zeta_N]$ and (2.6) imply

$$\lambda_1(k) + \lambda_2(k) = -\frac{4 + e^{\frac{2\pi k}{N}i} + e^{-\frac{2\pi k}{N}i}}{3} \in Q[\zeta_N]. \tag{2.8}$$

Combining $Z[\zeta_N] = A \cap Q[\zeta_N]$ with (2.7) and (2.8) gives

$$\lambda_1(k) + \lambda_2(k) \in Z[\zeta_N].$$

Especially, we now focus on $k = 1$ case:

$$\lambda_1(1) + \lambda_2(1) = -\frac{4 + e^{\frac{2\pi}{N}i} + e^{-\frac{2\pi}{N}i}}{3} \in Z[\zeta_N].$$

Here, we should remark that

$$3e^{\frac{2\pi}{N}i}\left(-\frac{4 + e^{\frac{2\pi}{N}i} + e^{-\frac{2\pi}{N}i}}{3}\right) \in 3Z[\zeta_N]. \tag{2.9}$$

On the other hand, we have

$$3e^{\frac{2\pi}{N}i}\left(-\frac{4 + e^{\frac{2\pi}{N}i} + e^{-\frac{2\pi}{N}i}}{3}\right) = -1 - 4e^{\frac{2\pi}{N}i} - e^{\frac{4\pi}{N}i}. \tag{2.10}$$

Combining (2.9) with (2.10) implies

$$-1 - 4e^{\frac{2\pi}{N}i} - e^{\frac{4\pi}{N}i} = -\zeta_N^0 - 4\zeta_N^1 - \zeta_N^2 \in 3Z[\zeta_N]. \tag{2.11}$$

In general, $x \in Z[\zeta_n]$ is uniquely expressed by a linear combination of ζ_n^0 to $\zeta_n^{\phi(n)-1}$, i.e.,

$$x = \sum_{j=0}^{\phi(n)-1} z_j \zeta_n^j,$$

where $\{z_j\}$ is a sequence of integer numbers and $\phi(n)$ is Euler's totient function. If $\phi(N) > 2$, then a contradiction occurs in the assumption $U_N^T = I_{3N}$, because the coefficients of ζ_N^0, ζ_N^1 and ζ_N^2 in (2.11) do not belong to $3Z$. In the rest of the proof, we consider N with $\phi(N) \leq 2$ and $N \neq 3$, i.e., $N = 2, 4$, and 6 cases. For three cases, it follows from (2.11) and the following easily obtained results (i) to (iii) that a contradiction occurs in the assumption $U_N^T = I_{3N}$.

$$\begin{aligned}
&\text{(i)} \quad N = 2 \text{ case}: \ -\zeta_2^0 - 4\zeta_2^1 - \zeta_2^2 = \ \ 2\zeta_2^0 \ \notin 3Z[\zeta_2] \\
&\text{(ii)} \quad N = 4 \text{ case}: \ -\zeta_4^0 - 4\zeta_4^1 - \zeta_4^2 = -4\zeta_4^1 \ \notin 3Z[\zeta_4] \\
&\text{(iii)} \quad N = 6 \text{ case}: \ -\zeta_6^0 - 4\zeta_6^1 - \zeta_6^2 = -5\zeta_6^1 \ \notin 3Z[\zeta_6]
\end{aligned}$$

Therefore, the desired conclusion is given.

2.2.2 The Fourier walk

The Fourier walk is defined by the coin operator as the Fourier matrix, i.e.,

$$C = F(3) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi}{3}i} & e^{\frac{4\pi}{3}i} \\ 1 & e^{\frac{4\pi}{3}i} & e^{\frac{2\pi}{3}i} \end{bmatrix}.$$

Here, $F(n) = (e^{\frac{2(u-1)(v-1)\pi}{n}i}/\sqrt{n})_{u,v=1,2,\ldots,n}$ is the Fourier matrix with size n. Then, the characteristic polynomial of $\hat{U}(k)$ is

$$\det\left(\lambda(k)I_3 - \hat{U}(k)\right) = \lambda(k)^3 - \frac{\sqrt{3}}{3}\left(e^{\frac{2\pi k}{N}i} + e^{\left(\frac{-2\pi k}{N}+\frac{2\pi}{3}\right)i} + e^{\frac{2\pi}{N}i}\right)\lambda(k)^2 - \frac{1}{3}\left(1 + e^{\frac{2\pi k}{N}i} + e^{\left(\frac{-2\pi k}{N}+\frac{2\pi}{3}\right)i} - e^{\frac{2\pi}{3}i} - e^{\left(\frac{2\pi k}{N}+\frac{2\pi}{3}\right)i} - e^{\left(\frac{-2\pi k}{N}+\frac{4\pi}{3}\right)i}\right)\lambda(k) + i. \tag{2.12}$$

As in the case of the Grover walk, the following result on the periodicity for the Fourier walk can be obtained by (2.12).

