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ON THE QUANTUM COMPLEXITY OF COMPUTING THE MEDIAN OF CONTINUOUS DISTRIBUTIONS

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We study the approximation of the median of an absolutely continuous distribution with respect to the Lebesgue measure given by a probability density function f. We assume that f has r continuous derivatives, with derivative of order r being Hölder continuous with the exponent ρ. We study the quantum query complexity of this problem. We show that the ε-complexity up to a logarithmic factor is of order $\varepsilon^{-1/(r+\rho+1)}$.
We also extend the results to the problem of computing the vector of quantiles of an absolutely continuous distribution.

Keywords: median, quantiles, quantum algorithms, optimality, complexity

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1 Introduction

The quantum complexity of discrete problems has been extensively studied. Speed-ups of quantum algorithms over deterministic and randomized algorithms have been shown for many problems. One of the first papers showing major speed-up of quantum computers was the paper of Shor [1], dealing with the problem of factorization. Another was by Grover [2] dealing with database search. Also the problem of computing other discrete statistics, such as discrete summation, computation of the mean, median and quantiles was investigated, see e.g. [3],[4],[5],[6],[7].
The interesting task is the investigation of the quantum complexity of continuous analogs of these problems. The first paper dealing with the quantum complexity of a continuous problem was the work of Novak [8], where the integration of a function from the Hölder class is considered. Integration in Lebesgue and Sobolev classes of functions was also investigated, see [9],[10]. The problem of function approximation on a quantum computer was studied by Heinrich [11],[12]. The problem of maximization of a function from the Hölder class was investigated in [13] and the problem of finding the root of a function was studied in [14]. Also path integration [15] and differential equations [16] on a quantum computer were investigated, and speed-ups were established.
In this paper, we consider the problem of approximating the median of an absolutely continuous distribution with respect to the Lebesgue measure. This problem is a continuous equivalent of the problem of finding the median of the discrete sequence which is investigated

in [7]. We assume that the distribution is given by a density function which belongs to a classical Hölder class of r times continuously differentiable functions with r-th derivatives being Hölder functions with exponent ρ. We present almost matching upper and lower complexity bounds in the quantum setting, and for the comparison, in the classical deterministic and the randomized setting. We show that the quantum query ε-complexity of this problem is up to a logarithmic factor of order $\varepsilon^{-1/(r+\rho+1)}$. The ε-complexity in the deterministic setting is shown to be of order $\varepsilon^{-1/(r+\rho)}$ and in the randomized setting of order $\varepsilon^{-1/(r+\rho+1/2)}$. Thus, quantum computation yields a speed-up relative to deterministic and randomized algorithms over the entire range of class parameters.
We also extend the problem to the problem of computing the vector of quantiles of a continuous distribution and show matching upper and lower complexity bounds for this problem.
In Section 2 the problem of computing the median is formulated and necessary definitions are presented. The complexity bounds for this problem are presented in Section 4. In Section 3 we recall results on the approximation of the discrete mean and integration, and on the solution of initial-value problems. These will be used to prove our complexity bounds. Section 5 deals with the complexity of computing a vector of quantiles.

2 Problem formulation and basic definitions

Let $f : [0,1] \to \mathbf{R}$ be a real-valued, nonnegative function which integrates to 1. We are interested in approximating the median ξ of the distribution with density function f and with precision $\varepsilon > 0$ in the sense of absolute or residual error criterion. Recall that ξ is defined as a real number from $[0,1]$ that solves the equation $\int_0^{\xi} f(t)dt = 1/2$. We wish to design an algorithm that computes a point $\hat{\xi}$ which will be "close" to the exact value of ξ with respect to a certain error models. In the absolute error criterion model the aim is to find the approximation $\hat{\xi}$, such that $|\xi - \hat{\xi}| \leq \varepsilon$, and for the residual criterion, such that $\left| \int_0^{\hat{\xi}} f(t)dt - 1/2 \right| \leq \varepsilon$.
We will use the Hölder class of functions given by

$$F^{r,\rho} = \Big\{ f : [0,1] \to \mathbf{R} \mid f \in C^r([0,1]), |f^{(i)}(x)| \leq D,\ i = 0,1,\ldots,r,$$
$$|f^{(r)}(x) - f^{(r)}(y)| \leq H|x-y|^{\rho} \ \text{ for } x,y \in [0,1] \Big\},$$

where $C^r([0,1])$ is a class of r times continuously differentiable functions on $[0,1]$, $r \geq 0$, $0 < \rho \leq 1$ and D, H are positive constants. Since f is a probability density function, we need additional assumptions. Let us define the class of probability density functions

$$F_1^{r,\rho} = \left\{ f \in F^{r,\rho} \mid f(x) \geq 0 \text{ for } x \in [0,1],\ \int_0^1 f(t)dt = 1 \right\}.$$

To study the absolute error criterion we need the additional assumption that the function f is separated form zero. Thus, for $\gamma > 0$ consider the following subclass $F_{1,\gamma}^{r,\rho}$ of $F_1^{r,\rho}$

$$F_{1,\gamma}^{r,\rho} = \{ f \in F_1^{r,\rho} \mid f(x) \geq \gamma \text{ for } x \in [0,1] \}.$$

Our aim is to analyze the problem of approximating the median of an absolutely continuous distribution with respect to the Lebesgue measure in the quantum model of computation, and to compare the results to those in the classical deterministic and randomized settings. Let us shortly describe these models. To compute the approximate solution $\hat{\xi}$ in all settings we need some information about the probability density functions $f \in F$, where F stands for $F_1^{r,\rho}$ or $F_{1,\gamma}^{r,\rho}$. In the deterministic setting, we allow only the classical information given by the values of f in some points in $[0,1]$, which may be chosen adaptively. The function values are obtained by oracle calls. In the randomized setting, the information is gathered by computing the values of f in randomly chosen points in $[0,1]$. The number of points in which the information is computed may also be a random number. Thus, the information is a family of operators $\{N^\omega\}_{\omega\in\Omega}$ on some probabilistic space $(\Omega, \Sigma, \mathbf{P})$.

