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THE DELTA GAME

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We introduce a game related to the I_{3322} game and analyze a constrained value function for this game over various families of synchronous quantum probability densities.

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1 Introduction

Suppose that two separated labs, but in a possibly entangled state, are conducting quantum experiments. In this case we let $p(i,j|v,w)$ denote the conditional probability that if the first lab conducts experiment v and the second lab conducts experiment w, then they get outcomes i and j, respectively. If each lab can conduct one of n experiments and each experiment has k outcomes then the set of all such conditional probability densities, $p(i,j|x,y)$, is a set of non-negative n^2k^2-tuples. A subject of a great deal of current research has been the study of various mathematical models for what should constitute the set of all such bipartite conditional quantum probabilities, also called quantum correlations, beginning with Tsirelson [1, 2] and continuing with [3, 4, 5, 6, 7, 8, 9, 10]. In particular, the *Tsirelson conjectures* ask whether or not several different mathematical models for these conditional quantum probabilities yield the same sets. Whether or not two of these models yield the same sets of densities is now known to be equivalent to *Connes' embedding problem* [3, 4, 5].

We will describe these models carefully later, but for now it is enough to know that if we let $C_q(n,k)$ denote the set of conditional probability densities $p(i,j|x,y)$ arising from the most commonly used (and smallest) model for n experiments with k outcomes each, and let $C_{qc}(n,k)$ denote the larger set of conditional probability given by what is called the quantum commuting model, then Connes' embedding conjecture is equivalent to determining if the closure of the smaller set is equal to the larger set for all n,k.

Recently, W. Slofstra [9, 10] has shown that for n about 100 the set $C_q(n, 8)$ is not closed. But it is still not known if its closure is equal to $C_{qc}(n, 8)$. Slofstra's proofs rely heavily on some very deep results in the theory of finitely presented groups, and give little information about the geometry of these sets, or what happens for smaller values of n and k.

Another topic of current interest is attempting to compute the quantum value of the I_{3322} game, which can be viewed as finding the supremum of a linear functional defined on $C_q(3, 2)$, and deciding if it is actually attained. If it was known that $C_q(3, 2)$ was closed, then, since this set is bounded, it would be compact, and we would at least know that the value is attained. This has been the subject of a great deal of research [11, 12, 13, 14]. The lack of a resolution to this problem illustrates how little we understand about the sets $C_q(n, k)$ even for relatively small values of n and k.

Many authors believe that I_{3322} does not attain its quantum value, over the set of quantum densities arising from this standard model for densities and hence that $C_q(3, 2)$ is not closed.

In this paper we introduce and study a game, brought to our attention by R. Cleve, that we call the Delta game. It is a simplification of the I_{3322} game, in the sense that the linear functional that one needs to maximize to find its value has more symmetries.

We reduce the number of parameters further by requiring that all of our quantum probability densities are *synchronous*. A conditional probability density is synchronous provided that whenever both labs conduct the same experiment, then they must get the same outcome, i.e., for every v, $p(i, j|v, v) = 0$ whenever $i \neq j$. The subsets of synchronous densities, which we denote, $C_q^s(n, k) \subseteq C_q(n, k)$ and $C_{qc}^s(n, k) \subseteq C_{qc}^s(n, k)$, respectively, are potentially easier to describe. Following [7], we refer to the maximum of the value function of a game over all synchronous densities as its *synchronous q-value* and *synchronous qc-value*, respectively.

Finally, we attempt to gain even more knowledge of the geometry of the set of synchronous correlations by adding on the constraint that certain marginal probability densities are a fixed value t. In this manner, we obtain two functions of t that correspond to the constrained synchronous q-value and synchronous qc-value of the Delta game.

Our main result is that these two functions are equal for all t and that for each t the supremum that defines the synchronous q-value is attained.

Our results lead us to believe that the set of all 3 input, 2 output synchronous quantum probability densities given by the standard model, $C_q^s(3, 2)$, is closed, while it is widely believed that $C_q(3, 2)$ is not closed. In fact, that it is believed that I_{3322} does not attain its value and, consequently, that $C_q(3, 2)$ is not closed.

Note added in proof: Since this paper was written, we have shown in [15], using essentially a constrained value function and an extension of this game, that the set of 5 input, 2 output quantum probability densities given by the standard model, is not closed.

2 Preliminaries

Recall that a general two person finite input-output game $\mathcal{G}$, involves two noncommunicating players, Alice (A) and Bob (B), and a Referee (R). The game is described by $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ where I_A, I_B, O_A, O_B are nonempty finite sets, representing Alice's inputs, Bob's inputs, Alice's outputs and Bob's outputs, respectively, and with $\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$ a function.

For each *round* of the game, Alice receives input $v \in I_A$ and Bob receives input $w \in I_B$

from the Referee and then Alice and Bob produce outputs $i \in O_A$ and $j \in O_B$, respectively. They win if $\lambda(v, w, i, j) = 1$ and lose if $\lambda(v, w, i, j) = 0$. The function λ is called the *rule* or *predicate* function.

Suppose that Alice and Bob have a random way to produce outputs. This is informally what is meant by a *strategy.* If we observe a strategy over many rounds we will obtain joint conditional probabilities $p(i, j|v, w)$ for the event that Alice outputs i on input v and Bob outputs j on input w. For this reason, any tuple $(p(i, j|v, w))_{i \in O_A, j \in O_B, v \in I_A, w \in I_B}$ satisfying

$$p(i, j|v, w) \geq 0 \ \text{ and } \sum_{i \in O_A, j \in O_B} p(i, j|v, w) = 1, \, \forall \, v \in I_A, w \in I_B,$$

will be called a *correlation.*

A correlation $(p(i, j|v, w))$ is called a *winning* or *perfect* correlation for $\mathcal{G}$ if

$$\lambda(v, w, i, j) = 0 \Rightarrow p(i, j|v, w) = 0,$$

that is, it produces disallowed outputs with zero probability.

If we also assume that the Referee chooses inputs according to a known probability distribution $\pi : I_A \times I_B \to [0, 1]$, that is,

$$\pi(v, w) \geq 0 \quad \text{ and } \quad \sum_{(v,w) \in I_A \times I_B} \pi(v, w) = 1,$$

then it is possible to assign a number to each correlation that measures the probability that Alice and Bob will win a round given their correlation. The *value* of the correlation $p = (p(i, j|v, w))$, corresponding to the distribution π on inputs, is given by

$$V(p, \pi) = \sum_{i,j,v,w} \lambda(v, w, i, j)\pi(v, w)p(i, j|v, w).$$

Note that a perfect correlation always has value 1 and, provided that $\pi(v, w) > 0$ for all v and w a correlation will have value 1 if and only if it is a perfect correlation.

The *value of the game* $\mathcal{G}$ with respect to a fixed probability density π on the inputs over a given set $\mathcal{F}$ of correlations is given by

$$\omega_{\mathcal{F}}(\mathcal{G}, \pi) = \sup\{V(p, \pi) : p \in \mathcal{F}\}.$$

Because the set of all correlations is a bounded set in a finite dimensional vector space, whenever $\mathcal{F}$ is a closed set, it will be compact and so this supremum over $\mathcal{F}$ will be attained.

A finite input-output game as above is called *synchronous* provided that $I_A = I_B := I$, $O_A = O_B := O$ and for all $v \in I$, $\lambda(v, v, i, j) = 0$ whenever $i \neq j$. This condition can be summarized as saying that whenever Alice and Bob receive the same input then they must produce the same output. A correlation $(p(i, j|v, w))$ is called *synchronous* provided that $p(i, j|v, v) = 0$ for all $v \in I_A$ and for all $i \neq j$. Note that when $\mathcal{G}$ is a synchronous game, then any perfect correlation must be synchronous.

