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NON-LOCAL UNIVERSAL GATES GENERATED WITHIN A RESONANT MAGNETIC CAVITY

FRANCISCO DELGADO[a]
Escuela de Ingeniería y Ciencias, Tecnológico de Monterrey, México
Atizapán, CP.52926, México

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Quantum Information is a quantum resource being advised as a useful tool to set up information processing. Despite physical components being considered are normally two-level systems, still the combination of some of them together with their entangling interactions (another key property in the quantum information processing) become in a complex dynamics needing be addressed and modeled under precise control to set programmed quantum processing tasks. Universal quantum gates are simple controlled evolutions resembling some classical computation gates. Despite their simple forms, not always become easy fit the quantum evolution to them. $SU(2)$ decomposition is a mechanism to reduce the dynamics on $SU(2)$ operations in composed quantum processing systems. It lets an easier control of evolution into the structure required by those gates by the adequate election of the basis for the computation grammar. In this arena, $SU(2)$ decomposition has been studied under piecewise magnetic field pulses. Despite, it is completely applicable for time-dependent pulses, which are more affordable technologically, could be continuous and then possibly free of resonant effects. In this work, we combine the $SU(2)$ reduction with linear and quadratic numerical approaches in the solving of time-dependent Schrödinger equation to model and to solve the controlled dynamics for two-qubits, the basic block for composite quantum systems being analyzed under the $SU(2)$ reduction. A comparative benchmark of both approaches is presented together with some useful outcomes for the dynamics in the context of quantum information processing operations.

Keywords: Gate design; Heisenberg-Ising anisotropic model; Non-local gates.

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1 Introduction

Gates based quantum computation (GBQC) has developed a robust theory to set quantum information processing by manipulating the quantum systems evolution in the form of archetypical structures barely imitating the classical computation gates. These quantum gates have been required to fulfill the DiVincenzo criteria [1] to set those tasks. The quest of this construction is the achievement of a universal purpose quantum computer in terms of generic or universal gates able to be complete and sufficient in the reproduction of any kind of information processing. This set has been attained in the form of the Boykin set of gates $\mathcal{B} \equiv \{S_{\pi/8}, S_{\pi/4}, \mathcal{H}, C^a NOT_b\}$ [2]. On this arena, quantum control should to provide concrete

[a]Carretera a Lago de Guadalupe km. 3.5, Atizapán, Estado de México, México. CP. 52926

forms for the dynamics in order to reproduce certain kinds of processing operations (gates) useful for the people developing quantum algorithms in terms either universal operations or dedicated operations (specialized operations solving certain common processing).

In order to have a more complete context of the current proposal, we review briefly the historic theoretical and experimental development of gate design in quantum information processing, particularly around of the C^aNOT_b gate, mainly responsible for the entangling operations in quantum processing. The first relevant fact is the theoretical proposal by Cirac and Zoller in 1994 [3] for the implementation of C^aNOT_b gate using cold trapped ions, which was experimentally achieved by [4, 5] in 1995 and 2003 respectively. This proposal required a complex control for the individual elements using sequences of laser pulses on them. After, in 1997, an alternative proposal used Nuclear Magnetic Resonance (NMR) to set several quantum gates on bulk matter [6]. Despite the technological control achieved on this technology, several limitations in the scaling were advised due to the diamagnetic shielding present in condensed matter. Then, [7] proposed quantum processing using electronic spin states on quantum dots, superseding the problems in the previous proposal. Despite, the preparation and control remain complex through of precise multi-component electronic devices. In 1998, [8] proposed quantum processing based on silicon nuclear spin by using individual phosphorus atoms. Nevertheless the possible simplification, this technology still requires lots of solid state elements and pulses, rapidly introducing decoherence. In 2002, [9] developed controlled-not gates using linear optical elements (polarization and path encoding) superseding the loss of non-linear effects in previous approaches. Despite this cleaner technology, quantum processing on matter is complementary necessary still. In 2009, [10] states the superconducting circuits approach for qubits, a current wide-spread technology in the current developments of quantum processing, despite the limited flexibility to set universal computation. In addition, instead of the sequenced approaches to quantum gates normally used in the overall previous technologies, this last work also introduced the Fourier approach to quantum gates problem in order to reduce the piecewise achievement of certain more elusive universal gates, particularly those controlled. Inspired on *in situ* processing for ion traps, [11] has proposed the use of traveling waves under the Fourier approach to quantum gates problem to enhance their implementation. In such proposal, the use of physical eigenstates remains as grammar.

The most common state basis to set the description of quantum information on specific composed systems is the computational basis. Last basis uses tensor products of basic states for two-level single systems or qubits. As instance, polarization states for light or spin states for matter. Nevertheless, this approach becomes complex when quantum systems are combined and the physical evolution should be addressed on the universal gates structure (being the source of multiple steps in their construction). Recently, $SU(2)$ decomposition [12] has been developed as a mathematical mechanism for certain two-level quantum systems architectures containing $2d$ qubits (with their dynamics ruled by the $U(2^{2d})$ group) to re-express and to translate the dynamics on momentary independent two-level quantum information subspaces (with a dynamics on $U(1) \times SU(2)$ each one) as equivalent quantum resources instead of the physical elements by themselves (with a dynamics on $U(2^{2d})$ group). In the $SU(2)$ decomposition for two qubits under the Heisenberg-Ising interaction [13], the control is achieved as driven operations on physical elements in the setup, but reflected on certain pure quantum information states as a grammar. Then, still the two-qubit case becomes relevant because it is the core of $SU(2)$ decomposition. Quantum processing information requires not only of local operations on each single system, instead of the interaction between these systems in terms of entangling operations. This single fact does that specific control to reduce the entire interaction dynamics become complex to fit into concrete quantum gates, particularly those in the set $\mathcal{B}$.