In the quantum setting, the information is obtained by applying a quantum query operator, which is a unitary operator that depends on f and provides the function values. For a detailed explanation of the quantum model of computation the reader is refereed to [9, 17].

The final approximation $\hat{\xi}$ is obtained by applying an algorithm ϕ. In the deterministic setting ϕ is a mapping that transforms the information $N(f)$ into a point in $[0,1]$, that is $\hat{\xi} = \phi(N(f))$. In the randomized setting an algorithm is a family of mappings $\phi = \{\phi^\omega\}_{\omega\in\Omega}$. Thus, $\hat{\xi} = \hat{\xi}^\omega = \phi^\omega(N^\omega(f))$ is assumed to be a random variable. In the quantum setting the algorithm is defined as a sequence of unitary operators including quantum queries applied to an initial state. The final quantum state is measured. The measurement outcome is transformed by a classical algorithm that produces the approximation $\hat{\xi} \in [0,1]$. Due to the randomness of the result of the measurement, the approximation $\hat{\xi} = \hat{\xi}^\omega$ is a random number.

We consider in this paper two error criteria: the absolute and the residual. First, we define the local error of the algorithm ϕ^ω. In the deterministic setting the superscript "ω" is redundant and may be omitted.

The local absolute error of an algorithm ϕ^ω is defined as

$$\mathrm{e}_{\mathrm{abs}}^{\omega}(f,\phi^\omega) = \left|\hat{\xi}^\omega - \xi\right|.$$

The local residual error is given by

$$\mathrm{e}_{\mathrm{res}}^{\omega}(f,\phi^\omega) = \left|\int_0^{\hat{\xi}^\omega} f(t)dt - 1/2\right|.$$

We assume that $\mathrm{e}_{\mathrm{abs}}^{\omega}(f,\phi^\omega)$ and $\mathrm{e}_{\mathrm{res}}^{\omega}(f,\phi^\omega)$ are random variables.

Let us now define the global error in the class F, where F stands for $F_1^{r,\rho}$ or $F_{1,\gamma}^{r,\rho}$ depending on whether we consider the relative or the absolute error, respectively. We are interested in the worst behaviour of the error for any function in the class F. Let $\mathrm{crit} \in \{\mathrm{abs}, \mathrm{res}\}$. The global error in the deterministic setting is defined by

$$\mathrm{e}_{\mathrm{crit}}^{\mathrm{det}}(F,\phi) = \sup_{f\in F} \mathrm{e}_{\mathrm{crit}}(f,\phi).$$

The global error in the randomized setting is defined in a standard way (similarly as in

[14, 16, 18, 19]) as

$$\mathrm{e}_{\mathrm{crit}}^{\mathrm{rand}}(F,\phi) = \sup_{f\in F}\left(\int_{\Omega}(\mathrm{e}_{\mathrm{crit}}^{\omega}(f,\phi^{\omega}))^2\,d\mathbf{P}(\omega)\right)^{1/2}. \tag{1}$$

Note that the L_1 norm may be used instead of the L_2 norm here. This will not change the obtained results. Only the constants may differ.
In the quantum setting the global error is defined probabilistically as

$$\mathrm{e}_{\mathrm{crit}}^{\mathrm{quant}}(F,\phi) = \sup_{f\in F}\ \inf\{\alpha : \mathbf{P}(\,\mathrm{e}_{\mathrm{crit}}^{\omega}(f,\phi^{\omega}) > \alpha\,) \le 1/4\}.$$

Hence, the success probability of ϕ is at least $3/4$. It may be increased by taking the median of a number of repetitions of the algorithm ϕ (see [9]). To get success at least $1-\delta$ one must make $O(\log(1/\delta))$ repetitions of the algorithm.
The cost of the algorithm ϕ is the number of its queries (i.e., oracle calls). Thus, in the deterministic setting it is defined as the number of computed function values. In the randomized setting by the expected number of function values. In the quantum setting we use the query cost, which is defined as number of applications of the quantum query operator. We denote the cost of an algorithm ϕ in the class F by $\mathrm{cost}(F,\phi)$ with suitable superscripts: "det", "rand" or "quant" according to the setting.
For $\varepsilon > 0$, the ε-complexity in the class F is defined as a minimal cost of an algorithm that approximates the median of a distribution with a density function from the class F with precision at most ε,

$$\mathrm{comp}_{\varepsilon,\mathrm{crit}}^{\#}(F) = \inf_{\phi}\{\,\mathrm{cost}^{\#}(F,\phi) :\ \ \mathrm{e}_{\mathrm{crit}}^{\#}(F,\phi) \le \varepsilon\,\},$$

where $\# \in \{\mathrm{det}, \mathrm{rand}, \mathrm{quant}\}$.

3 Useful results

In this section we recall results that will help us to prove the complexity bounds on the problem of approximating the median and the quantiles of a continuous distribution. In particular we present complexity bounds for the problem of computing the discrete mean, the integration problem and the problem of solving the initial-value problems. The results for the mean will be used to derive complexity lower bounds for the problem of computing the median, the integration results for the upper bounds of this problem, and the initial-value results will be used for the problem of computing the vector of quantiles.
We start with the problem of computing the mean of a sequence. We will use this result to prove the lower bounds. Let $g : \{0,1,\ldots,N-1\} \to [0,1]$ be a discrete function. We are interested in approximation of the mean of the sequence $(g(0), g(1), \ldots, g(N-1))$, that is the number $\dfrac{1}{N}\displaystyle\sum_{i=0}^{N-1} g(i)$ with precision $\varepsilon > 0$. Consider the deterministic, randomized and quantum settings. The definition of the error, the cost and the complexity are similar to those of Section 2. Denote the ε-complexity of this problem by $\mathrm{comp}_{\varepsilon}(N)$ with a suitable superscript indicating the setting. From the known complexity bounds for this problem (see [20, 21, 22] for classical results and [6, 7, 22] for quantum) we have that

- in the deterministic setting

$$\mathrm{comp}_\varepsilon^{\mathrm{det}}(N) = \Theta(N(1-2\varepsilon)); \tag{2}$$