In this paper we are interested in studying the Δ game, which is a synchronous game, and computing $\omega(\Delta, \mathcal{F})$ as we let $\mathcal{F}$ vary over the various mathematical models for synchronous quantum correlations. We now introduce these various models for quantum densities.

Recall that a set, $\{R_k\}_{k=1}^n$, of operators on some Hilbert space $\mathcal{H}$ is called a *positive operator valued measure* (POVM) provided $R_k \geq 0$, for each k, and $\sum_{k=1}^n R_k = I$. Also a set of projections, $\{P_k\}_{k=1}^n$, on some Hilbert space $\mathcal{H}$ is called a *projection valued measure* (PVM) provided $\sum_{k=1}^n P_k = I$. Thus every PVM is a POVM.

A *quantum* correlation for a game $\mathcal{G}$ means that Alice and Bob have finite dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For each input $v \in I$, Alice has a PVM – $\{P_{v,i}\}_{i\in O}$ on $\mathcal{H}_A$, and similarly for each input $w \in I$, Bob has a PVM – $\{Q_{w,j}\}_{j\in O}$ on $\mathcal{H}_B$. They also share a state $h \in \mathcal{H}_A \otimes \mathcal{H}_B$ ($\|h\| = 1$) such that

$$p(i,j|v,w) = \langle (P_{v,i} \otimes Q_{w,j})h, h\rangle .$$

The set of all $(p(i,j|v,w))$ arising from all choices of finite dimensional Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, all PVMs and all states h is called the set of *quantum correlations* denoted by $C_q(n,m)$.

Another family of correlations are the *commuting quantum* correlations. In this case there is a single (possibly infinite dimensional) Hilbert space $\mathcal{H}$ and for each input $v \in I$, Alice has a PVM $\{P_{v,i}\}_{i\in O}$, and similarly for each input $w \in I$, Bob has a PVM $\{Q_{w,j}\}_{j\in O}$, satisfying $P_{v,i}Q_{w,j} = Q_{w,j}P_{v,i}$ (hence the name commuting). They share a state $h \in \mathcal{H}$ ($\|h\| = 1$) such that

$$p(i,j|v,w) = \langle (P_{v,i}Q_{w,j})h, h\rangle .$$

The set of all $(p(i,j|v,w))$ arising this way is denoted by $C_{qc}(n,m)$ and is called the set of *commuting quantum correlations.*

Remark 1 *In the above definitions one could replace the PVM's with POVM's throughout, and this is used as the definitions of these sets in many references. Since there are more POVM's then PVM's one might obtain larger sets, say $\widetilde{C}_q(n,m)$ and $\widetilde{C}_{qc}(n,m)$. But, in fact, $\widetilde{C}_q(n,m) = C_q(n,m)$ and $\widetilde{C}_{qc}(n,m) = C_{qc}(n,m)$. The fact that $\widetilde{C}_q(n,m) = C_q(n,m)$ follows by a simple dilation trick. On the Hilbert space $\mathcal{H}_A$, one simply uses a Naimark dilation to enlarge the space to $\mathcal{K}_A$ and dilate the set of POVM's to a set of PVM's on $\mathcal{K}_A$. One similarly dilates Bob's POVM's to PVM's on $\mathcal{K}_B$ and then considers the tensor products of these PVM's on $\mathcal{K}_A \otimes \mathcal{K}_B$. The proof that $\widetilde{C}_{qc}(n,m) = C_{qc}(n,m)$ is somewhat more difficult and can be found in [4, Proposition 3.4], and also as Remark 10 of [3]. A third proof appears in [6]. We shall sometimes refer to this as the* disambiguation *of the two possible definitions.*

Remark 2 *By Theorem 5.3 in [8], $C_q(n,m) \subseteq C_{qc}(n,m)$, with $(p(i,j|v,w)) \in C_q(n,m)$ if and only if $(p(i,j|v,w)) \in C_{qc}(n,m)$ such that the Hilbert space $\mathcal{H}$ in its realization is finite dimensional.*

There is yet another correlation set denoted by $C_{vect}(n,m)$ that is often called the set of *vector correlations.* It is the set of all $(p(i,j|v,w))$ such that $p(i,j|v,w) = \langle x_{v,i}, y_{w,j}\rangle$ for sets of vectors $\{x_{v,i} : v \in I, i \in O\}, \{y_{w,j} : w \in I, j \in O\}$ in a Hilbert space $\mathcal{H}$ and a unit vector $h \in \mathcal{H}$, which satisfy

(a) $x_{v,i} \perp x_{v,j}$ and $y_{w,i} \perp y_{w,j}$ for all $i \neq j$ in O.
(b) $\sum_{i\in O} x_{v,i} = h = \sum_{j\in O} y_{w,j}$ for all $v, w \in I$.
(c) $\langle x_{v,i}, y_{w,j}\rangle \geq 0$ for all $v, w \in I$ and $i, j \in O$.

Since all of the inner products appearing in the above definition are real, there is no generality lost in requiring $\mathcal{H}$ to be a real Hilbert space as well. These correlations have been studied at other places in the literature, see for example [16] where they are referred to as *almost quantum correlations* and they can be interpreted as the first level of the NPA hierarchy [17]. Vector correlations also appear in [18] where they were used to approximate quantum values for unique games.

The above correlation sets are related in the following way

$$C_q(n,m) \subseteq C_{qc}(n,m) \subseteq C_{vect}(n,m) \subset \mathbb{R}^{n^2m^2}, \tag{1}$$

for all $n,m \in \mathbb{N}$ and they are all convex sets. It is known that the sets $C_{qc}(n,m)$ and $C_{vect}(n,m)$ are closed sets in $\mathbb{R}^{n^2m^2}$. Set $C_{qa}(n,m) = \overline{C_q(n,m)}$ so that

$$C_q(n,m) \subseteq C_{qa}(n,m) \subseteq C_{qc}(n,m), \tag{2}$$

for all $n,m \in \mathbb{N}$. W. Slofstra [10] recently proved that there exists an n and m such that $C_q(n,m)$ is not a closed set. Hence $C_q(n,m)$ is in general a proper subset of $C_{qa}(n,m)$, but whether or not they are different for all values of n,m is unknown. It also remains an open question to determine whether $C_{qa}(n,m) = C_{qc}(n,m)$ for all n and m or not. In [3], it was proven that if Connes' embedding conjecture is true then $C_{qa}(n,m) = C_{qc}(n,m)$ for all n and m. The converse was proven in [5]. Thus we know that $C_{qa}(n,m) = C_{qc}(n,m)$, $\forall n,m$ is equivalent to Connes' embedding conjecture.

For $t \in \{q, qa, qc, vect\}$, let $C_t^s(n,m)$ denote the subset of all synchronous correlations. The synchronous sets $C_t^s(n,m)$ are also convex for $t \in \{q, qa, qc, vect\}$. The set of synchronous commuting quantum correlations, $C_{qc}^s(n,m)$, may be characterized in the following way.

Let $\mathcal{A}$ be a unital C*-algebra. Recall that a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ is called a *tracial state* if τ is positive, $\tau(1) = 1$, and $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$.