In the last trend, [14] has proposed for magnetic systems, a set of gates closer to the evolution generated by the interactions being involved, $\mathcal{D} = \{(\mathbf{1}_1 \otimes S_{\pi/8_2})_\mathrm{B}, (\mathbf{1}_1 \otimes S_{\pi/4_2})_\mathrm{B}, (S_{\pi/8_1} \otimes \mathbf{1}_2)_\mathrm{B}, (S_{\pi/4_1} \otimes \mathbf{1}_2)_\mathrm{B}, (\mathbf{1}_1 \otimes \mathcal{H}_2)_\mathrm{B}, (\mathcal{H}_1 \otimes \mathbf{1}_2)_\mathrm{B}, (C^1NOT_2)_\mathrm{B}, (C^2NOT_1)_\mathrm{B}\}$, which is based on the Bell basis instead of computational basis. Despite the added complexity to manage non-local states, recent work goes in that direction [15]. As a result, those gates are non-local, then operating directly on the quantum information states instead on the states corresponding to physical observables for the single parts in the system.

Specific prescriptions for those gates have been reported for constant or piecewise control magnetic fields. Despite, these magnetic fields are difficult to reproduce in the practice compared with single oscillatory magnetic fields being present in the Fourier approach to quantum gates problem applied to traveling waves [11], thus enhancing the single pulse processing to reduce the resonant effects due to the sequence of pulses necessary in other approaches. Nevertheless, the dynamics of time-dependent Hamiltonians is difficult to be solved, they require to be analyzed under the Baker-Campbell-Hausdorff (BCH) formula which rarely provides closed analytical outcomes. Efficient numerical approaches are necessary to model the dynamics in a comprehensive way for general time-dependent Hamiltonians.

The aim of this work is to show how the combination of $SU(2)$ decomposition together with the Fourier approach to quantum gates problem can be effectively combined to reach the traveling wave processing in [11], thus getting prescriptions for the non-local gates depicted before when they are generated under time-dependent magnetic fields (it means, in the context of a time-dependent Hamiltonian). In the second section, we set the problem departing from [13] and [14], together, we develop the approximation theory to reach a computational approach to simulate and solve the problem, just avoiding the BCH approach. The third section sets the concrete conditions on the evolution operator to reach each universal gate in the set $\mathcal{D}$ [14]for single oscillatory magnetic pulses. Fourth section sets the final combined construction under the Fourier approach (for both blocks in the evolution matrix) for each quantum gate in $\mathcal{D}$, which can be achieved in magnetic resonant cavities to reach the traveling wave processing approach. Fifth section discusses some issues about the gates stability and fidelity, which become natural in the context of quantum error correction theory due to the $SU(2)$ decomposition. Finally, the conclusions are settled in the last section.

2 Gates in the time-dependent control regime

We will focus in the construction of the non-local gates obtained in [14] for magnetic systems. They are accomplished under the Hamiltonian:

$$H_h = \sum_{k=1}^{3} J_k \sigma_{1k} \otimes \sigma_{2k} - B_{1h}\sigma_{1h} \otimes \mathbf{1}_2 - B_{2h}\mathbf{1}_1 \otimes \sigma_{2h} \tag{1}$$

for a couple of qubits under the Heisenberg interaction with additional inhomogeneous magnetic fields (but in a fixed direction $h = 1, 2, 3$ or equivalently $h = x, y, z$) in their positions. There, σ_{sk} is the $k-$Pauli matrix operating on the Hilbert space for the spin states of qubit s. While, $\mathbf{1}_s$ is the identity matrix in such matrix space. This Hamiltonian and their derived dynamics has been studied in the Bell basis [13], showing that its Hilbert space splits in two subspaces as function of the direction h of the magnetic field: $\mathcal{H}_h^{\otimes 2} = \mathcal{H}_{h,1} \oplus \mathcal{H}_{h,2}$. This splitting generates a $SU(2)$ reduction of its dynamics originally in $SU(4)$ (properly in $U(1) \times SU(2)^2$: $U_h(t) = s_{h1} \oplus s_{h2}$ with s_{hj} as a $U(1) \times SU(2) \subset U(2)$ block on each $\mathcal{H}_{h,j}, j = 1, 2$; see Figure 1 in [13]). Last property fulfills independently if magnetic fields are time-dependent [16]. It recovers easier the computational forms required to set the universal gates for two-qubit processing despite the complexity introduced by the entangling operations letting to comprise several local control operations in addition.

Behind the single exact control in the proposal to be presented, $SU(2)$ decomposition lets a more direct construction of universal gates $\mathcal{B}$ in the form $\mathcal{D}$, being able to be fulfilled with few prescriptions, thus avoiding the stepwise approach necessary in other implementations. In addition, this reduction is compatible with the Fourier approach to quantum gates [10] as it will be seen. $SU(2)$ decomposition is a procedure derived from the Lie groups properties where quantum information dynamics on $SU(2^{2d})$ with $d \in \mathbf{Z}$ could be expressed in $SU(2)$ blocks [12], concretely on: $U(1)^{2^{2d-1}-1} \times SU(2)^{2d-1}$; see Figure 4 in [12]. Thus, for two-qubit processing, $SU(4)$, under Heisenberg-Ising operations, those blocks have been reported analytically in [13, 14] for stepwise magnetic fields, nevertheless, these fields are difficult to reproduce and to control in the practice, therefore, continuous fields are more able to be considered for the current purpose. Thus, in the approach used in this work, we analyze a possible implementation for the non-local gates in $\mathcal{D}$ using variable magnetic fields able to be generated in a resonant cavity with loss, combined with the Fourier approach to quantum gates.

