

## SUPER-ADDITIVITY AND ENTANGLEMENT ASSISTANCE IN QUANTUM READING

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Quantum information theory determines the maximum rates at which information can be transmitted through physical systems described by quantum mechanics. Here we consider the communication protocol known as quantum reading. Quantum reading is a protocol for retrieving the information stored in a digital memory by using a quantum probe, e.g., shining quantum states of light to read an optical memory. In a variety of situations using a quantum probe enhances the performance of the reading protocol in terms of fidelity, data density and energy efficiency. Here we review and characterize the quantum reading capacity of a memory model, defined as the maximum rate of reliable reading. We show that, like other quantities in quantum information theory, the quantum reading capacity is super-additive. Moreover, we determine conditions under which the use of an entangled ancilla improves the performance of quantum reading.

*Keywords:* Quantum communication, quantum reading, entanglement

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### 1 Introduction

The scope of quantum information theory is to determine how and how much information can be stored, processed, and transmitted through physical systems behaving according to the laws of quantum mechanics [1]. In particular, one is interested in transmitting classical or quantum information, possibly in the presence of physical constraints (e.g., limited energy or bandwidth) or additional resources (e.g., quantum entanglement or feedback communication) [2].

In the most common setting, one is given a quantum communication channel, that is, a physical process that transforms quantum states at the input into quantum states at the output, as for example an optical fiber does (see e.g., [3, 2]). A channel of this kind is also called a quantum-quantum (QQ) channel. Another kind of channel is the so-called classical-quantum (CQ) channel, which maps classical states (that is, a probability distribution over a set of symbols) into quantum states. To send classical information through a QQ channel, the sender (Alice) first encodes classical states into quantum states by applying a suitable CQ channel at the input of the QQ channel, as represented in Figure 1a. In this setting, the CQ

channel plays the role of an encoding map. At the other end of the QQ channel, the receiver (Bob) collects the output and measures it to decode the classical information sent by Alice.

Quantum reading (QR) is a communication protocol that is based on a different rationale [4, 5]. Instead of having a given communication channel and encoding classical information by choosing the input states, in QR the sender encodes information by choosing an element from a collection of QQ channels. Then, to decode, the receiver probes the QQ channels with a quantum state, collects the output, and measures it (see Figure 1b).

The prototypical example of QR is that of an optical memory, e.g., a CD or DVD, where information is encoded in a memory cell by means of the physical properties of the substrate, e.g., its reflectivity or phase. For this reason, QR has been mainly considered in the context of optical realizations [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. For example, a memory cell with low or high reflectivity may encode a logical "0" or "1". To read this information the receiver must shine a laser beam on the memory cell, and then collect the reflected beam (see Figure 2). There are proven advantages in using quantum states of light to perform this task, for instance increased fidelity, data density, and energy efficiency [4].

From a more abstract point of view, QR can be represented as shown in Figure 1b. The encoding of a symbol  $x$  belonging to an alphabet  $\mathcal{X}$  can be modeled as a controlled-QQ channel (a generalization of a controlled-unitary channel [1]) where the value of  $x$  determines which of the QQ channels in a set  $\Phi = \{\phi_x\}_{x \in \mathcal{X}}$  should be applied.

Following [5] we refer to the set  $\Phi = \{\phi_x\}_{x \in \mathcal{X}}$  as a "memory cell". One can define the *quantum reading capacity* of  $\Phi$  as the maximum rate (in bits per use of the memory cell) that can be reliably transmitted from the sender to the receiver using the encoding procedure specified by  $\Phi$ . Indeed, in previous works several notions of capacity have been defined according to which constraints are assumed or additional resources are allowed [5, 6, 9]. In this paper we present further results concerning quantum reading capacities. We show that QR capacity, like other quantities in quantum information theory [16, 17], is super-additive, and discuss under which conditions the assistance of entanglement enhances the QR capacity.

The paper proceeds as follows. In Section 2 we review a few basic notions and definitions. In Section 3 we analyze the case of noiseless QR. The property of super-additivity of QR is discussed in Section 4, and the case of noisy QR is considered in Section 5. Finally, Section 6 is devoted to zero-error QR capacity, and Section 7 is for conclusions.