Theorem 3 (Theorem 5.5, [8]) *Let $(p(i,j|v,w)) \in C_{qc}^s(n,m)$ be realized with PVMs $\{P_{v,i} : v \in I\}_{i\in O}$ and $\{Q_{w,j} : w \in I\}_{j\in O}$ on some Hilbert space $\mathcal{H}$ satisfying $P_{v,i}Q_{w,j} = Q_{w,j}P_{v,i}$ and with some unit vector $h \in \mathcal{H}$ so that $p(i,j|v,w) = \langle P_{v,i}Q_{w,j}h, h\rangle$. Then*

- *(a) $P_{v,i}h = Q_{v,i}h$ for all $v \in I, i \in O$;*
- *(b) $p(i,j|v,w) = \langle (P_{v,i}P_{w,j})h, h\rangle = \langle (Q_{w,j}Q_{v,i})h, h\rangle = p(j,i|w,v)$;*
- *(c) Let $\mathcal{A}$ be the C^*-algebra in $B(\mathcal{H})$ generated by the family $\{P_{v,i} : v \in I, i \in O\}$ and define $\tau : \mathcal{A} \to \mathbb{C}$ by $\tau(X) = \langle Xh, h\rangle$. Then τ is a tracial state on $\mathcal{A}$ and $p(i,j|v,w) = \tau(P_{v,i}P_{w,j})$.*

Conversely, let $\mathcal{A}$ be a unital C^*-algebra equipped with a tracial state τ and with $\{e_{v,i} : v \in I, i \in O\} \subset \mathcal{A}$ a family of projections such that $\sum_{i\in O} e_{v,i} = 1$ for all $v \in I$. Then $(p(i,j|v,w))$ defined by $p(i,j|v,w) = \tau(e_{v,i}e_{w,j})$ is an element of $C_{qc}^s(n,m)$. That is, there exists a Hilbert space $\mathcal{H}$, a unit vector $h \in \mathcal{H}$ and mutually commuting PVMs $\{P_{v,i} : v \in I\}_{i\in O}$ and $\{Q_{w,j} : w \in I\}_{j\in O}$ on $\mathcal{H}$ such that

$$p(i,j|v,w) = \langle (P_{v,i}Q_{w,j})h, h\rangle = \langle (P_{v,i}P_{w,j})h, h\rangle = \langle (Q_{w,j}Q_{v,i})h, h\rangle$$

This theorem and Remark 2 lead to the following characterization of $C_q^s(n,m)$:

Proposition 4 *We have that $(p(i,j|v,w)) \in C_q^s(n,m)$ if and only if there exists a finite dimensional C^*-algebra $\mathcal{A}$ with a tracial state τ and with a family of projections $\{e_{v,i} : v \in I, i \in O\} \subset \mathcal{A}$ such that $\sum_{i \in O} e_{v,i} = 1$ for all $v \in I$ and $p(i,j|v,w) = \tau(e_{v,i}e_{w,j})$ for all i,j,v,w.*

The set of synchronous vector correlations is described in the next proposition.

Proposition 5 *We have $(p(i,j|v,w)) \in C_{vect}^s(n,m)$ if and only if*

$$p(i,j|v,w) = \langle x_{v,i}, x_{w,j} \rangle$$

for a set of vectors $\{x_{v,i} : v \in I, i \in O\} \subset \mathcal{H}$ with $x_{v,i} \perp x_{v,j}$ when $i \neq j$, $\sum_{i=1}^m x_{v,i} = h$ for some unit vector $h \in \mathcal{H}$, and $\langle x_{v,i}, x_{w,j} \rangle \geq 0$.

The synchronous subsets satisfy inclusions as in expression 2,

$$C_q^s(n,m) \subseteq C_{qa}^s(n,m) \subseteq C_{qc}^s(n,m) \subseteq C_{vect}^s(n,m) \subseteq \mathbb{R}^{n^2m^2},$$

and since $C_{qa}(n,m)$, $C_{qc}(n,m)$, and $C_{vect}(n,m)$ are closed sets it is easy to see that their synchronous subsets are also closed. We can also ask the synchronous analogues of the questions described before. It is easy to see that, $C_t(n,m) = C_{t'}(n,m) \implies C_t^s(n,m) = C_{t'}^s(n,m)$, but there is no a priori reason that the converses should hold. It is shown in [7] that $\overline{C_q^s(n,m)} = C_{qc}^s(n,m)$ for all $n,m \in \mathbb{N}$ is equivalent to Connes' embedding conjecture. In [19] it is shown that $\overline{C_q^s(n,m)} = C_{qa}^s(n,m)$, i.e., that a synchronous density that is a limit of densities in $C_q(n,m)$ is a limit of synchronous densities in $C_q(n,m)$.

The questions described above can be formulated in terms of values of games. If we restrict $\omega_{\mathcal{F}}(\mathcal{G},\pi)$ to the synchronous subset $\mathcal{F}^s$ of $\mathcal{F}$, we obtain the *synchronous value* of the game $\mathcal{G}$ given the probability density π defined by

$$\omega_{\mathcal{F}}^s(\mathcal{G},\pi) = \sup\{V(p,\pi) : p \in \mathcal{F}^s\}.$$

As before we write this as $\omega_t^s(\mathcal{G},\pi)$ when $\mathcal{F} = C_t(n,m)$. The following proposition relates the synchronous values of a game to Connes' embedding conjecture.

Proposition 6 (Proposition 4.1, [7]) *If Connes' embedding conjecture is true, then we have $\omega_q(\mathcal{G},\pi) = \omega_{qc}(\mathcal{G},\pi)$ and $\omega_q^s(\mathcal{G},\pi) = \omega_{qc}^s(\mathcal{G},\pi)$ for every game $\mathcal{G}$ and every distribution π.*

Remark 7 *It is not known if the converse of any of these above implications is true. That is, for example, if $\omega_q(\mathcal{G},\pi) = \omega_{qc}(\mathcal{G},\pi), \forall\mathcal{G}, \forall\pi$, then must Connes' embedding conjecture be true?*

We now introduce the Delta game.

3 The Delta Game

The Delta (stylized as Δ) game is a nonlocal game with three inputs and two outputs. We have $I = \{0,1,2\}$ as the input set and $O = \{0,1\}$ as the output set (thus $n = 3, m = 2$). Out of the 36 possible tuples (v,w,i,j), allowed rules $(v,w,i,j) \in I \times I \times O \times O$ are

$$\begin{array}{cccccc} (0,0,0,0), & (0,1,0,1), & (1,1,0,0), & (1,2,0,1), & (2,2,0,0), & (2,0,0,1), \\ (0,0,1,1), & (0,1,1,0), & (1,1,1,1), & (1,2,1,0), & (2,2,1,1), & (2,0,1,0), \end{array}$$

whereas the disallowed rules are

$$\begin{array}{cccccc} (0,0,0,1), & (0,1,0,0), & (1,1,0,1), & (1,2,0,0), & (2,2,0,1), & (2,0,0,0), \\ (0,0,1,0), & (0,1,1,1), & (1,1,1,0), & (1,2,1,1), & (2,2,1,0), & (2,0,1,1). \end{array}$$

The remaining 12 tuples (v,w,i,j) are also all allowed.

The first 12 allowed rules may be visualized as in Figure 1. The allowed edges $(0,0),(1,1)$ and $(2,2)$ are shown with dashed lines while $(0,1),(1,2),(2,0)$ are shown with solid lines. The dashed lines are *even* while the solid lines are *odd*. This means that if Alice and Bob are given inputs joined by dashed lines then they return outputs with even sum; and in the other case they return outputs with odd sum.

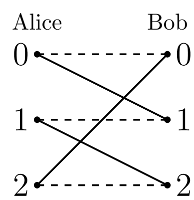

Fig. 1. Δ game rule function.