Theorem 2.3 *For any $N \geq 2$, the period of the Fourier walk is as follows.*

$$T_N = \begin{cases} 12 & (N = 3) \\ \infty & (N \neq 3) \end{cases}.$$

Proof: Firstly, we check $N = 3$ case. Then, (2.12) implies that

$$(\lambda_1(k), \lambda_2(k), \lambda_3(k)) = \begin{cases} (e^{\frac{2\pi}{4}i}, 1, -1) & (k = 0, 1) \\ (e^{\frac{2\pi}{3}i}, e^{\frac{5\pi}{3}i}, e^{\frac{7\pi}{6}i}) & (k = 2) \end{cases}.$$

Hence, Lemma 1.1 gives $T_3 = \mathrm{lcm}(4, 6) = 12$. Next, we want to show that $T_N = \infty$ if $N \neq 3$. As for $N \neq 2$ or 3^n ($n \in N$) cases, $T_N = \infty$ can be derived from the result proved by Saito [11] in our setting:

Theorem 2.4 [11] *For $N \neq 3^n$ ($N > 2$) with $n \in N$, the Fourier walk on C_N is not periodic.*

Therefore, we will prove only for $N = 2$ and 9 cases, since $\mathrm{Spec}(U_9) \subset \mathrm{Spec}(U_{3^n})$ ($n > 2$) and Lemma 1.1. As in the proof of Theorem 2.2, we focus on the following $\lambda_1(1) + \lambda_2(1) + \lambda_3(1)$ given by (2.12).

$$\lambda_1(1) + \lambda_2(1) + \lambda_3(1) = \frac{1}{3}\left(1 + e^{\frac{2\pi}{N}i} + e^{\left(-\frac{2\pi}{N}+\frac{2\pi}{3}\right)i} - e^{\frac{2\pi}{3}i} - e^{\left(\frac{2\pi}{N}+\frac{2\pi}{3}\right)i} - e^{\left(\frac{-2\pi}{N}+\frac{4\pi}{3}\right)i}\right). \tag{2.13}$$

Assume that $U_N^T = I_{3N}$. Then, Lemma 1.2 implies

$$\lambda_1(1) + \lambda_2(1) + \lambda_3(1) \in A \cap Q[\zeta_T]. \tag{2.14}$$

By definition of $Q[\zeta_N]$ and (2.13), we obtain

$$\lambda_1(1) + \lambda_2(1) + \lambda_3(1) \in Q[\zeta_{\mathrm{lcm}(3,N)}]. \tag{2.15}$$

Thus, Combining (2.14) with (2.15) gives

$$\frac{1}{3}\left(1+e^{\frac{2\pi}{N}i}+e^{\left(-\frac{2\pi}{N}+\frac{2\pi}{3}\right)i}-e^{\frac{2\pi}{3}i}-e^{\left(\frac{2\pi}{N}+\frac{2\pi}{3}\right)i}-e^{\left(\frac{-2\pi}{N}+\frac{4\pi}{3}\right)i}\right)\in Z[\zeta_{\mathrm{lcm}(3,N)}].$$

Then, we have

$$\begin{aligned}1&+e^{\frac{2\pi}{N}i}+e^{\left(-\frac{2\pi}{N}+\frac{2\pi}{3}\right)i}-e^{\frac{2\pi}{3}i}-e^{\left(\frac{2\pi}{N}+\frac{2\pi}{3}\right)i}-e^{\left(\frac{-2\pi}{N}+\frac{4\pi}{3}\right)i}\\&=\zeta_N^0+\zeta_N^1+\zeta_3^1\zeta_N^{-1}-\zeta_3^1-\zeta_3^1\zeta_N^1-\zeta_3^2\zeta_N^{-1}\in 3Z[\zeta_{\mathrm{lcm}(3,N)}].\end{aligned}\tag{2.16}$$

From (2.16) and the following results (i) and (ii), a contradiction occurs in the assumption $U_N^T=I_{3N}$.

(i) $N=2$ case : $1+e^{\frac{2\pi}{2}i}+e^{\left(-\frac{2\pi}{2}+\frac{2\pi}{3}\right)i}-e^{\frac{2\pi}{3}i}-e^{\left(\frac{2\pi}{2}+\frac{2\pi}{3}\right)i}-e^{\left(\frac{-2\pi}{2}+\frac{4\pi}{3}\right)i}$

$$=\zeta_6^0-2\zeta_6^1\notin 3Z[\zeta_{\mathrm{lcm}(3,2)}]=3Z[\zeta_6]$$

(ii) $N=9$ case : $1+e^{\frac{2\pi}{9}i}+e^{\left(-\frac{2\pi}{9}+\frac{2\pi}{3}\right)i}-e^{\frac{2\pi}{3}i}-e^{\left(\frac{2\pi}{9}+\frac{2\pi}{3}\right)i}-e^{\left(\frac{-2\pi}{9}+\frac{4\pi}{3}\right)i}$

$$=\zeta_9^0+\zeta_9^1+\zeta_9^2-\zeta_9^3-\zeta_9^4-\zeta_9^5\notin 3Z[\zeta_{\mathrm{lcm}(3,9)}]=3Z[\zeta_9]$$

Hence, desired conclusion is given.

3 Summary and discussion

In the present paper, we got some results about the periodicity of 3-state quantum walks on cycles by using a method of cyclotomic field. Especially, we completely determined the periodicity of two kinds of typical quantum walks, the Grover and Fourier walks, in Theorems 2.2 and 2.3, respectively. We should remark that there are two kinds of typical shifts for quantum walks, i.e., moving and flip-flop shifts. The shift of the walk considered here is moving shift. On the other hand, the corresponding flip-flop type is defined by

$$P=|\leftarrow\rangle\langle\rightarrow|C,\quad R=|\bullet\rangle\langle\bullet|C,\quad Q=|\rightarrow\rangle\langle\leftarrow|C.$$

In this type, it is easily derived from a similar method that

$$T_N\ (\text{Grover})=\begin{cases}4 & (N=3)\\ \infty & (N\neq 3)\end{cases},\qquad T_N\ (\text{Fourier})=\begin{cases}12 & (N=3)\\ \infty & (N\neq 3)\end{cases}.$$

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