## 2 Quantum reading capacities

A QR protocol comprises an encoding and a decoding stage. During the encoding stage, which is classical, the sender Alice encodes messages  $i = 1, 2, \dots, M$  using codewords of length  $n$ ,  $x^n(i) = x_1(i)x_2(i) \cdots x_n(i)$ , where  $x_k(i)$  belong to a given set of symbols  $\mathcal{X}$ . Each codeword identifies a corresponding sequence of quantum channels from the memory cell  $\Phi = \{\phi_x\}_{x \in \mathcal{X}}$ , e.g.,  $\phi_{x^n(i)}^n = \phi_{x_1(i)} \otimes \phi_{x_2(i)} \otimes \cdots \otimes \phi_{x_n(i)}$ . During the decoding stage, the receiver Bob prepares a state  $\rho^n$ , also called a transmitter, which is used to probe the sequence of quantum channels  $\phi_{x^n(i)}^n$ . Finally, Bob collects and measures the output to retrieve the encoded message. For the sake of simplicity we assume that the input and output spaces of the channels  $\phi_{x^n(i)}^n$  coincides, the extension to the most general case being straightforward.

In analogy with other quantum communication protocols we introduce the definitions (here and in the following  $\log \equiv \log_2$  denotes the binary logarithm):

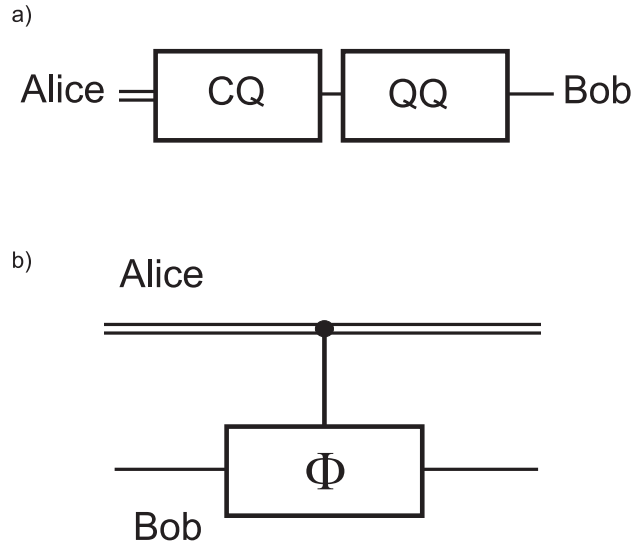


Fig. 1. a) Classical communication through a QQ channel, where the CQ channel plays the role of an encoding map. b) In quantum reading the encoding is represented by a control-QQ channel. The receiver Bob has access to both the input and output space of the memory cell  $\Phi$ . Single lines represent qubits, and double lines represent classical bits.

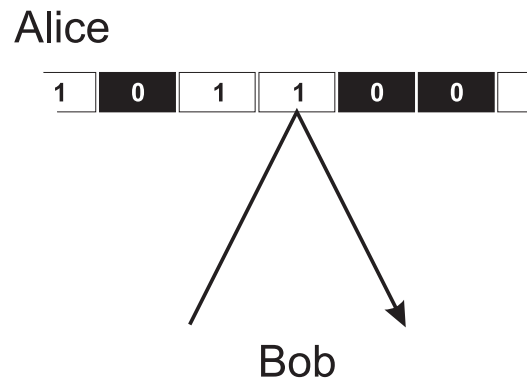


Fig. 2. An optical memory is the prototype of quantum reading. A memory cell with low or high reflectivity encodes a logical 0 or 1. To read this information the receiver shines a laser on the memory cell, and then collect the reflected beam.

**Definition 1 (Quantum Reading protocol)** A  $(M, n, \epsilon)$ -QR protocol for a memory cell  $\Phi = \{\phi_x\}_{x \in \mathcal{X}}$  is defined by an encoding map  $\mathcal{E}$  from  $i = 1, \dots, M$  to  $\mathcal{X}^{\times n}$ , a transmitter state  $\rho^n$ , and a measurement with POVM elements  $\{\Lambda(j)\}_{j \in \mathcal{J}}$ , such that the average probability of error in decoding is smaller than  $\epsilon$ , that is,

$$\frac{1}{M} \sum_i \left\{ 1 - \text{Tr} \left[ \Lambda(i) \phi_{\mathcal{E}(i)}^n(\rho^n) \right] \right\} < \epsilon.$$

The rate of the QR protocol is  $R = \frac{1}{n} \log M$ .