Alice and Bob receive inputs according to the uniform distribution $\pi = (\pi(v,w))$ on the set of inputs

$$E = \{(0,0),(1,1),(2,2),(0,1),(1,2),(2,0)\},$$

that is, $\pi(v,w) = \frac{1}{6}$ for all $(v,w) \in E$ (and zero otherwise). To compute the synchronous value of the game given the distribution π we first compute the value of a single correlation $p = (p(i,j|v,w))$, which is,

$$V(p,\pi) = \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right),$$

where the sum over i is done mod 2, while the sum over v is done mod 3. The value of the game then becomes,

$$\omega_t^s(\mathcal{G},\pi) = \sup\left\{\frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right) : p(i,j|v,w) \in C_t^s(3,2)\right\},$$

where $t \in \{q, qa, qc, vect\}$. Denote the expression inside the braces by,

$$\widetilde{\theta} = \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right).$$

We will use Theorem 3 and Proposition 5 to simplify $\widetilde{\theta}$ and to obtain expressions involving operators and vectors in the case of $t = qc$ and $t = vect$, respectively. Moreover, when $t = q$, by Remark 2 it suffices to proceed as in the case $t = qc$ using Theorem 3 to simplify $\widetilde{\theta}$, but restricting to the case of operators on finite dimensional Hilbert spaces.

We first handle the $t = qc$ case. By Theorem 3, a correlation $(p(i, j|v, w))$ is in $C^s_{qc}(3, 2)$ if and only if there exists a C*-algebra $\mathcal{A}$ of $B(\mathcal{H})$ generated by a family of projections $\{A_{v,i} : i = 0, 1 \text{ and } v = 0, 1, 2\}$ satisfying $A_{v,0} + A_{v,1} = I_{\mathcal{H}}$ for $v \in \{0, 1, 2\}$ and a tracial state $\tau : \mathcal{A} \to \mathbb{C}$ such that $p(i, j|v, w) = \tau(A_{v,i}A_{w,j}) = \langle (A_{v,i}A_{w,j})h, h\rangle$, for some unit vector $h \in \mathcal{H}$. For notational convenience we define

$$A_0 = A_{0,0}, \qquad A_1 = A_{1,0}, \qquad A_2 = A_{2,0}.$$

Then $A_{v,1} = I_{\mathcal{H}} - A_v = I_{\mathcal{H}} - A_{v,0}$ for $v \in \{0, 1, 2\}$. Using this we can rewrite $\widetilde{\theta}$ as

$$\begin{aligned}\widetilde{\theta} &= \frac{1}{6}\sum_{v=0}^{2}\sum_{i=0}^{1} p(i, i|v, v) + p(i, i+1|v, v+1)\\ &= \frac{1}{6}\sum_{v=0}^{2}\sum_{i=0}^{1} \tau(A_{v,i}A_{v,i}) + \tau(A_{v,i}A_{v+1,i+1})\\ &= \frac{1}{2} + \frac{1}{3}\tau(A_0 + A_1 + A_2) - \frac{1}{3}\sum_{v=0}^{2}\tau(A_vA_{v+1}).\end{aligned}$$

We now define a "parameter" θ by setting

$$\theta = \frac{1}{3}\tau(A_0 + A_1 + A_2) = \frac{1}{3}\sum_{v=0}^{2} p(0, 0|v, v),$$

which enables us to write $\widetilde{\theta}$ as

$$\widetilde{\theta} = \frac{1}{2} + \frac{1}{3}\tau(A_0 + A_1 + A_2) - \frac{1}{3}\sum_{v=0}^{2}\tau(A_vA_{v+1}) = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2}\tau(A_vA_{v+1}). \tag{3}$$

Similarly, in the $t = vect$ case, using Proposition 5 and proceeding as in the previous paragraph, writing x_i for $x_{i,0}$, we see that $\widetilde{\theta}$ is given by

$$\widetilde{\theta} = \frac{1}{2} + \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle - \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle,$$

for some set of vectors $\{x_0, x_1, x_2, h\}$ in some Hilbert space $\mathcal{H}$ satisfying $\|h\| = 1$ and, for all v and w,

$$x_v \perp (h - x_v), \qquad \langle x_v, x_w\rangle \geq 0, \qquad \langle x_v, h - x_w\rangle \geq 0, \qquad \langle h - x_v, h - x_w\rangle \geq 0.$$

Notice that $x_v \perp (h - x_v)$ implies $\langle x_v, h\rangle = \langle x_v, x_v\rangle$. Again letting $\theta = \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle$, we may write

$$\widetilde{\theta} = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle. \tag{4}$$

For each $t \in \{q, qa, qc, vect\}$, let Θ_t^s denote the set of all points $(\theta, \widetilde{\theta}) \in \mathbb{R}^2$ that can be obtained from correlations $(p(i,j|v,w)) \in C_t^s(n,m)$ in the manner described above. Since the above equations defining θ and $\widetilde{\theta}$ involve linear combinations of $p(i,j|v,w)$, the map sending $(p(i,j|v,w)) \in \mathbb{R}^{36}$ to $(\theta, \widetilde{\theta}) \in \mathbb{R}^2$ will be linear.

We want to see how Θ_t^s behaves under different values of t. It is easy to verify that each Θ_t^s is a convex set since it is the image of the convex set $C_t^s(n,m)$. Moreover, since $C_t^s(n,m)$ is compact for $t \in \{qa, qc, vect\}$, it follows that Θ_t^s is also compact (and hence closed). To find Θ_t^s, it is enough to compute the following two functions for each θ,

$$f_t^u(\theta) = \sup\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta_t^s\}, \qquad f_t^l(\theta) = \inf\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta_t^s\},$$

where u and l stand for *upper* and *lower*, respectively. We also need to determine if the supremum and the infimum are attained or not. Notice that in the qc case, in order to find the supremum (resp., infimum) of $\widetilde{\theta} = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2} \tau(A_v A_{v+1})$, we need to find the infimum (resp., supremum) of the quantity $\sum_{v=0}^{2} \tau(A_v A_{v+1})$. A similar statement holds for the *vect* case.

In the qc case, notice that since A_v's are projections and τ is a state we get, $0 \leq \frac{1}{3}\tau(A_0 + A_1 + A_2) \leq 1$. Similarly in the *vect* case, by the Cauchy-Schwarz inequality we get $0 \leq \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle \leq 1$. Hence $0 \leq \theta \leq 1$. Conversely, if $\theta \in [0,1]$, then we can always find projections A_0, A_1, A_2 in some C*-algebra with a tracial state τ, such that $\frac{1}{3}\tau(A_0 + A_1 + A_2) = \theta$.

It is evident that $\Theta_q^s \subseteq \Theta_{qa}^s \subseteq \Theta_{qc}^s \subseteq \Theta_{vect}^s$.

Theorem 8 *For $t \in \{q, qa, qc\}$, we have*

$$f_t^l(\theta) = \frac{1}{2}, \qquad f_t^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \leq \theta \leq \frac{1}{3} \\ \frac{3+\theta}{4} & \text{for } \frac{1}{3} \leq \theta \leq \frac{1}{2} \\ \frac{4-\theta}{4} & \text{for } \frac{1}{2} \leq \theta \leq \frac{2}{3} \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \leq \theta \leq 1. \end{cases} \tag{5}$$

Moreover, we have

$$f_{vect}^l(\theta) = \frac{1}{2}, \qquad f_{vect}^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \leq \theta \leq \frac{1}{3} \\ \frac{1+3\theta-3\theta^2}{2} & \text{for } \frac{1}{3} \leq \theta \leq \frac{2}{3} \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \leq \theta \leq 1. \end{cases} \tag{6}$$

In all of the these cases, the infimum and supremum are attained by both f_t^u and f_t^l. Since $(\theta, \widetilde{\theta}) \in \Theta_t^s$ if and only if $0 \leq \theta \leq 1$ and $f_t^l(\theta) \leq \widetilde{\theta} \leq f_t^u(\theta)$, we see that Θ_t^s is a closed set in $\mathbb{R}^2$ for each $t \in \{q, qa, qc, vect\}$. In particular, we have

$$\Theta_q^s = \Theta_{qa}^s = \Theta_{qc}^s \subsetneq \Theta_{vect}^s. \tag{7}$$

The functions as obtained in Theorem 8 are shown in Figure 2.