2.1 Gates in a non-local basis

The non-local gates proposed in $\mathcal{D}$ are achieved switching the direction of magnetic field. All of them are expressed in terms of the Bell states basis. The Hamiltonian (1) has been written in the Bell basis in [13] as:

$$\begin{aligned} H_h = \quad & H_{1,h} \oplus H_{2,h} \\ \text{with}: \quad & H_{k,h} \equiv \tilde{H}^0_{k,h} + \tilde{H}_{k,h} \\ & \tilde{H}^0_{k,h} = -s_0 J_h \sigma_0^{(h,k)} \\ & \tilde{H}_{k,h} = s_1 J_{\{h\}_{s_0}} \sigma_3^{(h,k)} + s_2 B_{h-s_0} \sigma_q^{(h,k)} \\ & s_0 = (-1)^{h+k+1}, s_1 = s_0^p, s_2 = (-1)^p s_0^{p+q} \\ & p = 1 + \frac{1}{2}(h-1)(h-2), q = 2 - h \bmod 2 \end{aligned} \tag{2}$$

reflecting the splitting in the Hilbert space in two subspaces. For the dynamics, each $H_{k,h}$ is the block $k = 1, 2$ in the Hamiltonian whose structure will be inherited to $U_h(t)$. While, the parameters $J_h, J_{\{h\}_{s_0}}, B_{h-s_0}$ are physical parameters obtained from the anisotropic Heisenberg interaction strengths and the magnetic fields as they are depicted in [13]. s_0, s_1, s_2, p

and q depend entirely on the election of h, k. In addition:

$$\sigma_q^{(h,k)} = \begin{pmatrix} 0 & (-i)^{q-1} \\ i^{q-1} & 0 \end{pmatrix}, \quad \text{for} \quad q = 1, 2 \tag{3}$$

note in addition any matrix $\sigma_q^{(h,k)}, q = 1, 2$ has the same form of the correspondent Pauli matrices but they are understood for certain pairs of Bell states in agreement with the election of h, k. The correct value of $h = 1, 3$ should be selected in order to generate each one of the gates included in $\mathcal{D}$ (noting then the only case of interest involves $q = 1$). In any case, after of a convenient rearrangement in the order of the Bell states as they are settled in [13], the gates in $\mathcal{D}$ adopt one of the following forms:

$$(\mathbf{1}_a \otimes S_{\chi_b})_B : \quad \begin{pmatrix} e^{-i\chi} & 0 & 0 & 0 \\ 0 & e^{i\chi} & 0 & 0 \\ 0 & 0 & e^{-i\chi} & 0 \\ 0 & 0 & 0 & e^{i\chi} \end{pmatrix} \tag{4}$$

$$(C^a NOT_b)_B : \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

$$(\mathbf{1}_a \otimes \mathcal{H}_b)_B : \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{6}$$

2.2 A numerical approach to reproduce the dynamics

As has been presented in [17], by departing from the Schrödinger equation:

$$H_h U_h(t) = i\hbar \frac{\partial U_h(t)}{\partial t} \tag{7}$$

then, assuming the $SU(2)$ decomposition in two subspaces explained in the previous section:

$$\bigoplus_{k=1}^{2} \left(H_{k,h} s_{hk} - i\hbar \frac{\partial s_{hk}}{\partial t} \right) = 0 \tag{8}$$

thus, if any state is split in their respective components in the two subspaces $|\psi(t)\rangle = \alpha_1 |\psi_1(t)\rangle + \alpha_2 |\psi_2(t)\rangle$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$ and $|\psi_k(t)\rangle = s_{hk} |\psi_k(0)\rangle$. In addition, because $\tilde{H}^0_{k,h}$ commutes with $\tilde{H}_{k,h}$, we can define $s_{hk} \equiv e^{-i\frac{\tilde{H}^0_{k,h} t}{\hbar}} s_{hk}^0$. Then, it is easy demonstrate that:

$$\tilde{H}_{k,h} s_{hk}^0 = i\hbar \frac{\partial s_{hk}^0}{\partial t} \tag{9}$$

Thus, we can work with $\tilde{H}_{k,h}$ and ${s_h}_k^0$. Note ${s_h}_k^0 \in SU(2)$ due its generators are only $\sigma_j^{(h,k)}, j = 1, 2, 3$. By defining the differential evolution operator:

$$|\psi_k(t_0 + \delta t)\rangle = {s_h}_k^0(t_0 + \delta t, t_0)\,|\psi_k(t_0)\rangle \tag{10}$$

and noting that ${s_h}_k^0(t_0, t_0) = \mathbf{1}_k = \sigma_0^{(h,k)}$ when $\delta t \approx 0$, we arrive to the approximation:

$${s_h}_k^0(t_0 + \delta t, t_0) \approx \mathbf{1}_k - \frac{i}{\hbar}\tilde{H}_{k,h}(t_0)\delta t - \frac{i}{2\hbar}\left(\frac{\partial \tilde{H}_{k,h}(t_0)}{\partial t} + \frac{\tilde{H}^2_{k,h}(t_0)}{i\hbar}\right)\delta t^2 + ... \tag{11}$$