We remark that both the transmitter state  $\rho^n$  and the output states  $\phi_{\mathcal{E}(i)}^n(\rho^n)$  are under the control of the receiver Bob (see Figure 1a).

**Definition 2 (Quantum Reading capacity)** The QR capacity of a given memory cell  $\Phi$  is defined as

$$C(\Phi) = \sup \left\{ \liminf_n \frac{1}{n} \log M_n : \limsup_n \epsilon_n = 0 \right\},$$

where the sup is over all sequences of  $(M_n, n, \epsilon_n)$ -QR protocols.

We now introduce some notions of constrained QR capacities. Instead of considering a generic state for the transmitter, we can restrict to the family of QR protocols for which the transmitter has the form,  $\rho^n = \rho^{\otimes n}$ , that is, it is a tensor-power state across different uses of the memory cell. The maximum QR rate that can be achieved under this constraint is defined as  $C^1(\Phi)$ . We have

$$C^1(\Phi) = \max_{\rho} C^1(\Phi|\rho), \quad (1)$$

where  $C^1(\Phi|\rho)$  denotes the maximum QR rate achievable for a given  $\rho$ . Applying well known results of quantum information theory [18, 19], the latter can be expressed as:

$$C^1(\Phi|\rho) = \max_{\{p_x\}_{x \in \mathcal{X}}} \chi(\{p_x, \phi_x(\rho)\}_{x \in \mathcal{X}}) \quad (2)$$

where the maximum is over all probability distributions on the alphabet  $\mathcal{X}$ ,

$$\chi(\{p_x, \phi_x(\rho)\}_{x \in \mathcal{X}}) = S \left( \sum_{x \in \mathcal{X}} p_x \phi_x(\rho) \right) - \sum_{x \in \mathcal{X}} p_x S(\phi_x(\rho)), \quad (3)$$

is the Holevo information, and  $S(\sigma) = -\text{Tr}(\sigma \log \sigma)$  denotes the von Neumann entropy. It can be easily shown that the maximum is obtained when  $\rho$  is a pure state [5].

Similarly we can define the QR capacities  $C^k(\Phi)$ , for  $k = 1, 2, \dots$ , where the transmitter is a product state across pairs, triplets, etc., of different uses of the memory cell, that is,  $\rho^n = \rho^{k \otimes (n/k)}$  (for  $n$  multiple of  $k$ ). We have

$$C^k(\Phi|\rho^k) = \frac{1}{k} \max_{\{p_{x^k}\}_{x^k \in \mathcal{X}^{\times k}}} \chi(\{p_{x^k}, \phi_{x^k}(\rho^k)\}_{x^k \in \mathcal{X}^{\times k}}). \quad (4)$$

Clearly  $C(\Phi) \geq C^k(\Phi) \geq C^h(\Phi)$  for  $k > h$ . If at least one of these inequality is strict, that is,  $C^k(\Phi) > C^h(\Phi)$  for some  $k$ , we say that the  $C(\Phi)$  is super-additive.

More generally, the transmitter state can be chosen to be entangled with an ancilla, which Bob retains and measures jointly (see Figure 3). Following well known ideas from quantum

information theory [20], in this case we speak of entanglement-assisted QR. Notice that an entanglement-assisted QR protocol for a memory cell  $\Phi = \{\phi_x\}$  is equivalent to an unassisted protocol for the extended memory cell  $\Phi \otimes \text{id}_d = \{\phi_x \otimes \text{id}_d\}$ , where  $\text{id}_d$  denotes the identity channel acting on an ancillary system of dimensions  $d$ . From the Schmidt decomposition it follows that the dimensions of the ancillary system can be assumed, without loss of generality, not larger than the dimension of the Bob's space, denoted as  $|B|$ . The entanglement-assisted QR capacity is hence defined as

$$C_{EA}(\Phi) = \max_d C(\Phi \otimes \text{id}_d) = C(\Phi \otimes \text{id}_{|B|}). \tag{5}$$