- in the randomized setting

$$\mathrm{comp}_\varepsilon^{\mathrm{rand}}(N) = \Theta(\min\{N, \varepsilon^{-2}\}); \tag{3}$$

- in the quantum setting

$$\mathrm{comp}_\varepsilon^{\mathrm{quant}}(N) = \Theta(\min\{N, \varepsilon^{-1}\}). \tag{4}$$

We will also need the upper bounds on the complexity of approximating the integral, that we will use to show the upper bounds. Let $g : [0,1] \to \mathbf{R}$ be a function from the Hölder class $F^{r,\rho}$. Our aim is to approximate $\int_0^1 g(x)dx$ with a given precision ε in the sense of the absolute error, i.e. the local error of the algorithm ϕ is defined as $|\int_0^1 g(x)dx - \phi(g)|$ and the global errors in the class $F^{r,\rho}$ are defined as in Section 2. Denote the complexity of this problem by $\mathrm{comp}_\varepsilon(Int, F^{r,\rho})$ (with suitable superscript). It is known (see [20, 22] for classical results and [8, 22] for the quantum), that the complexity of this problem is bounded by

- in the deterministic setting

$$\mathrm{comp}_\varepsilon^{\mathrm{det}}(Int, F^{r,\rho}) = O(\varepsilon^{-1/(r+\rho)}); \tag{5}$$

- in the randomized setting

$$\mathrm{comp}_\varepsilon^{\mathrm{rand}}(Int, F^{r,\rho}) = O(\varepsilon^{-1/(r+\rho+1/2)}); \tag{6}$$

- in the quantum setting

$$\mathrm{comp}_\varepsilon^{\mathrm{quant}}(Int, F^{r,\rho}) = O(\varepsilon^{-1/(r+\rho+1)}). \tag{7}$$

In the randomized setting, besides of the average error, we will also need to know the distribution of the error. Let ϕ^{opt} be the optimal randomized algorithm. Using the upper complexity bounds and Markov's inequality we get

$$\mathbf{P}\left(\left|\int_0^1 g(x)dx - \phi^{opt}(g)\right| > 2\varepsilon\right) \le \frac{\mathbf{E}\left((\int_0^1 g(x)dx - \phi^{opt}(g))^2\right)}{(2\varepsilon)^2} \le \frac{\varepsilon^2}{4\varepsilon^2} = \frac{1}{4}$$

with cost bounded by $C\varepsilon^{-1/(r+\rho+1/2)}$ for some positive constant C. Using the standard technique for boosting the success probability by repeating the algorithm and taking the median of the results we get that there exist an algorithm ϕ and a positive constant $\tilde{C}$ such that for any $g \in F^{r,\rho}$, $\varepsilon > 0$ and $\delta \in (0, 1/2)$ we have

$$\mathbf{P}\left(\left|\int_0^1 g(x)dx - \phi(g)\right| > \varepsilon\right) \le \delta$$

with cost

$$\mathrm{cost}^{\mathrm{rand}}(\phi) \leq \tilde{C}\varepsilon^{-1/(r+\rho+1/2)} \log(1/\delta). \tag{8}$$

The proof of this statement is very similar to [9, Lemma 3].
We also need some results on the approximate solution of initial-value problems of the form

$$\begin{cases} z'(x) = g(z(x)), & x \in [a,b] \\ z(a) = \eta \end{cases}, \tag{9}$$

where $a < b$, $g : \mathbf{R}^d \to \mathbf{R}^d$, $z : [a,b] \to \mathbf{R}^d$. The input of an algorithm is a function g which belongs to the Hölder class

$$G_d^{r,\rho} = \{f : \mathbf{R}^d \to \mathbf{R}^d |\ f \in C^r(\mathbf{R}^d), \left| \frac{\partial^{|\alpha|}}{\partial x_i^{\alpha_1} \dots \partial x_d^{\alpha_d}} f(x) \right| \leq D,\ |\alpha| = 0, 1, \dots r,$$
$$\left\| \frac{\partial^{|\alpha|}}{\partial x_i^{\alpha_1} \dots \partial x_d^{\alpha_d}} f(x) - \frac{\partial^{|\alpha|}}{\partial x_i^{\alpha_1} \dots \partial x_d^{\alpha_d}} f(y) \right\| \leq H\|x-y\|^\rho,\ \ |\alpha| = r,\ x, y \in \mathbf{R}^d\},$$

where $r \geq 0$, $\rho \in (0,1]$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $|\alpha| = \alpha_1 + \dots \alpha_d$, D and H are positive constants, and $\|\cdot\| = \|\cdot\|_\infty$. The output of an algorithm is a bounded function approximating z. Denote the ε-complexity of this problem by $\mathrm{comp}_\varepsilon(IVP, G_d^{r,\rho})$. There are known complexity bounds of this problem (see [23] for the deterministic setting, [18, 19] for the randomized setting and [16] for the quantum setting):

- in the deterministic setting

$$\mathrm{comp}_\varepsilon^{\mathrm{det}}(IVP, G_d^{r,\rho}) = O(\varepsilon^{-1/(r+\rho)}); \tag{10}$$

- in the randomized setting

$$\mathrm{comp}_\varepsilon^{\mathrm{rand}}(IVP, G_d^{r,\rho}) = O(\varepsilon^{-1/(r+\rho+1/2)}); \tag{11}$$

- in the quantum setting

$$\mathrm{comp}_\varepsilon^{\mathrm{quant}}(IVP, G_d^{r,\rho}) = O(\varepsilon^{-1/(r+\rho+1-\gamma)}), \tag{12}$$

for arbitrary $\gamma \in (0,1)$ (constant in big-O notation in the quantum setting may depend on γ).

4 Complexity bounds for median approximation

4.1 Upper bounds

First we will present a theorem that states the upper complexity bounds on the approximation of the median of an absolutely continuous distribution in the deterministic, randomized and quantum setting for both: the absolute and the residual error criterion.