Thus, by splitting $[0, t]$ in n sub-intervals $[0, \delta t] \cup [\delta t, 2\delta t] \cup ... \cup [(n-1)\delta t, n\delta t = t]$:

$${s_h}_k^0 = {s_h}_k^0(t, 0) \approx \prod_{i=1,\ldots,n}^{\leftarrow} {s_h}_k^0(i\delta t, (i-1)\delta t), \quad \text{if}: \quad \delta t \approx 0 \tag{12}$$

where $\leftarrow$ means factors stack on the left. Approximation becomes exact if $\delta t \to 0$. By identifying $J_h = \hbar\mathcal{J}_0, -s_1 J_{\{h\}_{s_0}} = \hbar\mathcal{J}_{s_0}, -s_2 B_{h-s_0} = \hbar\mathcal{B}_{-s_0}$ and using the equation (2) into the last expression, we get:

$$\begin{aligned} {s_h}_k^0((s+1)\delta t, s\delta t) \quad &\approx \quad \sigma_0^{(h,k)} + i\left(\mathcal{J}_{s_0}\sigma_3^{(h,k)} + \mathcal{B}_{-s_0}(s\delta t)\sigma_q^{(h,k)}\right)\delta t \\ &\quad -\frac{1}{2}\left((\mathcal{J}^2_{s_0} + \mathcal{B}^2_{-s_0}(s\delta t))\sigma_0^{(h,k)} - i\mathcal{B}'_{-s_0}(s\delta t)\sigma_q^{(h,k)}\right)\delta t^2 \end{aligned} \tag{13}$$

Last expression (13) is a second order approximation for ${s_h}_k^0((s+1)\delta t, s\delta t)$. Because the first term has the common parameter J_h for each block and as the remaining parameters are independent among them, we can reduce the dependence of $s_{h\,k}$ for $k = 1, 2$ on the unrestricted parameters and functions $\mathcal{J}_0, \mathcal{J}_\pm, \mathcal{B}_\pm(t)$ defined above, together with s_0 and q defining the block and the h-direction being selected. Moreover, the complete $s_{h\,k}$ becomes:

$$s_{h\,k} \approx e^{is_0\mathcal{J}_0 t}\prod_{i=1,\ldots,n}^{\leftarrow} s^0_{h\,k}(i\delta t, (i-1)\delta t), \quad \text{if}: \quad \delta t \approx 0 \tag{14}$$

as in the independent time case, $\mathcal{J}_0$ only is responsible from the weak link between the two $SU(2)$ blocks through the $U(1)$ phase factor $e^{is_0\mathcal{J}_0 t}$ in the indirect product $U(1) \times SU(2)^2$. In addition, $s^0_{h\,k}((s+1)\delta t, s\delta t) \in SU(2)$.

For the numerical implementation of $s_{h\,k}$, a benchmark of the performance was made by [17] showing the effect by the inclusion of the second order term in (13), improving the precision around of two figures, letting reduce n from 5×10^4 into 10^2 with respect to the linear approximation. This implementation (second order and $n = 100$) reaches at least five figures of precision and it will be used as numerical algorithm in the following.

The $SU(2)$ reduction formalism is not only an important simplification procedure to understand the quantum information dynamics, also it lets to reduce the analysis on easier

Table 1. General restrictions to reduce $SU(2)$ blocks into the forms required by $\mathcal{D}$.

Gate	Block type	A	ϕ	θ	φ	q	$\mathcal{J}_\pm$	$\mathcal{A}_\mp$
$(\mathbf{1}_a \otimes S_{\chi_b})_B$	S_χ	1	$-\chi$	-	$0, \pi$	1	$\mathcal{C}_{\mathcal{J}_\pm}$	$\mathcal{C}_{\mathcal{A}_\mp}$
	S_χ	1	$-\chi$	-	$0, \pi$	1	$\mathcal{C}_{\mathcal{J}_\pm}$	$\mathcal{C}_{\mathcal{A}_\mp}$
$(\mathbf{1}_a \otimes S_{\chi_b})_B$	S_χ	1	$-\chi$	-	χ	1	$\mathcal{C}_{\mathcal{J}_\pm}$	$\mathcal{C}_{\mathcal{A}_\mp}$
	S_χ	1	χ	-	$-\chi$	1	$\mathcal{C}_{\mathcal{J}_\pm}$	$\mathcal{C}_{\mathcal{A}_\mp}$
$(C^a NOT_b)_B$	σ_0	1	0	-	$\frac{\pi}{4}$	1	0	0
	σ_1	0	-	$\frac{\pi}{2}$	$-\frac{\pi}{4}$	1	0	2.464
$(C^a NOT_b)_B$	σ_0	1	π	-	$\frac{\pi}{4}$	1	0	0
	σ_1	0	-	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	1	0	-2.464
$(\mathbf{1}_a \otimes \mathcal{H}_b)_B$	$\mathcal{H}$	$\frac{1}{\sqrt{2}}$	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{2}$	$0, \pi$	1	± 1.240	± 1.532
	$\mathcal{H}$	$\frac{1}{\sqrt{2}}$	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{2}$	$0, \pi$	1	± 1.240	± 1.532
$(\mathbf{1}_a \otimes \mathcal{H}_b)_B$	$\mathcal{H}$	$\frac{1}{\sqrt{2}}$	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{2}$	$-\frac{\pi}{2}, \frac{\pi}{2}$	1	± 1.240	± 1.532
	$\mathcal{H}$	$\frac{1}{\sqrt{2}}$	$\mp\frac{\pi}{2}$	$\mp\frac{\pi}{2}$	$\frac{\pi}{2}, -\frac{\pi}{2}$	1	∓ 1.240	∓ 1.532

problems as in the current case. The block decomposition states an easier analysis to reduce the dynamics into concrete desired outcomes. Combined with the quadratic approximation given by (13), it becomes sufficient to deal with this kind of problems involving several physical parameters in an affordable time without specialized computer resources.