Similarly, we can define the assisted QR capacities  $C_{EA}^k(\Phi)$  by constraining the transmitter to be a product state across groups of  $k$  uses of the memory cell. Clearly, we have  $C_{EA}(\Phi) \geq C(\Phi)$ , and  $C_{EA}^k(\Phi) \geq C^k(\Phi)$ . If, for some value of  $k$ , this inequality is strict we say that the assistance of entanglement enhances the QR capacity  $C^k(\Phi)$  of the memory cell.

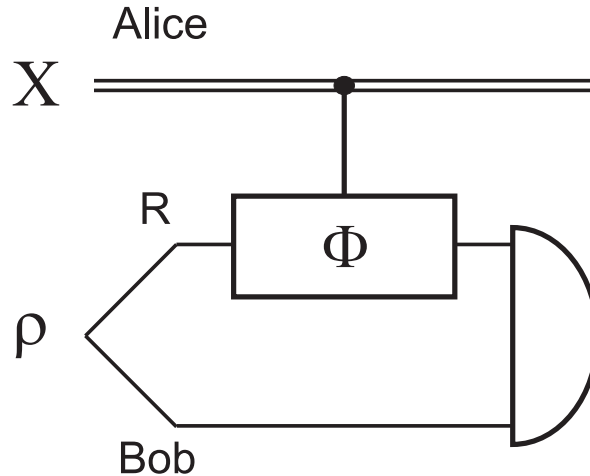


Fig. 3. Diagram for entanglement-assisted quantum reading.

### 3 Noiseless quantum reading

We first consider a noiseless setting in which the QQ channels in the memory cell are unitary transformations, that is,  $\phi_x(\rho) = U_x \rho U_x^\dagger$ . For the sake of simplicity we consider the binary setting,  $x \in \{0, 1\}$ , with the unitaries acting on a finite-dimensional Hilbert space of dimension  $d$ .

Let us consider the Holevo information,

$$\chi \left( \{p_x, U_x |\psi\rangle\langle\psi| U_x^\dagger\}_{x=0,1} \right) = S \left[ p U_0 |\psi\rangle\langle\psi| U_0^\dagger + (1-p) U_1 |\psi\rangle\langle\psi| U_1^\dagger \right] \tag{6}$$

$$= S \left[ p |\psi\rangle\langle\psi| + (1-p) U |\psi\rangle\langle\psi| U^\dagger \right], \tag{7}$$

where  $p = p_0$ , and  $U = U_0^\dagger U_1$ . From Eq. (2) we obtain  $C^1$  by maximization of the Holevo information. This is equivalent to maximizing the von Neuman entropy of the state  $\sigma =$

$p|\psi\rangle\langle\psi| + (1-p)U|\psi\rangle\langle\psi|U^\dagger$ . In order to do that, it is convenient to introduce a unit vector  $|\psi_\perp\rangle$  such that  $\langle\psi|\psi_\perp\rangle = 0$  and

$$U|\psi\rangle = \alpha|\psi\rangle + \sqrt{1-|\alpha|^2}|\psi_\perp\rangle. \quad (8)$$

In the system of orthonormal vectors  $|\psi\rangle, |\psi_\perp\rangle$ , the state  $\sigma$  is represented by the density matrix:

$$\tilde{\sigma} = \begin{pmatrix} p + (1-p)|\alpha|^2 & (1-p)\alpha\sqrt{1-|\alpha|^2} \\ (1-p)\alpha^*\sqrt{1-|\alpha|^2} & (1-p)(1-|\alpha|^2) \end{pmatrix}. \quad (9)$$