Theorem 1 *There exist positive constants $C^{\mathrm{quant}}_{\mathrm{abs}}$, $C^{\mathrm{det}}_{\mathrm{abs}}$ and $C^{\mathrm{rand}}_{\mathrm{abs}}$ which depend on r, ρ, D and γ, and $C^{\mathrm{quant}}_{\mathrm{res}}$, $C^{\mathrm{det}}_{\mathrm{res}}$ and $C^{\mathrm{rand}}_{\mathrm{res}}$ which depend on r, ρ and D, such that for any $\varepsilon > 0$ the ε-complexity of the approximation of the median satisfies*

$$\mathrm{comp}^{\mathrm{quant}}_{\varepsilon,\mathrm{abs}}(F^{r,\rho}_{1,\gamma}) \le C^{\mathrm{quant}}_{\mathrm{abs}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon},$$

$$\mathrm{comp}^{\mathrm{det}}_{\varepsilon,\mathrm{abs}}(F^{r,\rho}_{1,\gamma}) \le C^{\mathrm{det}}_{\mathrm{abs}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)} \log\frac{1}{\varepsilon},$$

$$\mathrm{comp}^{\mathrm{rand}}_{\varepsilon,\mathrm{abs}}(F^{r,\rho}_{1,\gamma}) \le C^{\mathrm{rand}}_{\mathrm{abs}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log^2\frac{1}{\varepsilon},$$

$$\mathrm{comp}^{\mathrm{quant}}_{\varepsilon,\mathrm{res}}(F^{r,\rho}_{1}) \le C^{\mathrm{quant}}_{\mathrm{res}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon},$$

$$\mathrm{comp}^{\mathrm{det}}_{\varepsilon,\mathrm{res}}(F^{r,\rho}_{1}) \le C^{\mathrm{det}}_{\mathrm{res}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)} \log\frac{1}{\varepsilon},$$

$$\mathrm{comp}^{\mathrm{rand}}_{\varepsilon,\mathrm{res}}(F^{r,\rho}_{1}) \le C^{\mathrm{rand}}_{\mathrm{res}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log^2\frac{1}{\varepsilon}.$$

Proof: To prove this theorem we construct general algorithms ϕ^{det}, ϕ^{rand} and ϕ^{quant}. These algorithms combine the bisection method for solving the nonlinear equation $G(x) = 0$ for $G(x) = \int_0^x f(t)dt - 1/2$ and the numerical integration. For the positive parameter ε and x_i, $i = 1, 2, \ldots$ being the successive bisection points let G_i be the approximation of $G(x_i)$ computed by optimal deterministic, randomized or quantum integration algorithm with precision ε with probability at least $1-\delta^{\mathrm{rand/quant}}$ in the randomized and quantum setting, respectively, where $\delta^{\mathrm{rand}} = \varepsilon^2/\lceil\log\varepsilon^{-1}\rceil$ and $\delta^{\mathrm{quant}} = 1/(4\lceil\log\varepsilon^{-1}\rceil)$. That is

$$|G_i - G(x_i)| \le \varepsilon \text{ with probability at least } 1 - \delta^{\mathrm{rand/quant}}. \tag{13}$$

The successive interval is chosen based on these approximate values G_i. Note that the integration problem in Section 3 is defined on interval $[0,1]$. Here, to compute $G(x_i)$, we need to compute the integral on $[0, x_i]$ for $x_i \in [0,1]$. However, we may pass to the interval $[0,1]$ by substitution of variable and integrate the function $f(x_i t)$ instead of $f(t)$. Such defined function still belongs to $F^{r,\rho}$ and bounds on the cost and the error of the algorithm will not change.
The bisection algorithm works in the following steps:

1. Set $\alpha = 0$, $\beta = 1$ and $i = 1$.

2. Set $x_i = \frac{\alpha+\beta}{2}$. Compute G_i by the optimal integration algorithm in the respective setting.
   If $|G_i| \le \varepsilon$ or $i = i_{max} := \lceil\log\varepsilon^{-1}\rceil$ then finish and return $\hat{\xi} = x_i$.

3. If $G_i > 0$ then set $\alpha := \alpha$ and $\beta := x_i$.
   If $G_i < 0$ then set $\alpha := x_i$ and $\beta := \beta$.
   Set $i := i + 1$. Go to step 2.

Note that the bisection procedure finishes when for some i_0, $|G_{i_0}| \leq \varepsilon$ or $i_0 = i_{max} = \lceil \log \varepsilon^{-1} \rceil$. The algorithm returns the value $\hat{\xi} = x_{i_0}$. In all previous steps $i = 1, 2, \ldots, i_0 - 1$ we have $|G_i| > \varepsilon$. So, due to (13) if $G_i \geq 0$ then $G(x_i) = G_i + G(x_i) - G_i \geq G_i - \varepsilon \geq 0$ (with probability at least $1 - \delta^{\mathrm{rand/quant}}$ in the randomized and quantum settings). Similarly, if $G_i < 0$, then with the same probability we have $G(x_i) < 0$. So, with probability at least $(1 - \delta^{\mathrm{rand/quant}})^{i_{max}}$ the last bisection interval contains the median. If the algorithm finishes when $i_0 = i_{max}$, then with probability at least $(1 - \delta^{\mathrm{rand/quant}})^{i_{max}}$ we have $|\xi - \hat{\xi}| \leq \varepsilon$. With the same probability, if the algorithm finishes when $|G_{i_0}| \leq \varepsilon$, then

$$|G(x_{i_0})| \leq |G(x_{i_0}) - G_{i_0}| + |G_{i_0}| \leq 2\varepsilon.$$

So, we have

$$|G(\hat{\xi}) - G(\xi)| \leq 2\varepsilon \text{ or } |\xi - \hat{\xi}| \leq \varepsilon \text{ with probability at least } (1 - \delta^{\mathrm{rand/quant}})^{i_{max}}. \tag{14}$$

We now state the error bounds. Consider first the residual error. Due to (14), $|G(\hat{\xi}) - G(\xi)| \leq 2\varepsilon$ or $|G(\hat{\xi}) - G(\xi)| \leq \sup_{x \in [0,1]} |G'(x)| \, |\hat{\xi} - \xi| \leq D|\hat{\xi} - \xi| \leq D\varepsilon$. Thus,

$$e_{\mathrm{res}}(f, \phi^{\mathrm{det/rand/quant}}) \leq \max\{2, D\}\varepsilon.$$