2.3 Prescriptions to generate each gate in a resonant cavity

First, we are interested in the reduction of s_{h_k} into the forms $\sigma_0, \sigma_1, \mathcal{H}$ and $S_\phi = \cos\phi\sigma_0 + i\sin\phi\sigma_3$. By considering that $s_{h_k}^0$ has the generic form [17]:

$$s_{h_k} = e^{i\varphi} s_{h_k}^0 \equiv e^{i\varphi} \begin{pmatrix} Ae^{i\phi} & Be^{i\theta} \\ -Be^{-i\theta} & Ae^{-i\phi} \end{pmatrix} \tag{15}$$

with $A^2 + B^2 = 1$. Then, that task is affordable by imposing restrictions on A or B, and some concrete prescriptions for ϕ, θ and φ. Table 1 gathers the restrictions to get each gate and their constituent blocks in the context of $\mathcal{D}$ in terms of the matrices (4). The set of prescriptions for each gate is contained between each horizontal double line section of the table. Two possibilities for the implementation of all gates are reported separately between a single line giving alternative but equivalent prescriptions. Only the election of $h = 1, 3$ (in anyway $q = 1$) is open as a function of how the information states should be combined in agreement with [14]. Thus, φ is reported in the perspective of blocks could be combined among them as in (4). Clearly, φ is only related with the restrictions imposed on J_h through the relation $\varphi = s_0\mathcal{J}_0 t$ and the restriction of this phase appears in both blocks with opposite signs.

3 Gates under a time-dependent magnetic field inside a resonant magnetic fields

In the present section, we will fix the magnetic field into a easier model based on a magnetic resonant cavity of width d operating in a TM mode, it means without phase change under the reflection. These arrangements have been simulated in order to understand their control [18, 19]. Together, a similar approach has given for qubits as superconducting circuits [20]. Figure 1a shows the setup, there, field is shown in the z direction (vertical) but it could be directed in the x direction too (horizontal), in agreement with the requirements to generate the gates. Cavity modes are identified by $m = 1, 2, ...$, thus, taking the mode m for our immediate purposes, by defining $t' = \frac{mc}{d} t$ and doing the identification $\mathcal{A}'_\pm = \mathcal{A}_\pm \frac{d}{mc}, \mathcal{J}'_\pm = \mathcal{J}_\pm \frac{d}{mc}$, we will assume an effective field given by:

$$B_{h\pm}(t') = \mathcal{A}_\pm \sin(\pi t') \tag{16}$$

which is the most basic form experimentally achievable and it is in agreement with the Fourier approach to quantum gates. This approach to quantum gates eases the control implementation but only combined with the use of Bell states basis as grammar under the $SU(2)$decomposition. The precise relation with this approach will be seen after.

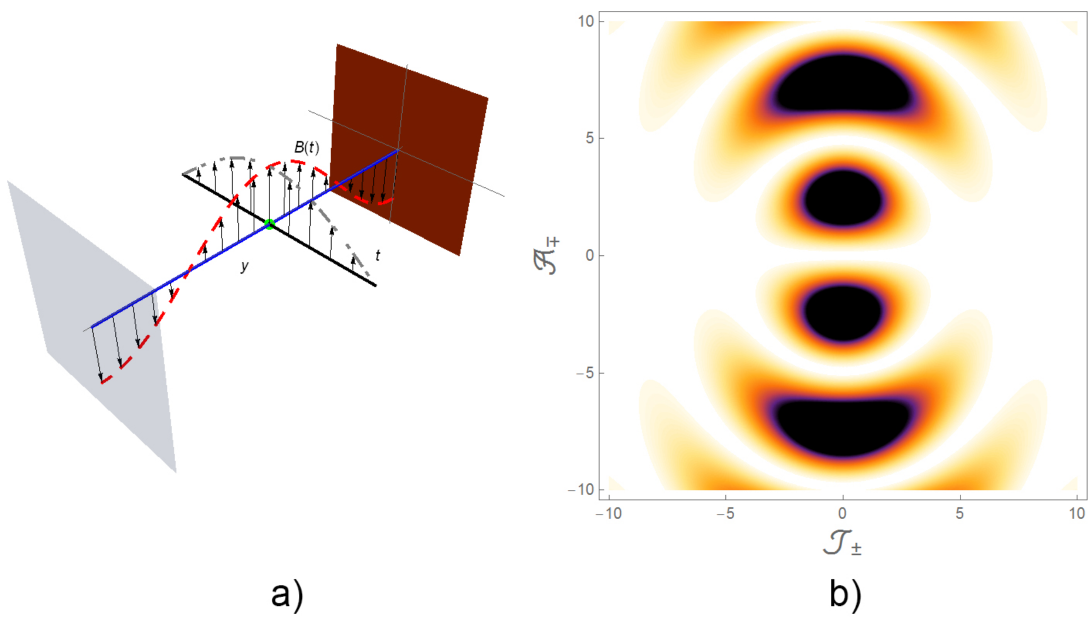

Fig. 1. a) A qubit inside of a rectangular magnetic cavity in a TM mode traveling through the direction y with the time evolution of the field represented in the transverse one-half cycle. b) Contour plot for A in the $\mathcal{J}_\pm - \mathcal{A}_\mp$ space.

The general solutions have been obtained by [17] for this field, we report here the outcomes useful for our concrete purposes in this work. We will comeback with the specification of m. In the meanwhile development, we will remove the apostrophe in t'. Thus, we will focus in the generation of the gates depicted before by imposing the restrictions of Table 1 on A in the equation (15) while we use together the model (16) during a time $t' \in [0, 1]$ (it means, in the original time scale we want reach the gate in the interval $[0, \frac{d}{mc}]$, one-half of the $\sin(\frac{2\pi c}{\lambda} t)$ cycle because in the complete cycle only the identity is achieved). Figure 1b shows a contour plot for A in the $\mathcal{J}_\pm - \mathcal{A}_\mp$ space for both cases $q = 1, 2$. This graph exhibits the existence of the necessary full set of solutions for $A \in [0, 1]$, nevertheless the achievement of the correct value combinations for ϕ and θ should be still demonstrated.