The fact that the functions f_{vect}^u and f_{qc}^u are different allows us to deduce the following.

Corollary 9 *We have that $C_{qc}^s(3,2) \subsetneq C_{vect}^s(3,2)$ and consequently, $C_{qc}(3,2) \subsetneq C_{vect}(3,2)$.*

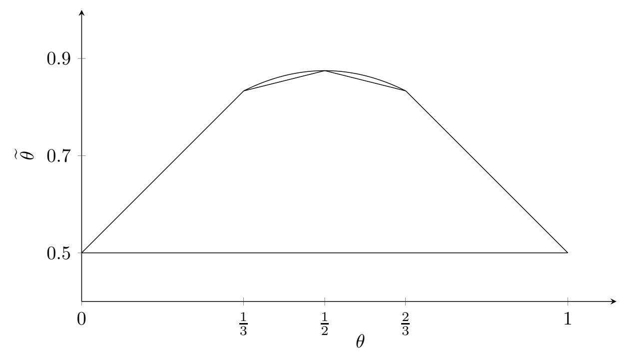

Fig. 2. Plots of $f_t^l = f_{vect}^l$, f_t^u and f_{vect}^u from Theorem 8

Remark 10 *There is another larger set of correlations that we could have considered, the* nonsignalling correlations. *For a definition, see [20]. If we let $C_{ns}^s(n,k)$ denote the set of synchronous nonsignalling correlations, then it is shown in [20] that the set $C_{ns}^s(n,2)$ is a polytope. If we let f_{ns}^u denote the analogous function obtained by taking the supremum over the set of synchronous nonsignalling correlations, then the fact that the set of such correlations is a polytope implies that f_{ns}^u would be piecewise linear. Hence, $f_{vect}^u \neq f_{ns}^u$ and we can conclude that $C_{vect}^s(3,2) \subsetneq C_{ns}^s(3,2)$.*

4 The Case of $t = vect$.

In this section, we compute f_{vect}^l and f_{vect}^u to prove (6) in Theorem 8. We will employ the symmetrization provided by the next lemma.

Lemma 11 Θ_{vect}^s *is equal to the set of pairs $(\theta, \widetilde{\theta})$ with $0 \leq \theta \leq 1$, such that there exist vectors x_0, x_1, x_2, h in a Hilbert space with the properties:*

(*a*) $\|h\| = 1$,
(*b*) $\forall v \ \langle x_v, h\rangle = \langle x_v, x_v\rangle = \theta$,
(*c*) $\forall v \ \langle x_v, x_{v+1}\rangle = \beta \ \geq 0$, *where* $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$ *and* $2\theta - 1 \leq \beta \leq \theta$.

Proof. By Proposition 2.4, and the discussion in Section 3, Θ_{vect}^s is the set of pairs $(\theta, \widetilde{\theta})$ such that there exist vectors x_0, x_1, x_2, h in a Hilbert space $\mathcal{H}$ with the properties that $\|h\| = 1$, for all v and w, we have

$$\langle x_v, h\rangle = \langle x_v, x_v\rangle, \quad \langle x_v, x_w\rangle \geq 0, \quad \langle x_v, h - x_w\rangle \geq 0, \quad \langle h - x_v, h - x_w\rangle \geq 0$$

and, moreover,

$$\frac{1}{3}\sum_{v=0}^{2}\langle x_v, h\rangle = \theta, \qquad \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle = \beta,$$

where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$. The conditions appearing in the lemma are precisely these, but with the additional requirement that the quantities $\langle x_v, h\rangle$ and $\langle x_v, x_{v+1}\rangle$ are the same for all $v \in \{0, 1, 2\}$. However, given x_0, x_1, x_2, h satisfying these weaker conditions and considering

$$\widetilde{h} = \frac{1}{\sqrt{3}}(h \oplus h \oplus h), \qquad \widetilde{x}_v = \frac{1}{\sqrt{3}}(x_v \oplus x_{v+1} \oplus x_{v+2})$$

in the Hilbert space $\mathcal{H}^{\oplus 3}$, we see that $\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2, \widetilde{h}$ satisfy the stronger conditions and yield the same pair $(\theta, \widetilde{\theta})$. $\square$.

We now prove the part of Theorem 8 involving the case $t = vect$.

Theorem 12 *The functions*

$$f^l_{vect}(\theta) = \inf\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta^s_{vect}\}, \qquad f^u_{vect}(\theta) = \sup\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta^s_{vect}\} \tag{8}$$

are given by

$$f^l_{vect}(\theta) = \frac{1}{2}, \qquad f^u_{vect}(\theta) = \begin{cases} \frac{1}{2} + \theta & \textit{for } 0 \le \theta \le \frac{1}{3} \\ \frac{1+3\theta-3\theta^2}{2} & \textit{for } \frac{1}{3} \le \theta \le \frac{2}{3} \\ \frac{3}{2} - \theta & \textit{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{9}$$

Moreover, both the infimum and supremum are attained, for all values of $\theta \in [0, 1]$.

Proof. Fix $\theta \in [0, 1]$. By Lemma 11, we are interested in the set of β such that there exist vectors x_0, x_1, x_2, h in some Hilbert space satisfying the conditions listed there. Let $y_v = h - x_v$. Consider the Gramian matrix G associated with the seven vectors $h, x_0, x_1, x_2, y_0, y_1, y_2$. The conditions of Lemma 11 imply that this is the 7×7 matrix

$$G = \begin{bmatrix} 1 & \theta & \theta & \theta & 1-\theta & 1-\theta & 1-\theta \\ \theta & \theta & \beta & \beta & 0 & \theta-\beta & \theta-\beta \\ \theta & \beta & \theta & \beta & \theta-\beta & 0 & \theta-\beta \\ \theta & \beta & \beta & \theta & \theta-\beta & \theta-\beta & 0 \\ 1-\theta & 0 & \theta-\beta & \theta-\beta & 1-\theta & 1+\beta-2\theta & 1+\beta-2\theta \\ 1-\theta & \theta-\beta & 0 & \theta-\beta & 1+\beta-2\theta & 1-\theta & 1+\beta-2\theta \\ 1-\theta & \theta-\beta & \theta-\beta & 0 & 1+\beta-2\theta & 1+\beta-2\theta & 1-\theta \end{bmatrix}$$

and furthermore, that G is positive semidefinite and

$$\max(0, 2\theta - 1) \le \beta \le \theta. \tag{10}$$

Conversely, assume that we are given a positive semidefinite 7×7 matrix. Then it is the Gramian of some set of vectors, $h, x_0, x_1, x_2, y_0, y_1, y_2$ and we claim that these vectors satisfy the relations given in Lemma 11. To see this note that $\|h\|^2 = 1$, while $\|x_i\|^2 = \theta$ and $\|y_i\|^2 = 1 - \theta$. The zeros in the matrix yield that $x_i \perp y_i$. Thus, $\|x_i + y_i\| = 1$. The fact that $\langle h, x_i + y_i\rangle = w + (1 - w) = 1$, together with Cauchy-Schwarz inequality yields that $h = x_i + y_i$. The rest of the relations follow similarly. Thus, we are interested in the set of β that satisfy (10) and yield a positive semidefinite matrix G given above.