The maximum von Neumann entropy of  $\sigma$  is achieved in correspondence to the maximum determinant of the matrix  $\tilde{\sigma}$ . We have,

$$\det \tilde{\sigma} = p(1-p)(1-|\alpha|^2), \quad (10)$$

which achieves its maximum for  $p = 1/2$  and in correspondence of the minimum value of  $|\alpha|^2 = |\langle\psi|U|\psi\rangle|^2$ . Let us denote as  $\{|j\rangle\}_{j=0,\dots,d-1}$  the eigenvectors of  $U$ , and as  $e^{i\theta_j}$  the corresponding eigenvalues. We expand  $|\psi\rangle$  in the basis of eigenvectors,  $|\psi\rangle = \sum_j \psi_j|j\rangle$ , which yields  $\alpha = \sum_j |\psi_j|^2 e^{i\theta_j}$ . The reading capacity is hence obtained by putting  $\alpha = \alpha_{\min}$ , with

$$|\alpha_{\min}|^2 = \min_{\{\psi_j\} | \sum_j |\psi_j|^2 = 1} \sum_{jj'} |\psi_j|^2 |\psi_{j'}|^2 e^{i(\theta_j - \theta_{j'})}, \quad (11)$$

which finally yields

$$C^1 = h\left(\frac{1-|\alpha_{\min}|}{2}\right), \quad (12)$$

where  $h(x) = -x \log x - (1-x) \log(1-x)$  is the binary entropy.

Let us now consider the entanglement-assisted QR capacity  $C_{EA}^1$ . To compute  $C_{EA}^1$  we can repeat the reasoning of above with  $U$  replaced by  $U \otimes I$ . Notice that  $U \otimes I$  has the same eigenvalues of  $U$  (but with higher multiplicity). We can then consider a system of eigenvectors of  $U \otimes I$ , denoted as  $\{|jk\rangle\}$ , where  $|jk\rangle$  are the eigenvectors with eigenvalue  $e^{i\theta_j}$ . Expanding the transmitter state  $|\psi\rangle$  in this basis we obtain  $|\psi\rangle = \sum_{jk} \psi_{jk}|jk\rangle$ , which yields  $\alpha = \sum_{jk} |\psi_{jk}|^2 e^{i\theta_j} = \sum_j (\sum_k |\psi_{jk}|^2) e^{i\theta_j}$ . We then obtain the same expression for  $|\alpha_{\min}|^2$  as in Eq. (11) upon replacing  $\sum_k |\psi_{jk}|^2 \rightarrow |\psi_j|^2$ . In conclusion, we have obtained that  $C^1 = C_{EA}^1$ , that is, the assistance of entanglement does not enhance the QR capacity  $C^1$  in the noiseless setting.

As an example, let us consider the case of qubit unitaries. We have  $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$  and

$$|\alpha_{\min}|^2 = \min_{\{\psi_0, \psi_1\} : |\psi_0|^2 + |\psi_1|^2 = 1} |\psi_0|^4 + |\psi_1|^4 + 2|\psi_0|^2|\psi_1|^2 \cos(\delta\theta), \quad (13)$$

with  $\delta\theta = |\theta_1 - \theta_0|$ . The minimum is hence obtained for  $|\psi_0|^2 = |\psi_1|^2 = 1/2$  and yields

$$|\alpha_{\min}| = \sqrt{\frac{1 + \cos(\delta\theta)}{2}} = |\cos(\delta\theta/2)|. \quad (14)$$

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<sup>a</sup>This follows from the fact that the eigenvalues of a  $2 \times 2$  matrix  $A$  are  $\lambda_\pm = \frac{1}{2} [\text{Tr}A \pm \sqrt{(\text{Tr}A)^2 - 4\text{Det}A}]$ , with  $\text{Tr}A = 1$  and  $\text{Det}A \geq 0$  for  $A$  a density matrix. This implies that  $\lambda_+ \geq \lambda_- \geq 0$ . The von Neumann entropy  $S = -\lambda_- \log \lambda_- - \lambda_+ \log \lambda_+$  is maximum in correspondence of the maximum value of  $\lambda_-$ , which in turn corresponds to the maximum value of  $\text{Det}A$ .

Finally from (12) we obtain

$$C^1 = h(\sin^2(\delta\theta/4)) . \tag{15}$$

#### 4 Super-additivity

Let us consider the case of a binary memory cell composed of two qubit unitary transformations,  $U_0$  and  $U_1$ . We now show that this cell exhibits the phenomenon of super-additivity.