This holds with probability at least $(1 - \delta^{\mathrm{rand/quant}})^{i_{max}} \geq 1 - i_{max}\, \delta^{\mathrm{rand/quant}}$. Recall, that $i_{max} = \lceil \log \varepsilon^{-1} \rceil$, $\delta^{\mathrm{rand}} = \varepsilon^2 / \lceil \log \varepsilon^{-1} \rceil$ and $\delta^{\mathrm{quant}} = 1/(4\lceil \log \varepsilon^{-1} \rceil)$. Thus, in the randomized setting $1 - i_{max}\, \delta^{\mathrm{rand}} = 1 - \lceil \log \varepsilon^{-1} \rceil \, \varepsilon^2 / \lceil \log \varepsilon^{-1} \rceil = 1 - \varepsilon^2$ and $1 - i_{max}\, \delta^{\mathrm{quant}} = 1 - \lceil \log \varepsilon^{-1} \rceil / (4\lceil \log \varepsilon^{-1} \rceil) = 3/4$ in the quantum settings.
Hence we have

$$\mathrm{e}_{\mathrm{res}}^{\mathrm{quant}}(F_1^{r,\rho}, \phi^{\mathrm{quant}}) \leq \max\{2, D\}\varepsilon \tag{15}$$

and

$$\mathrm{e}_{\mathrm{res}}^{\mathrm{det}}(F_1^{r,\rho}, \phi^{\mathrm{det}}) \leq \max\{2, D\}\varepsilon. \tag{16}$$

To get the residual error bound in the randomized setting, note that always $e_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}) \leq 1/2$. So we have in the randomized setting

$$\begin{aligned}
\left(\mathrm{e}_{\mathrm{res}}^{\mathrm{rand}}(F_1^{r,\rho}, \phi)\right)^2 &= \sup_{f \in F_1^{r,\rho}} \int_{\Omega} (\mathrm{e}_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}))^2 d\mathbf{P}(\omega) \\
&= \sup_{f \in F_1^{r,\rho}} \Bigg(\int_{\mathrm{e}_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}) \leq \max\{2,D\}\varepsilon} (\mathrm{e}_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}))^2 d\mathbf{P}(\omega) + \\
&\qquad \int_{\mathrm{e}_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}) > \max\{2,D\}\varepsilon} (\mathrm{e}_{\mathrm{res}}^{\omega}(f, \phi^{\mathrm{rand}}))^2 d\mathbf{P}(\omega) \Bigg) \leq \left(\max\{2, D\}\right)^2 \varepsilon^2 + 1/4\varepsilon^2.
\end{aligned}$$

Thus,

$$e_{\mathrm{res}}^{\mathrm{rand}}(F_1^{r,\rho}, \phi) \leq \sqrt{(\max\{2, D\})^2 + 1/4}\, \varepsilon. \tag{17}$$

Let us pass to the absolute error. Recall that we have an additional assumption, that $f(x) \geq \gamma$ for all $x \in [0,1]$. Note that $G(\hat{\xi}) - G(\xi) = f(c)(\hat{\xi} - \xi)$ for some $c \in [0,1]$. Hence, we have $|\hat{\xi} - \xi| \leq |G(\hat{\xi}) - G(\xi)|/\gamma$. So, due to (14), $|\hat{\xi} - \xi| \leq \max\{1, 2/\gamma\}\varepsilon$ with probability at least $1 - i_{max}\,\delta^{\mathrm{rand/quant}}$ in the randomized and quantum settings. Hence, as in the case of the residual criterion, we have

$$\mathrm{e}^{\mathrm{quant}}_{\mathrm{abs}}(F^{r,\rho}_{1,\gamma}, \phi^{\mathrm{quant}}) \leq \max\{1, 2/\gamma\}\varepsilon, \tag{18}$$

$$\mathrm{e}^{\mathrm{det}}_{\mathrm{abs}}(F^{r,\rho}_{1,\gamma}, \phi^{\mathrm{det}}) \leq \max\{1, 2/\gamma\}\varepsilon \tag{19}$$

and

$$\mathrm{e}^{\mathrm{rand}}_{\mathrm{abs}}(F^{r,\rho}_{1,\gamma}, \phi^{\mathrm{rand}}) \leq \sqrt{(\max\{1, 2/\gamma\})^2 + 1}\,\varepsilon. \tag{20}$$

Let us state the cost bounds. In the quantum setting, cost of one bisection step is the cost of computing one integral in $G(x_i)$. Recall that $\delta^{\mathrm{quant}} = 1/(4\lceil \log \varepsilon^{-1} \rceil)$. Due to (7) this cost is for both criteria bounded by (the cost bound here is multiplied by a factor of order $\log(1/\delta^{\mathrm{quant}})$ to get the success probability at least $1 - \delta^{\mathrm{quant}}$)

$$\begin{aligned} O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log(1/\delta^{\mathrm{quant}})\right) &= O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log(4\lceil \log \varepsilon^{-1} \rceil)\right) \\ &= O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log\log \varepsilon^{-1}\right). \end{aligned}$$

The maximum number of bisection steps is at most $i_{max} = \lceil \log \varepsilon^{-1} \rceil$. So, the total cost is

$$\mathrm{cost}^{\mathrm{quant}}(F^{r,\rho}_1, \phi) \asymp \mathrm{cost}^{\mathrm{quant}}(F^{r,\rho}_{1,\gamma}, \phi) = O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \log \varepsilon^{-1} \log\log \varepsilon^{-1}\right). \tag{21}$$

In the deterministic setting, due to (5), the cost of one bisection step (cost of computing the approximation G_i of $G(x_i)$) for both criteria is of order $O\left((1/\varepsilon)^{1/(r+\rho)}\right)$. The number of steps is at most $i_{max} = \lceil \log \varepsilon^{-1} \rceil$. Thus, the total cost is bounded by

$$\mathrm{cost}^{\mathrm{det}}(F^{r,\rho}_1, \phi^{\mathrm{det}}) \asymp \mathrm{cost}^{\mathrm{det}}(F^{r,\rho}_{1,\gamma}, \phi^{\mathrm{det}}) = O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)} \log \varepsilon^{-1}\right). \tag{22}$$