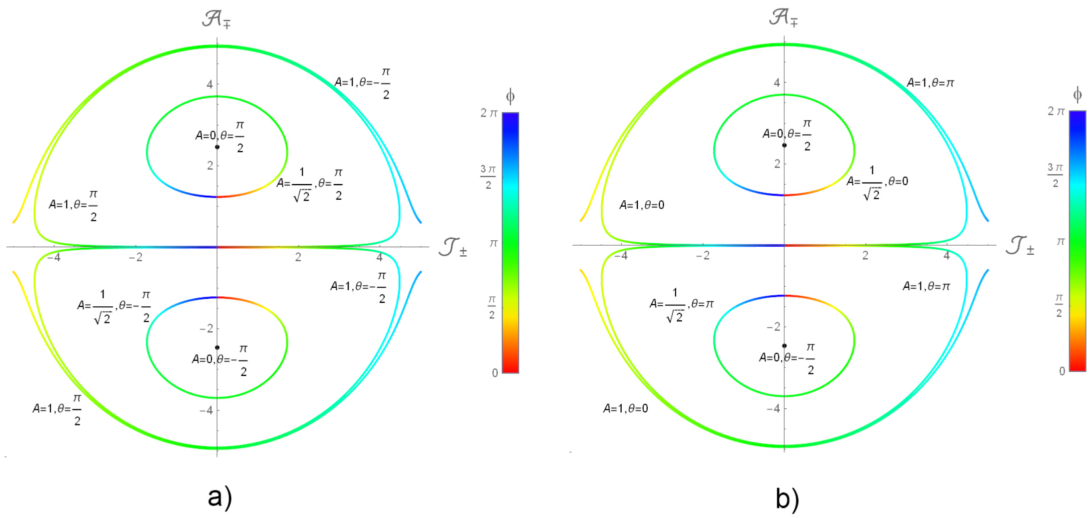

Fig. 2. Solutions for the parameters $\mathcal{A}_{\mp}, \mathcal{J}_{\pm}$ with ϕ indicated in colors in agreement with the color chart and θ as an inset for each curve depicting the cases $A = 0, \frac{1}{\sqrt{2}}, 1$. Two cases a) $q = 1$, and b) $q = 2$ are reported.

3.1 *Concrete prescriptions to get each gate*

The concrete prescriptions to get the blocks needed in the gates construction are reported in Table 1 as a result of the analysis of Figure 2 (a reproduction of the plot reported in [17] only for the cases interesting here). This figure was obtained by sweeping the region $[-5, 5] \times [-5, 5]$ in the plane $\mathcal{A}_{\pm}, \mathcal{J}_{\mp}$ to seek the values $A = 0, \frac{1}{\sqrt{2}}, 1$ for the two cases $q = 1$ (Fig. 2a) and $q = 2$ (Fig. 2b). This last value is included despite it is not used in our discussion. Each curve then presents the ϕ values in color in agreement with the color-bar on the right. Several insets are allocated to specify the corresponding values of A and θ (which are unique for each curve). Values of θ for $A = 1$ are obtained as a limit case. For the case of S_{χ}, the solutions for $\mathcal{A}_{\pm}, \mathcal{J}_{\mp}$ are located in the entire curves $\mathcal{C} = (\mathcal{C}_{\mathcal{J}_{\pm}}, \mathcal{C}_{\mathcal{A}_{\mp}})$ for the case $A = 1$ as function of χ. In the Table 1, pairs of signs reported in some columns are always corresponding with another pair for the same block type in the row. While, values separated by colon are both possible for any combination of other values. Thus, φ is reported as it should appear in the global context to reproduce the entire gates in $\mathcal{D}$ and (4).

4 The time-dependent magnetic field within a resonant magnetic cavity

Fourier approach to the quantum gate problem was stated with the superconducting circuit approach to quantum processing as physical system in order to avoid the multi-pulse approach to the gates construction and then the resonant effects. For a magnetic cavity with length d operating without loss, the resonant magnetic field inside could be modeled for a single mode as:

$$
\begin{aligned}
B_h(t) &= B_h^0\left(\sin(\frac{2\pi}{\lambda}(y-ct))+\sin(\frac{2\pi}{\lambda}(-(y-2d)-ct))\right)\\
&= -2B_h^0\cos(\frac{2\pi}{\lambda}y)\sin(\frac{2\pi c}{\lambda}t) \qquad (17)
\end{aligned}
$$

where $\lambda_m = \frac{2d}{m}$ for the m-mode, $m = 1, 2, ...$ and $h = 1, 3$. Then a complete set of functions depicting the field inside are: $\cos(\frac{2\pi}{\lambda_m}y)\cos(\frac{2\pi c}{\lambda_m}t), m = 0, 1, ...$ and $\cos(\frac{2\pi}{\lambda_m}y)\sin(\frac{2\pi c}{\lambda_m}t), m = 1, 2, ...$ In the last section it was analyzed the procedure to reproduce the universal set of gates in $\mathcal{D}$ through a single semi-harmonic pulse (16). Thus, we can realize a quantum computational algorithm as a train of N magnetic semi-harmonic pulses in the directions y and z (with only one turned-on at the time) and a tight control of the Heisenberg interaction strengths, J_i. Each pulse will represent a gate in $\mathcal{D}$ and it will have the form (16) with the prescriptions given in the Table I as it will be required:

$$
B_{h\pm}(t) = \sum_{j=1}^{N} A_{h\pm}^j \sin(\frac{\pi Nc}{d}t)\delta_{\frac{Nc}{d}}[j](t) \qquad (18)
$$