For given  $k > 1$ , let us consider the Holevo information  $\chi^k = \chi(\{p_{x^k}, \phi_{x^k}(|\psi\rangle\langle\psi|\})$  in Eq. (4) with a probability distribution such that  $p_{00\dots 0} = p_{11\dots 1} = 1/2$ . That is, we are only considering, with equal probability, the unitary transformations  $U_0^{\otimes k}$  and  $U_1^{\otimes k}$ . The Holevo information then reads

$$\chi^k = S\left(\frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}U^{\otimes k}|\psi\rangle\langle\psi|U^{\otimes k\dagger}\right) , \tag{16}$$

with  $U = U_0^\dagger U_1$ .

Let us denote the eigenvectors of  $U$  as  $|0\rangle$  and  $|1\rangle$ , with corresponding eigenvalues  $e^{i\theta_0}$  and  $e^{i\theta_1}$ . As a transmitter state we choose the entangled state  $|\psi\rangle = (|0\rangle^{\otimes k} + |1\rangle^{\otimes k})/\sqrt{2}$ . We then obtain

$$C^k \geq \frac{1}{k} h(\sin^2(k\delta\theta/4)) . \tag{17}$$

Comparing with  $C^1$  in (15) we find that for any  $k$  there are values of  $\delta\theta$  such that  $C^k > C^1$ , that is, QR is super-additive. We can also write a lower bound on the ultimate QR capacity, that is,

$$C \geq \sup_k \frac{1}{k} h(\sin^2(k\delta\theta/4)) . \tag{18}$$

See Figure 4 for a comparison among different QR capacities.

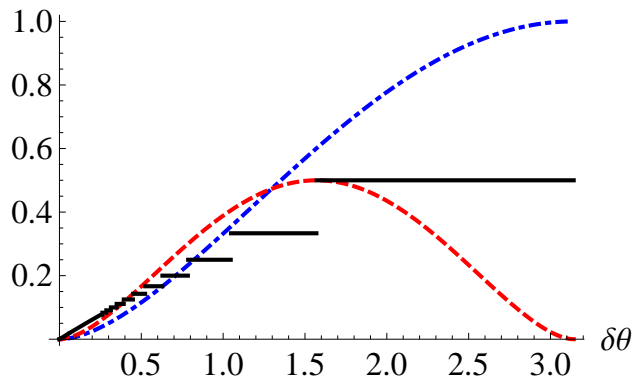


Fig. 4. QR capacities for a binary qubit unitary memory cell versus the angular separation  $\delta\theta$ . Dot-dashed blue line:  $C^1$  from Eq. (15). Dashed red line: lower bound on  $C^2$  from Eq. (17). Solid black line: zero-error capacity  $C^0$  from Eq. (31)

## 5 Noisy quantum reading

Going beyond the case of noiseless quantum reading, we consider a simple yet physically motivated example of noisy binary quantum reading where the two encoding maps are of the form

$$\phi_x(\rho) = (1 - q) U_x \rho U_x^\dagger + q \rho_0, \quad (19)$$

where for  $x = 0, 1$ ,  $U_x$ 's are unitary matrices in  $SU(d)$ ,  $\rho_0$  is the maximally mixed state in  $d$  dimensions, and  $q \in [0, 1]$  is the probability of depolarization.

Let us first consider the QR capacity  $C^1$ . Putting  $U = U_0^\dagger U_1$ , the Holevo information in Eq. (2) reads

$$\chi = S((1 - q)(p_0|\psi\rangle\langle\psi| + p_1U|\psi\rangle\langle\psi|U^\dagger) + q\rho_0) - S((1 - q)U|\psi\rangle\langle\psi|U^\dagger + q\rho_0) \quad (20)$$

$$= S((1 - q)(p_0|\psi\rangle\langle\psi| + p_1U|\psi\rangle\langle\psi|U^\dagger) + q\rho_0) - \eta(1 - q + q/d) - (d - 1)\eta(q/d) \quad (21)$$