In the randomized setting, due to (8), since $\delta^{\mathrm{rand}} = \varepsilon^2/\lceil \log \varepsilon^{-1} \rceil$, the cost of one bisection step is

$$\begin{aligned} &O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log(1/\delta^{\mathrm{rand}})\right) = O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log \frac{\log \varepsilon^{-1}}{\varepsilon^2}\right) \\ &= O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} (\log\log \varepsilon^{-1} + \log \varepsilon^{-2})\right) = O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log \varepsilon^{-1}\right). \end{aligned}$$

So, the total cost in the randomized setting is

$$\mathrm{cost}^{\mathrm{rand}}(F^{r,\rho}_1, \phi^{\mathrm{rand}}) \asymp \mathrm{cost}^{\mathrm{rand}}(F^{r,\rho}_{1,\gamma}, \phi^{\mathrm{rand}}) = O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \log^2 \varepsilon^{-1}\right). \tag{23}$$

The bounds (15–20) show that in all cases the accuracy is $O(\varepsilon)$. Thus (21–23) prove the theorem statement. □

4.2 Lower bounds

The following theorem presents the lower bounds on the complexity of approximating the median in the deterministic, randomized and quantum settings.

Theorem 2 *There exist positive constants* $c_{\mathrm{abs}}^{\mathrm{quant}}$, $c_{\mathrm{abs}}^{\mathrm{det}}$, $c_{\mathrm{abs}}^{\mathrm{rand}}$, $c_{\mathrm{res}}^{\mathrm{quant}}$, $c_{\mathrm{res}}^{\mathrm{det}}$, $c_{\mathrm{res}}^{\mathrm{rand}}$ *and* ε_0, *such that for any* $\varepsilon \in (0, \varepsilon_0)$

$$\mathrm{comp}_{\varepsilon,\mathrm{res}}^{\mathrm{quant}}(F_1^{r,\rho}) \geq c_{\mathrm{res}}^{\mathrm{quant}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)},$$

$$\mathrm{comp}_{\varepsilon,\mathrm{res}}^{\mathrm{det}}(F_1^{r,\rho}) \geq c_{\mathrm{res}}^{\mathrm{det}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)},$$

$$\mathrm{comp}_{\varepsilon,\mathrm{res}}^{\mathrm{rand}}(F_1^{r,\rho}) \geq c_{\mathrm{res}}^{\mathrm{rand}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)},$$

$$\mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{quant}}(F_{1,\gamma}^{r,\rho}) \geq c_{\mathrm{abs}}^{\mathrm{quant}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)},$$

$$\mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{det}}(F_{1,\gamma}^{r,\rho}) \geq c_{\mathrm{abs}}^{\mathrm{det}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)},$$

$$\mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{rand}}(F_{1,\gamma}^{r,\rho}) \geq c_{\mathrm{abs}}^{\mathrm{rand}} \left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)}.$$

Proof: We will prove the theorem above by reducing the problem of approximating the mean of the discrete sequence to the problem of approximating the median of the distribution. Let $\varepsilon_1 > 0$ be a parameter to be specified later on. The class $F^{r,\rho}$ contains $n = \Theta\left(\varepsilon_1^{-1/(r+\rho)}\right)$ functions h_i, $i = 1, 2, \ldots, n$, with disjoint supports in the interval $[0, 1/4]$ and g_i, $i = 1, 2, \ldots, n$ with disjoint supports in the interval $[3/4, 1]$ such that

$$\int_0^1 h_i(x)dx = \int_0^1 g_i(x)dx = \varepsilon_1^{1+1/(r+\rho)}, \quad \max_{x\in[0,1]} h_i(x) = \max_{x\in[0,1]} g_i(x) = c\,\varepsilon_1, \quad i = 1, 2, \ldots n$$

for some constant c (see [20], p. 35). The support of h_i ($i = 1, 2, \ldots, n$) is the interval $\left(\frac{i-1}{4n}, \frac{i}{4n}\right)$ and the support of g_i is $\left(\frac{3}{4} + \frac{i-1}{4n}, \frac{3}{4} + \frac{i}{4n}\right)$. Thus all functions h_i for $i = 1, 2, \ldots, n$ are equal to 0 outside the interval $[0, 1/4]$ and all functions g_i are equal to 0 outside the interval $[3/4, 1]$. Let $x_1, x_2, \ldots, x_n$ be a sequence of real numbers in $[0, 1]$. Then the function $f_{\varepsilon_1}(x) = 1 + \sum_{i=1}^{n} x_i h_i(x) - \sum_{i=1}^{n} x_i g_i(x)$ belongs to the class $F_{1,\gamma}^{r,\rho} \subset F_1^{r,\rho}$.

Note that for sufficiently small ε_1 we have $2/3 \leq f_{\varepsilon_1}(x) \leq 2$ for all $x \in [0, 1]$. Let ξ be the

median of f_{ε_1}. Since $1/2 = \int_0^{\xi} f_{\varepsilon_1}(x)dx = \xi\, f_{\varepsilon_1}(\eta)$ for some $\eta \in [0,1]$, we have $\xi = 1/(2f_{\varepsilon_1}(\eta))$ and $1/4 \le \xi \le 3/4$. This yields that on the interval $[0,\xi]$ all functions g_i are equal to 0. Thus,

$$1/2 = \int_0^{\xi} f_{\varepsilon_1}(x)dx = \int_0^{\xi} \left(1 + \sum_{i=1}^{n} x_i h_i(x)\right) dx = \int_0^{1/4} \left(1 + \sum_{i=1}^{n} x_i h_i(x)\right) dx + \int_{1/4}^{\xi} 1dx$$
$$= 1/4 + \varepsilon_1^{1+1/(r+\rho)} \sum_{i=1}^{n} x_i + \xi - 1/4.$$