We apply one step of the Cholesky algorithm, and conclude that the 7×7 matrix G is positive semidefinite if and only if the following 6×6 matrix G' is positive semidefinite:

$$G' = \begin{bmatrix} \theta-\theta^2 & \beta-\theta^2 & \beta-\theta^2 & \theta^2-\theta & \theta^2-\beta & \theta^2-\beta \\ \beta-\theta^2 & \theta-\theta^2 & \beta-\theta^2 & \theta^2-\beta & \theta^2-\theta & \theta^2-\beta \\ \beta-\theta^2 & \beta-\theta^2 & \theta-\theta^2 & \theta^2-\beta & \theta^2-\beta & \theta^2-\theta \\ \theta^2-\theta & \theta^2-\beta & \theta^2-\beta & \theta-\theta^2 & \beta-\theta^2 & \beta-\theta^2 \\ \theta^2-\beta & \theta-\theta^2 & \theta^2-\beta & \beta-\theta^2 & \theta-\theta^2 & \beta-\theta^2 \\ \theta^2-\beta & \theta^2-\beta & \theta^2-\theta & \beta-\theta^2 & \beta-\theta^2 & \theta-\theta^2 \end{bmatrix}.$$

This matrix G' partitions into a block matrix of the form $\begin{bmatrix} A & -A \\ -A & A \end{bmatrix}$, where

$$A = \begin{bmatrix} a & x & x \\ x & a & x \\ x & x & a \end{bmatrix},$$

with $a = \theta - \theta^2$ and $x = \beta - \theta^2$. Thus the matrix G' is positive semi-definite if and only if $A \geq 0$. Using the determinant criteria we see that $A \geq 0$ if and only if $|x| \leq a$ and $2x^3 - 3ax^2 + a^3 \geq 0$. Simplifying we see that $A \geq 0$ if and only if $-\frac{a}{2} \leq x \leq a$. Substituting the values of a and x, we find that the Gramian matrix G is positive semidefinite if and only if

$$\frac{3\theta^2 - \theta}{2} \leq \beta \leq \theta.$$

Thus, the set of all possible β is the set satisfying

$$\max\left\{\frac{3\theta^2 - \theta}{2}, 2\theta - 1, 0\right\} \leq \beta \leq \theta.$$

This becomes

$$\begin{aligned} 0 \leq \beta \leq \theta \quad & \text{for } 0 \leq \theta \leq \tfrac{1}{3} \\ \tfrac{3\theta^2-\theta}{2} \leq \beta \leq \theta \quad & \text{for } \tfrac{1}{3} \leq \theta \leq \tfrac{2}{3} \\ 2\theta - 1 \leq \beta \leq \theta \quad & \text{for } \tfrac{2}{3} \leq \theta \leq 1. \end{aligned}$$

Thus, we obtain the values (9) and we have that the infimum and supremum in (8) are attained. □.

5 The Cases $t \in \{q, qa, qc\}$.

In this section, we compute f_t^l and f_t^u when $t \in \{q, qa, qc\}$ to prove (5) in Theorem 8. We begin with a symmetrization lemma, analogous to Lemma 11.

Lemma 13 *The set Θ_{qc}^s (resp., Θ_q^s), is equal to the set of pairs $(\theta, \widetilde{\theta})$ with $0 \leq \theta \leq 1$, such that there exists a C^*-algebra $\mathcal{A}$ (resp., a finite dimensional C^*-algebra, $\mathcal{A}$) with a faithful tracial state τ and with projections $A_0, A_1, A_2 \in \mathcal{A}$ such that for all v,*

$$\tau(A_v) = \theta, \qquad \tau(A_v A_{v+1}) = \beta, \tag{11}$$

where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$.

Proof. By Theorem 3 and the discussion in Section 3, $(\theta, \widetilde{\theta})$ belongs to Θ_{qc}^s (respectively, Θ_q^s) if and only if there is a C*-algebra $\mathcal{A}$ (respectively, a finite dimensional C*-algebra $\mathcal{A}$), with a faithful tracial state τ and projections A_0, A_1, A_2 such that

$$\frac{1}{3}\sum_{v=0}^{2} \tau(A_v) = \theta, \qquad \frac{1}{3}\sum_{v=0}^{2} \tau(A_v A_{v+1}) = \beta,$$

where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$. But if such exist, then we can consider the C*-algebra $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$ with the trace $\widetilde{\tau} = \frac{1}{3}\tau \oplus \frac{1}{3}\tau \oplus \frac{1}{3}\tau$, and projections $\widetilde{A}_v = A_v \oplus A_{v+1} \oplus A_{v+2}$ that satisfy the stronger requirements of the lemma that include (11). □.

We now have some C*-algebra results.

Proposition 14 *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ. Let A and P be hermitian elements in $\mathcal{A}$. If $AP - PA \neq 0$, then there exists $H = H^* \in \mathcal{A}$ such that, letting $f(t) = \tau(A(e^{iHt}Pe^{-iHt}))$ for $t \in \mathbb{R}$, we have $f'(0) > 0$.*

Proof. If $H \in \mathcal{A}$ is hermitian, then

$$f'(0) = i\tau(AHP - APH) = i\tau((PA - AP)H),$$

where we used the fact that τ is a tracial state. Supppose $AP - PA \neq 0$. Let $H = i(PA - AP)$. Then H is hermitian and $f'(0) = \tau(|PA - AP|^2) > 0$, where the strict inequality follows beacuse $AP - PA \neq 0$ and τ is a faithful state. □.

Corollary 15 *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ. Fix $\theta \in [0, 1]$. Let*

$$\beta = \inf \left\{ \frac{1}{3}\tau\left(AB + BC + CA\right) : A, B, C \in \mathcal{A} \text{ projections}, \tau(A) = \tau(B) = \tau(C) = \theta \right\}.$$

If there exist projections A_0, B_0, C_0 in $\mathcal{A}$ such that $\tau(A_0) = \tau(B_0) = \tau(C_0) = \theta$ and $\beta = \frac{1}{3}\tau(A_0B_0 + B_0C_0 + C_0A_0)$, then

$$[A_0, B_0 + C_0] = [B_0, C_0 + A_0] = [C_0, A_0 + B_0] = 0.$$

Proof. We will show that A_0 commutes with B_0+C_0 and the other commutation relations follow by symmetry. Let $P = B_0 + C_0$. Suppose, for contradiction, that $[A_0, P] \neq 0$. Then, by Proposition 14, there exists $H = H^* \in \mathcal{A}$ such that if $f(t) = \tau(A_0(e^{iHt}Pe^{-iHt}))$, then $f'(0) > 0$. Fix some small and negative t such that $f(t) < f(0)$. Letting $B_t = e^{iHt}B_0e^{-iHt}$ and $C_t = e^{iHt}C_0e^{-iHt}$, we see that B_t and C_t are themselves projections in $\mathcal{A}$ and $\tau(B_t) = \tau(C_t) = \theta$. But then for our value of t,

$$\begin{aligned}
\tau(A_0B_t + B_tC_t + C_tA_0) &= \tau(A_0(B_t + C_t) + B_tC_t) \\
&= \tau(A_0(e^{iHt}Pe^{-iHt})) + \tau((e^{iHt}B_0e^{-iHt})(e^{iHt}C_0e^{-iHt})) \\
&= f(t) + \tau(B_0C_0) < f(0) + \tau(B_0C_0) = 3\beta,
\end{aligned}$$

which implies that β is not the infimum, contrary to hypothesis. Thus, A_0 commutes with $B_0 + C_0$. $\square$.