where $\delta_a[\zeta](t) = \theta_{\frac{\zeta-1}{a}}(t) - \theta_{\frac{\zeta}{a}}(t)$ is the discrete unit impulse function defined in terms of the Heaviside step function $\theta_\zeta(t) \equiv \theta(t-\zeta)$. Note there is magnetic field in only one direction at the time to fulfill the requisites. Thus, when $A_{h\pm}^j \neq 0$ for some h, j and for at least one $-s_0 \in \{+,-\}$, then $A_{h'\pm}^j = 0$ for any $h, h' \in \{x, z\}, h \neq h'$. The magnetic field on each qubit position y_i is:

$$
B_{ih}(t) = B_h(y_i, t) = \frac{1}{2}(B_{h+}(t) - (-1)^i B_{h-}(t)) \qquad (19)
$$

where this function could be achieved as a Fourier series of fundamental modes pulses within the resonant magnetic cavity:

$$
B_h(y,t) = \sum_{k=0}^{\infty} \alpha_{h}{}_i^k \cos(\frac{\pi k}{d}x)\cos(\frac{\pi ck}{d}t) + \sum_{k=1}^{\infty}\beta_{h}{}_i^k\cos(\frac{\pi k}{d}x)\sin(\frac{\pi ck}{d}t) \qquad (20)
$$

Due to the properties of the last set of functions:

$$
\begin{aligned}
\alpha_{h}{}_i^k &= \frac{c}{d\cos(\frac{\pi k}{d}y_i)}\int_0^{\frac{2d}{c}} B_{ih}(t)\cos(\frac{\pi ck}{d}t)dt \qquad (21)\\
\beta_{h}{}_i^k &= \frac{c}{d\cos(\frac{\pi k}{d}y_i)}\int_0^{\frac{2d}{c}} B_{ih}(t)\sin(\frac{\pi ck}{d}t)dt \qquad (22)
\end{aligned}
$$

In the following we assume each qubit is located in independent magnetic fields with different amplitudes, meaning they are not necessarily collinear on the same y line. This aspect clearly is a challenge on the current design cavities in spite of the effective distances for the Heisenberg-Ising interactions. In addition, the fields (20) still exhibits discontinuities in their derivatives, so it induces overshoots or undershoots in the magnetic signal, thus in the desired target behavior of the system [21]. Despite, they are lower to those presented in the step-magnetic fields case as it was considered in [14].

5 Gates fidelity issues and quantum error analysis

$SU(2)$ decomposition lets to address the quantum error correction in a very similar way that for traditional constructions using physical eigenstates for the spin. Thus, in this section we will evaluate the impact on s_{hk} of uncontrollable deviations on prescriptions in $\mathcal{A}_\pm$ and $\mathcal{J}_\mp$. In agreement with the decomposition (9), it is possible to set a comparison between a block $s_{h}{}^{0}_{k\mathbf{p}}$ with the exact prescriptions in Table 1 and another block $s_{h}{}^{0}_{k}$ obtained as some deviation $\delta\mathbf{p} = (\delta\mathcal{J}_\mp, \delta\mathcal{A}_\pm)$ from those prescriptions. Then, by defining $\Delta_{h}{}^{0}_{k\mathbf{p}} = s_{h}{}^{0}_{k} s_{h}{}^{0}_{k\mathbf{p}}{}^{-1}$, we have:

$$s_{h}{}^{0}_{k} \left|\psi_k\right\rangle = \Delta_{h}{}^{0}_{k\mathbf{p}} s_{h}{}^{0}_{k\mathbf{p}} \left|\psi_k\right\rangle \tag{23}$$

thus, the operator $\Delta_{h}{}^{0}_{k\mathbf{p}}$ states the deviation from the final desired quantum state $s_{h}{}^{0}_{k\mathbf{p}} \left|\psi_k\right\rangle$. Because this operator underlies in $SU(2)$:

$$\Delta_{h}{}^{0}_{k\mathbf{p}} = \alpha_I \sigma_0 + \alpha_X \sigma_1 + \alpha_Z \sigma_3 + \alpha_Y \sigma_1 \sigma_3 \tag{24}$$

which can be understood as a combination of quantum error syndromes on the corresponding block k (bit flipping σ_1, phase flipping σ_3, or both $\sigma_1\sigma_3$, corresponding with the quantum states being related by $s_{h}{}^{0}_{k}$). While, the quantities:

$$p_S \equiv |\alpha_S|^2 = \frac{1}{4}|\mathrm{Tr}(\sigma_S \Delta_{h}{}^{0}_{k\mathbf{p}})|^2, \quad \sigma_S \in \{\sigma_I = \sigma_0, \sigma_X = \sigma_1, \sigma_Y = \sigma_1\sigma_3, \sigma_Z = \sigma_3\} \tag{25}$$

could be understood as the probabilities to get certain error syndrome. Here, it is better use p_S to measure the impact of variations than the quantum state fidelity, because each gate (and their blocks) works on a wide variety of quantum states. Thus, p_I can be understood as the probability to get the correct quantum state as it was planned under the precise gate application. In the following, we report p_I as a measure of the key blocks fidelity reported in Table 1.

Figure 3 shows the values of p_I for each block: a) $s_{hk} \sim \sigma_0$, b) $s_{hk} \sim \sigma_1$, c) $s_{hk} \sim \mathcal{H}$, and d) $s_{hk} \sim S_\chi$. The first three figures show a contour map of p_I around of prescriptions for each gate with variations of $(\delta\mathcal{J}_\mp, \delta\mathcal{A}_\pm) \in [-0.5, 0.5] \times [-0.5, 0.5]$, it means around of 10% of values considered in the Figure 2. Only the representative cases for $q = 1$ with positive parameters $\mathcal{J}_\mp$ and $\mathcal{A}_\pm$ are reported, other cases are identical or similar with reflexions through the axis of the variables $\delta\mathcal{J}_\mp, \delta\mathcal{A}_\pm$. p_I values are higher than $0.69, 0.85$ and 0.70 respectively. Note for σ_1 the best stability. For the S_χ case, we have several prescriptions as function of the associated χ value. Then we report the lower p_I value for the region $(\delta\mathcal{J}_\mp, \delta\mathcal{A}_\pm) \in [-0.5, 0.5] \times [-0.5, 0.5]$ on each prescription given in the Figure 2. Assigned color on each prescriptions line gives that worst p_I value. For any curve, the worst values sets between 0.68 and 0.85. Despite aspects related with the errors introduced by the control