Thus

$$\sum_{i=1}^{n} x_i = \frac{1/2 - \xi}{\varepsilon_1^{1+1/(r+\rho)}}.$$

Suppose that algorithm ϕ (deterministic, randomized or quantum) computes the median of the distribution with a density function f with error at most ε and cost Q, for any function $f \in F_{1,\gamma}^{r,\rho}$, and in particular for $f = f_{\varepsilon_1}$. Denote the result of the algorithm ϕ for function f_{ε_1} by $\hat{\xi}$. Note that

$$\left| \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1/2 - \hat{\xi}}{n\varepsilon_1^{1+1/(r+\rho)}} \right| = \frac{|\hat{\xi} - \xi|}{n\varepsilon_1^{1+1/(r+\rho)}}.$$

For the residual error using $f_{\varepsilon_1}(x) \ge 2/3$ we have

$$\left| \int_0^{\hat{\xi}} f_{\varepsilon_1}(x)dx - \int_0^{\xi} f_{\varepsilon_1}(x)dx \right| = |f_{\varepsilon_1}(\eta)|\ |\hat{\xi} - \xi| \ge \frac{2}{3}|\hat{\xi} - \xi|,$$

for some $\eta \in [0,1]$. Hence, algorithm ϕ also computes the mean $\dfrac{1}{n}\sum_{i=1}^{n} x_i$ with error at most $\dfrac{C\varepsilon}{n\varepsilon_1^{1+1/(r+\rho)}} = \Theta\left(\dfrac{\varepsilon}{\varepsilon_1}\right)$, where $C = 1$ for the absolute criterion and $C = 3/2$ when the residual criterion is used.

We now use lower complexity bounds for the problem of computing the mean of n real numbers from $[0,1]$ presented in Section 3.

Consider first the quantum setting. Recall that $n = \Theta\left(\varepsilon_1^{-1/(r+\rho)}\right)$. We take $\varepsilon_1 = \varepsilon^{(r+\rho)/(r+\rho+1)}$. From (4) the cost Q of the algorithm ϕ satisfies

$$Q = \Omega\left(\min\left\{n, \varepsilon_1/\varepsilon\right\}\right) = \Omega\left(\varepsilon^{-1/(r+\rho+1)}\right).$$

Let us pass to the deterministic setting. We have that $n \ge G\varepsilon_1^{-1/(r+\rho)}$ for some $G > 0$. Take $\varepsilon_1 = 4C\varepsilon/G$. From (2) we have that the cost of algorithm ϕ is bounded by

$$Q = \Omega\left(n\left(1 - \frac{2C\varepsilon}{n\varepsilon_1^{1+1/(r+\rho)}}\right)\right) = \Omega\left(n - \frac{2C\varepsilon}{\varepsilon_1^{1+1/(r+\rho)}}\right)$$
$$= \Omega\left(G\varepsilon_1^{-1/(r+\rho)} - \frac{1}{2}G\varepsilon_1\varepsilon_1^{-1-1/(r+\rho)}\right) = \Omega\left(\varepsilon^{-1/(r+\rho)}\right).$$

In the randomized setting, we take $\varepsilon_1 = \varepsilon^{(r+\rho)/(r+\rho+1/2)}$. From (3) we have

$$Q = \Omega\left(\min\left\{n, (\varepsilon_1/\varepsilon)^2\right\}\right) = \Omega\left(\varepsilon^{-\frac{2}{2(r+\rho)+1}}\right) = \Omega\left(\varepsilon^{-\frac{1}{r+\rho+1/2}}\right).$$

The cost of any algorithm for computing the median must be at least Q. This yields the desired lower bounds on the complexity in the deterministic, randomized and quantum settings. □
Note that, since the lower and the upper complexity bounds match up to a logarithmic factor, the algorithms presented in Section 4.1 are almost optimal.

5 Complexity bounds for computing of quantiles

5.1 Problem formulation

In this section we consider the problem of approximation of the vector of quantiles of an absolute continuous distribution. Let f be a density function. Suppose that for $k \in \mathbf{N}$ we are given a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in [0,1]^k$. Our aim is to approximate the vector of quantiles $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$, such that $\int_0^{\xi_1} f(x)dx = \alpha_1$, $\int_0^{\xi_2} f(x)dx = \alpha_2, \ldots, \int_0^{\xi_k} f(x)dx = \alpha_k$. We assume here that function f is separated from zero and belongs to the Hölder class $F_{1,\gamma}^{r,\rho}$ defined in Section 2. We use the absolute error criterion, thus the local error of algorithm ϕ approximating the vector of quantiles ξ for the density function f is defined by:

$$\mathrm{e}(f,\phi) = \max_{i=1,\ldots,k} |\hat{\xi}_i - \xi_i|, \tag{24}$$

where $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)$ is the approximation returned by the algorithm ϕ. The global error in the class $F_{1,\gamma}^{r,\rho}$, the cost and the complexity are defined similarly as in Section 2.

5.2 Complexity bounds

The following theorem presents the complexity bounds for the problem of approximation of the vector of the quantiles.

Theorem 3 *There exist positive constants C^{quant}, C^{det}, C^{rand}, c^{quant}, c^{det}, c^{rand} and ε_0 which depend on r, ρ, D and γ, such that for any $k \in \mathbf{N}$, $\alpha \in [0,1]^k$ and $\varepsilon \in (0, \varepsilon_0)$ the ε-complexity of quantiles approximation problem satisfies*

$$c^{\mathrm{quant}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1)} \le \mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{quant}}(F_{1,\gamma}^{r,\rho}) \le C^{\mathrm{quant}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1-\delta)},$$

$$c^{\mathrm{det}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)} \le \mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{det}}(F_{1,\gamma}^{r,\rho}) \le C^{\mathrm{det}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho)},$$

$$c^{\mathrm{rand}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)} \le \mathrm{comp}_{\varepsilon,\mathrm{abs}}^{\mathrm{rand}}(F_{1,\gamma}^{r,\rho}) \le C^{\mathrm{rand}}\left(\frac{1}{\varepsilon}\right)^{1/(r+\rho+1/2)}$$

for an arbitrary $\delta \in (0,1)$ (C^{quant} may depend on δ).