We now consider the universal unital C*-algebra $\mathfrak{A}$ generated by self-adjoint projections A, B, and C satisfying the commutator relations

$$[A, B + C] = [B, A + C] = [C, A + B] = 0. \tag{12}$$

This is the C*-algebra that one obtains in the following manner. First form the universal unital complex algebra $\mathcal{A}$ generated by three noncommuting variables A, B and C. Each time that we have a set of three self-adjoint projections on a Hilbert space $\mathcal{H}$ satisfying the above equations, they induce a representation of this algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$. Setting $|||u||| = \sup \|\pi(u)\|$, where the supremum is over all such representations defines a seminorm on $\mathcal{A}$. The elements of norm 0 are a 2-sided ideal, $\mathcal{J}$, and this seminorm induces a norm on $\mathcal{A}/\mathcal{J}$. The completion of $\mathcal{A}/\mathcal{J}$ is what we mean by $\mathfrak{A}$. It has the universal property that given three projections P_A, P_B, P_C on a Hilbert space, $\mathcal{H}$, satisfying the relations, then there exists a unique *-homomorphism $\pi : \mathfrak{A} \to B(\mathcal{H})$ with $\pi(A+\mathcal{J}) = P_A, \pi(B+\mathcal{J}) = P_B, \pi(C+\mathcal{J}) = P_C$.

Proposition 16 *The universal C*-algebra $\mathfrak{A}$ described above is isomorphic to $\mathbb{C}^8 \oplus \mathbb{M}_2$, where $\mathbb{M}_2$ is the space of 2×2 complex matrices, and wherein*

$$A = 0 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \oplus \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix},$$

$$C = 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

Proof. We will describe all irreducible $*$-representations of $\mathfrak{A}$ on Hilbert spaces. Let

$$Y = 2(B + C) - (B + C)^2 \in \mathfrak{A}.$$

By the commutation relations (12), Y commutes with A. We also note that $Y = B + C - BC - CB$ and

$$BY = B - BCB = YB,$$

namely, that Y commutes with B. Similarly, Y commutes with C. Hence Y lies in the center of $\mathfrak{A}$. Thus, under any irreducible $*$-representation π, Y must be sent to a scalar multiple of the identity operator. In other words, we have

$$\pi(B + C - BC - CB) = \pi(Y) = \lambda\pi(1)$$

for some $\lambda \in \mathbb{C}$, so that

$$\pi(CB) \in \operatorname{span}\pi\big(\{1, B, C, BC\}\big).$$

Similarly, we have

$$\pi(CA) \in \operatorname{span}\pi\big(\{1, A, C, AC\}\big), \qquad \pi(BA) \in \operatorname{span}\pi\big(\{1, A, B, AB\}\big).$$

Since $\mathfrak{A}$ is densely spanned by the set of all words in the idempotents A, B and C, we see

$$\pi(\mathfrak{A}) = \operatorname{span}\pi\big(\{1, A, B, C, AB, AC, BC, ABC\}\big).$$

This implies that $\dim \pi(\mathfrak{A}) \le 8$. Since $\pi(\mathfrak{A})$ is finite dimensional and acts irreducibly on a Hilbert space $\mathcal{H}_\pi$, it must be equal to a full matrix algebra. Considering dimensions, we must have $\dim \mathcal{H}_\pi \le 2$.

The irreducible representations π of $\mathfrak{A}$ for which $\dim \mathcal{H}_\pi = 1$ are easy to describe. They are the eight representations that send A, B and C variously to 0 and 1. We will now characterize the irreducible representations π of $\mathfrak{A}$ for which $\dim \mathcal{H}_\pi = 2$, up to unitary equivalence. Let π be such a representation. From the commutation relations (12), we see that, if $\pi(A)$ and $\pi(B)$ commute, then also $\pi(C)$ commutes with $\pi(A)$ and with $\pi(B)$, and the entire algebra $\pi(\mathfrak{A})$ is commutative. This would require $\dim \mathcal{H}_\pi = 1$. By symmetry we conclude that no two of $\pi(A)$, $\pi(B)$ and $\pi(C)$ can commute. In particular, each must be a projection of rank 1. After conjugation with a unitary, we must have

$$\pi(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi(B) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$$

for some $0 < t < 1$. Since $\pi(B) + \pi(C)$ must commute with $\pi(A)$, we must have

$$\pi(C) = \begin{pmatrix} c_{11} & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & c_{22} \end{pmatrix},$$

for some $c_{11}, c_{22} \ge 0$. Since $\pi(C)$ is a projection, the only possible choices are (i) $c_{11} = t$ and $c_{22} = 1-t$ and (ii) $c_{11} = 1-t$ and $c_{22} = t$. But in Case (ii), we have $\pi(C) = I_{\mathcal{H}_\pi} - \pi(B)$, which violates the prohibition against $\pi(C)$ and $\pi(B)$ commuting. Thus, we must have

$$\pi(C) = \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Now, using that $\pi(A) + \pi(B)$ and $\pi(C)$ commute, we see that we must have $t = \frac{1}{4}$ and we easily check that this does provide an irreducible representation of $\mathfrak{A}$.

To summarize, up to unitary equivalence, there are exactly nine different irreducible representations of $\mathfrak{A}$, one of them is two-dimensional and the others are one-dimensional. Thus, $\mathfrak{A}$ is finite dimensional and is isomorphic to the direct sum of the images of its irreducible representations, namely to $\mathbb{C}^8 \oplus \mathbb{M}_2$, with A, B and C as indicated. $\square$.

We now prove Theorem 8 for the cases $t \in \{q, qa, qc\}$.

Theorem 17 *For $t \in \{q, qa, qc\}$, the functions*

$$f_t^l(\theta) = \inf\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta_t^s\}, \qquad f_t^u(\theta) = \sup\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta_t^s\} \tag{13}$$

are given by

$$f_t^l(\theta) = \frac{1}{2}, \qquad f_t^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \le \theta \le \frac{1}{3} \\ \frac{3+\theta}{4} & \text{for } \frac{1}{3} \le \theta \le \frac{1}{2} \\ \frac{4-\theta}{4} & \text{for } \frac{1}{2} \le \theta \le \frac{2}{3} \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{14}$$

Moreover, both the infimum and supremum are attained, for all values of $\theta \in [0, 1]$.

Proof. Fix $\theta \in [0,1]$. From the inclusions (7), we conclude

$$f^l_{qc}(\theta) \le f^l_{qa}(\theta) \le f^l_q(\theta) \le f^u_q(\theta) \le f^u_{qa}(\theta) \le f^u_{qc}(\theta).$$

To find $f^l_{qc}(\theta)$, by Lemma 13, we should find the supremum of values β such that there exists a C*-algebra $\mathcal{A}$ with faithful tracial state τ and with projections A_0, A_1, A_2 such that

$$\forall v, \quad \tau(A_v) = \theta, \qquad \tau(A_v A_{v+1}) = \beta. \tag{15}$$

By Cauchy-Schwarz, $\beta \le \theta$. But taking $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ with $A_v = 1 \oplus 0$ and an appropriate trace τ shows that $\beta = \theta$ occurs, and in a finite dimensional example. Thus, we find $f^l_{qc}(\theta) = f^l_q(\theta) = \frac{1}{2}$.