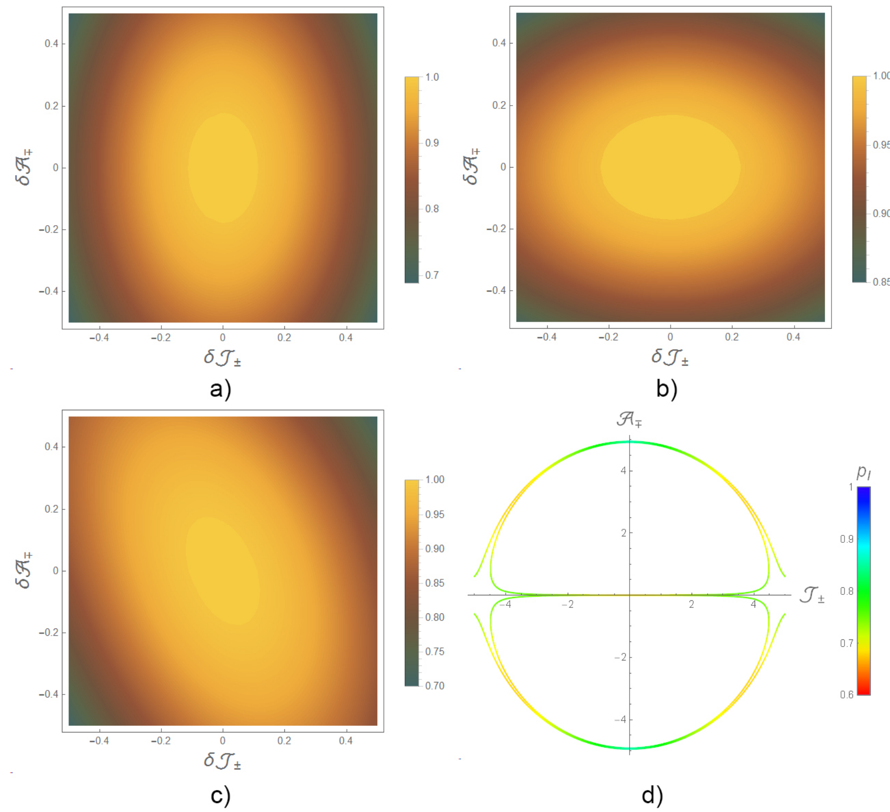

Fig. 3. p_I-values for possible deviations $(\delta\mathcal{J}_\pm, \delta\mathcal{A}_\mp) \in [-0.5, 0.5] \times [-0.5, 0.5]$ around the prescriptions in $\mathcal{A}_\mp$ and $\mathcal{J}_\pm$ for blocks a) $s_{h\,k} \sim \sigma_0$, b) $s_{h\,k} \sim \sigma_1$, c) $s_{h\,k} \sim \mathcal{H}$, and d) $s_{h\,k} \sim S_\chi$. In the last case the worst p_I-value is reported for each solution.

on $\mathcal{J}_0$ and the entire gate construction departing from the blocks, this brief analysis shows a good stability for the gates corresponding to errors until 10% in the values of $\delta\mathcal{J}_\mp$ and $\delta\mathcal{A}_\pm$.

6 Conclusions

Several approaches to the quantum gates design are being explored in the experimental arena. The task is not always easy, because it is highly dependent on the kind of systems on which the implementation is being considered and of course on innumerable aspects of experimental issues in control. Clearly, this aspect is still independent of the proper technological restrictions added, as decoherence or noise. Approaches based on gate factorization [22] reduce that task to be able to control the form and prescriptions for the factor gates, a mathematical idea presented in [23], which fits perfectly into the $SU(2)$ decomposition outcomes. Otherwise, the use of universal gates becomes convenient in terms of the development of a set of well-defined and tuned gates with multiple and repeated use. In any case, those constructions will have an efficiency and fidelity depending on the number of basic gates being used.

The use of an adequate basis for specific systems could to help in the reduction of the number of factor gates, as in the case presented here, where each gate is reached in one single operation. Still for no universal processing approach, for arbitrary gates on demand involving more than two qubits (a scenario not strictly necessary because two-qubit processing is universal), the current Fourier approach could be extended under the most general $SU(2)$ decomposition [12] than for the $SU(4)$ case. Together, improved solutions using optimal control for $SU(2)$ processing could be implemented [24, 25]. Despite, in the proposal there are other technological issues to be addressed, as the tight control required on the entangled states working as a grammar. Nevertheless, the $SU(2)$ decomposition approach to construct alternative gates becomes very precise in terms of the improvement provided by the use of natural basis of quantum information states resembling the properties of the system where they are settled [14]. In addition, the use of time-dependent fields as driven elements is an open possibility to integrate fields technologically more affordable [17] and not only the case exploited in the current work.

The proposal based on continuous multichannel fields (in terms of their alternative direction) has been exploited in other works [11]. These magnetic fields, as in our case, could to work as traveling agents providing control in the generation of programmed quantum information processing on condensed matter. These ideas should be still matured into a more robust design, inclusively proposing more optimal solutions in terms of the avoidance of superposition modes as in our approach. The numerical approach introduced in [17], opens the possibility to explore alternative designs where the qubits exposure to the waves becomes in a design parameter, letting the possibility to contribute in still more simple solutions for the induction of quantum processing on matter.

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