Proof: Let $f \in F_{1,\gamma}^{r,\rho}$. Define the function $F : [0,1] \to [0,1]$ by $F(x) = \int_0^x f(t)dt$. Let $G = F^{-1}$ be the inverse of F. Then, $\xi = (\xi_1, \xi_2, \ldots, \xi_k) = (G(\alpha_1), G(\alpha_2), \ldots, G(\alpha_k))$. Note that G is the solution of the initial-value problem

$$G'(x) = \frac{1}{f(G(x))}, \quad x \in [0,1], \quad G(0) = 0. \tag{25}$$

Since $f \in F_{1,\gamma}^{r,\rho}$, the function $1/f(x)$ is also Hölder continuous with the same parameters r and ρ, but with different constant $\tilde{D}$ dependent on D, r and γ. Let l be the approximation of G returned by the optimal algorithm solving initial-value problems in the Hölder class (deterministic, randomized or quantum). Then, the approximation of ξ is defined by

$$\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k) = (l(\alpha_1), l(\alpha_2), \ldots, l(\alpha_k)).$$

The upper bounds on the complexity of the problem of approximating of the vector of quantiles follow directly from the upper bounds on the problem of solving the initial-value problems presented in Section 3.
The lower bounds follow from the lower bounds on the complexity of approximating of the median presented in Section 4.2. □

6 Remarks

Note that the upper bounds for the problem of approximating the vector of quantiles improve slightly the bounds on the problem of approximating the median in the deterministic and randomized setting for the absolute error criterion (by a logarithmic factor). This is the result of different proof techniques. However, the algorithm for computing the median presented in Section 4.1 is much simpler. These bounds match the lower complexity bounds. All the other bounds are almost sharp.

The information cost of the algorithm for computing the vector of quantiles (and so the complexity upper bounds for this problem) does not depend on the parameter k. Only the combinatory cost grows with the length k of the vector of quantiles.

Note that in the definition of the error in the randomized setting any L_p norm for $1 \leq p < \infty$ instead of L_2 norm may be used and the result will not change significantly. The upper bounds for the residual error follow from the local residual error and the derivation of equation (17). The upper bounds for the absolute error follow similarly. The lower bounds for the error in L_p norm can be derived similarly as in Section 4.2 using the complexity of computing the mean subject to the choice of the norm.

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Appendix A

We use in this paper standard definitions of the error in the randomized and quantum settings

– the probabilistic error in the quantum setting and the expected error in the randomized setting. These definitions are not equivalent. The definition in the randomized setting seems to be more demanding. However, for the problems considered in this paper one may simply pass from one definition to the other. To justify the speed-up of a quantum computer it is sufficient to show suitable lower complexity bounds in the randomized setting using the bounded probability error (as in the quantum setting) or upper complexity bounds in the quantum setting using the expected error. It is quite straightforward to prove that the bounded probability error cannot significantly reduce the randomized complexity. Suppose that we have the optimal randomized algorithm having error ε with probability $p > 3/4$. Using the standard technique of boosting the success probability by repeating the algorithm $O(\log(\varepsilon^{-1}))$ times and taking the median of the results (see [9, Lemma 3]) we get an algorithm with error ε with probability $1 - \varepsilon^2$. Since the error is always deterministically bounded by 1 we can show similarly as (17) that the average error of this algorithm is bounded by $O(\varepsilon)$. Thus, the cost of the boosted randomized algorithm is bounded from below by the complexity bounds from Theorem 2 and the cost of the original randomized algorithm may be reduced only by a factor proportional to $(\log(\varepsilon^{-1}))^{-1}$.
On the other hand, it is easy to show that when we define the error in the quantum setting similarly as in the randomized setting the upper complexity bounds for the approximation of the median will grow only by a logarithmic factor. It is sufficient to take the failure probability parameter $\delta^{\mathrm{quant}} = \varepsilon^2/\lceil \log \varepsilon^{-1} \rceil$, similarly as in the randomized setting, in the definition of the algorithm and repeat the reasoning leading to (17).

Note that for absolute error the assumption that the function is bounded away from zero is necessary at least in some neighbourhood of the median. Without this assumption, in all settings, the ε-complexity of approximating the median in infinite for sufficiently small ε. Consider the the deterministic setting. Let $f_{m,\delta} \in F^{r,\rho}$ be the function with support in the interval $(m-\delta/2, m+\delta/2)$ and with the median equal to m. Let $a = \int_0^1 f_{m,\delta}(t)dt$. Let $g \in F^{r,\rho}$ be the function, such that $g(x) = 0$ for all $x \in [1/4, 3/4]$, $\int_0^{1/4} g(t)dt = \int_{3/4}^1 g(t)dt = 1/2 - a/2$. Consider the deterministic algorithm ϕ that uses n function values. Let $x_1, x_2, \ldots x_n$ be the information points of algorithm ϕ for function g. For $\delta < 1/(8(n+1))$ there exist points $m_1 \in [1/4, 3/8]$ such that $f_{m_1,\delta}(x_i) = 0$ for $i = 0, 1, \ldots, n$, and $m_2 \in [5/8, 3/4]$ such that $f_{m_2,\delta}(x_i) = 0$ for $i = 0, 1, \ldots, n$. Define two functions $h_1 = g + f_{m_1,\delta}$ and $h_2 = g + f_{m_2,\delta}$ belonging to $F_1^{r,\rho}$. Both functions share the same information and are indistinguishable by the algorithm ϕ. The medians of these functions are equal to m_1 and m_2, respectively. Since $m_2 - m_1 \geq 1/4$, the error of algorithm ϕ is at least $1/8$. Thus, the problem cannot be solved with precision less than $1/8$.
We can prove in the similar way (see the proof of Thm 1 in [14]) that the complexity in the randomized setting is infinite for sufficiently small ε. In the quantum setting the statement follows from the fact that the quantum qubit ε-complexity (the minimal number of qubits needed to solve the problem with precision ε) is bounded from below by the deterministic 2ε-complexity (see [24, Thm 8.1]). Thus, when the problem is unsolvable in the deterministic setting, it remains unsolvable in the quantum setting.