To find $f^u_{qc}(\theta)$, again using Lemma 13, we should find the infimum β_0 of values β as described above. Since Θ^s_{qc} is closed, this infimum is attained. Thus, there exists a C*-algebra $\mathcal{A}$ with tracial state τ and projections A_0, A_1, A_2 such that (15) holds with $\beta = \beta_0$. Morover, by the proof of Lemma 13, we have that β_0 equals the infimum of $\frac{1}{3}\tau(AB+BC+CA)$ over all projections A, B, C in some C*-algebra with faithful tracial state τ such that $\tau(A) = \tau(B) = \tau(C) = \theta$. Thus, Corollary 15 applies and the commutation relations

$$[A_0, A_1 + A_2] = [A_1, A_0 + A_2] = [A_2, A_0 + A_1] = 0$$

hold. Thus, there is a representation of the universal C*-algebra $\mathfrak{A}$ considered in Proposition 16, sending A to A_0, B to A_1 and C to A_2. So, using Gelfand–Naimark–Segal representations, in order to find β_0, it suffices to consider tracial states (faithful or not) on $\mathfrak{A}$. In particular, β_0 is the minimum of all values of $\beta \ge 0$ for which there exists a tracial state τ on $\mathfrak{A}$ satisfying

$$\tau(A) = \tau(B) = \tau(C) = \theta, \quad \tau(AB) = \tau(AC) = \tau(BC) = \beta. \tag{16}$$

Since $\mathfrak{A}$ is finite dimensional, we get $f^u_{qc}(\theta) = f^u_q(\theta)$.

An arbitrary tracial state of $\mathfrak{A}$ is of the form

$$\tau\left(\lambda_1 \oplus \cdots \oplus \lambda_8 \oplus \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \left(\sum_{j=1}^{8} t_j \lambda_j\right) + \frac{s}{2}(x_{11} + x_{22}),$$

for some $t_1, \ldots, t_8, s \ge 0$ satisfying $t_1 + \cdots + t_8 + s = 1$. The conditions (16) applied on A, B and C as in Proposition 16 become

$$t_5 + t_6 + t_7 + t_8 + \frac{s}{2} = t_3 + t_4 + t_7 + t_8 + \frac{s}{2} = t_2 + t_4 + t_6 + t_8 + \frac{s}{2} = \theta,$$
$$t_7 + t_8 + \frac{s}{8} = t_6 + t_8 + \frac{s}{8} = t_4 + t_8 + \frac{s}{8} = \beta.$$

These are equivalent to

$$\begin{aligned} t_1 &= 1 + 3\beta - 3\theta + \frac{s}{8} - t_8 \\ t_2 = t_3 = t_5 &= \theta - 2\beta - \frac{s}{4} + t_8 \\ t_4 = t_6 = t_7 &= \beta - \frac{s}{8} - t_8. \end{aligned}$$

Thus, writing $t = t_8$, β_0 is the minimum value of β such that there exist $s, t \geq 0$ such that the inequalities

$$1 + 3\beta - 3\theta + \frac{s}{8} - t \geq 0, \qquad \theta - 2\beta - \frac{s}{4} + t \geq 0, \qquad \beta - \frac{s}{8} - t \geq 0.$$

hold. This is a linear programming problem. We solved it by hand using the simplex method and also (to check) by using the Mathematica software platform [21]. The solution is,

$$\beta_0 = \begin{cases} 0, & 0 \leq \theta \leq \frac{1}{3} \\ \frac{3\theta-1}{4}, & \frac{1}{3} \leq \theta \leq \frac{1}{2} \\ \frac{5\theta-2}{4}, & \frac{1}{2} \leq \theta \leq \frac{2}{3} \\ 2\theta - 1, & \frac{2}{3} \leq \theta \leq 1, \end{cases}$$

which, using $f^u_{qc}(\theta) = \frac{1}{2} + \theta - \beta_0$, yields the values given in (14). $\square$.

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References

1. B. S. Tsirelson (1993), *Some results and problems on quantum Bell-type inequalities,* Hadronic J. Suppl., Vol. 8, No. 4, pp. 329 – 345.
2. B. S. Tsirelson (2006), *Bell inequalities and operator algebras,* available at http://qig.itp.uni-hannover.de/qiproblems/33. Accessed: 2018-03-18.
3. M. Junge, M. Navascues, C. Palazuelos, V. B. Scholz, and R. F. Werner (2011), *Connes' embedding problem and Tsirelson's problem,* J. Math. Phys., Vol. 52, No. 1, pp. 012102, 12.
4. T. Fritz (2012), *Tsirelson's problem and Kirchberg's conjecture,* Rev. Math. Phys., Vol. 24, No. 5, pp. 1250012, 67.
5. N. Ozawa (2013), *About the Connes' embedding conjecture,* Jpn. J. Math., Vol. 8, No. 1, pp. 147–183.
6. V. I. Paulsen, and I. G. Todorov (2015), *Quantum chromatic numbers via operator systems,* Q. J. Math., Vol. 66, No. 2, pp. 677–692.
7. K. Dykema, and V. I. Paulsen (2016), *Synchronous correlation matrices and Connes' embedding conjecture,* J. Math. Phys., Vol. 57, No. 1, pp. 015214, 12.
8. V. I. Paulsen, S. Severini, S. Stahlke, I. G. Todorov, and A. Winter (2016), *Estimating quantum chromatic numbers,* J. Funct. Anal., Vol. 270, No. 6, pp. 2188–2222.
9. W. Slofstra (2016), *Tsirelson's problem and an embedding theorem for groups arising from non-local games,* arXiv:1606.03140.
10. W. Slofstra (2017), *The set of quantum correlations is not closed,* arXiv:1703.08618.
11. M. Froissart (1981), *Constructive generalization of Bell's inequalities,* Nuovo Cimento B (11), Vol. 64, No. 2, pp. 241–251.
12. D. Collins, and N. Gisin (2004), *A relevant two qubit Bell inequality inequivalent to the CHSH inequality,* J. Phys. A, Vol. 37, No. 5, pp. 1775–1787.
13. K. F. Pál, and T. Vértesi (2010), *Maximal violation of a bipartite three-setting, two-outcome bell inequality using infinite-dimensional quantum systems,* Phys. Rev. A, Vol. 82, No. 2, pp. 022116.
14. T. Vidick, and S. Wehner (2011), *More nonlocality with less entanglement,* Phys. Rev. A, Vol. 83, No. 5, pp. 052310.

15. K. Dykema, V. I. Paulsen, and J. Prakash (2017), *Non-closure of the set of quantum correlations via graphs,* arXiv:1709.05032
16. M. Navascués, Y. Guryanova, M. J. Hoban, and A. Acín (2015), *Almost quantum correlations,* Nat. Commun. 6:6288.
17. M. Navascués, S. Pironio, and A. Acín (2008), *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations,* New J. Phys., Vol. 10, No. 7, pp. 73013.
18. J. Kempe, O. Regev, and B. Toner (2010), *Unique games with entangled provers are easy,* SIAM J. Comput., Vol. 39, No. 7, pp. 3207–3229.
19. S. J. Kim, V. I. Paulsen, and C. Schafhauser (2017), *A synchronous game for binary constraint systems,* arXiv:1707.01016.
20. B. Lackey, and N. Rodrigues (2017), *Nonlocal games, synchronous correlations, and Bell inequalities,* arXiv:1707.06200.
21. Wofram Research Inc. (2016), *Mathematica, Version 11.0.* Wolfram Research Inc., Champaign, IL