where  $\eta(y) = -y \log y$ . The maximization of the Holevo information is thus reduced to the maximization of the von Neumann entropy of the state  $\sigma = (1 - q)(p_0|\psi\rangle\langle\psi| + p_1U|\psi\rangle\langle\psi|U^\dagger) + q\rho_0$ . It is convenient to expand this state in a basis defined by the vector  $|\psi\rangle$ , the vector  $|\psi_\perp\rangle$  (such that  $\langle\psi|\psi_\perp\rangle = 0$  and  $U|\psi\rangle = \alpha|\psi\rangle + \sqrt{1 - |\alpha|^2}|\psi_\perp\rangle$ ), and any other complementary set of  $d - 2$  orthogonal vectors. In this basis the state  $\sigma$  is represented by the density matrix

$$\tilde{\sigma} = \begin{pmatrix} (1 - q)[p + (1 - p)|\alpha|^2] + \frac{q}{d} & (1 - q)(1 - p)\alpha\sqrt{1 - |\alpha|^2} \\ (1 - q)(1 - p)\alpha^*\sqrt{1 - |\alpha|^2} & (1 - q)(1 - p)(1 - |\alpha|^2) + \frac{q}{d} \end{pmatrix} \bigoplus_{j=2}^{d-1} \left(\frac{q}{d}\right). \quad (22)$$

The maximum von Neumann entropy of  $\sigma$  corresponds to the maximum determinant of  $\tilde{\sigma}$ , where

$$\det \tilde{\sigma} = \left( p(1 - p)(1 - q)^2(1 - |\alpha|^2) + \frac{q(1 - q)}{d} + \frac{q^2}{d^2} \right) \left( \frac{q}{d} \right)^{d-2}. \quad (23)$$

For any given  $q$ , the determinant is maximized for  $p = (1 - p) = 1/2$  and in correspondence of the minimum value of  $|\alpha|^2$ . We hence obtain

$$C^1 = \eta\left(\frac{q}{d} + \frac{(1 - q)(1 + |\alpha_{\min}|)}{2}\right) + \eta\left(\frac{q}{d} + \frac{(1 - q)(1 - |\alpha_{\min}|)}{2}\right) - \eta(1 - q + q/d) - \eta(q/d), \quad (24)$$

where  $|\alpha_{\min}|$  is as in Eq. (11).

For example, in the case  $d = 2$ , using Eq. (14), we obtain

$$C^1 = \eta\left(\frac{q}{2} + (1 - q)\cos^2(\delta\theta/4)\right) + \eta\left(\frac{q}{2} + (1 - q)\sin^2(\delta\theta/4)\right) - \eta(1 - q/2) - \eta(q/2). \quad (25)$$

### 5.1 Entanglement assisted reading

Unlike the noiseless case, the assistance of entanglement can be beneficial in the noisy setting. We consider the entanglement-assisted QR capacity  $C_{EA}^1$ , which can be computed by maximizing the Holevo information

$$\begin{aligned} \chi &= S((1 - q)(p_0|\psi\rangle\langle\psi| + p_1(U \otimes I)|\psi\rangle\langle\psi|(U^\dagger \otimes I)) + q\rho_0 \otimes \rho_R) \\ &\quad - S((1 - q)(U \otimes I)|\psi\rangle\langle\psi|(U^\dagger \otimes I) + q\rho_0 \otimes \rho_R), \end{aligned} \quad (26)$$



where  $|\psi\rangle$  is a joint pure state for the  $BR$  system, comprising both Bob's output and an ancillary system  $R$ , and  $\rho_R = \text{Tr}_B(|\psi\rangle\langle\psi|)$  denotes the reduced state of the ancilla. Without loss of generality we can assume that the dimension of the ancilla  $R$  equals that of Bob system  $B$ . Moreover, as an example we take  $|\psi\rangle$  to be a maximally entangled state in the  $BR$  system, which implies  $\rho_R = \rho_0 = I/d$ . We then obtain

$$\begin{aligned}\chi &= S\left((1-q)(p_0|\psi\rangle\langle\psi| + p_1(U \otimes I)|\psi\rangle\langle\psi|(U^\dagger \otimes I)) + q\frac{I}{d^2}\right) \\ &\quad - S\left((1-q)(U \otimes I)|\psi\rangle\langle\psi|(U^\dagger \otimes I) + q\frac{I}{d^2}\right)\end{aligned}\quad (27)$$

$$\begin{aligned}&= S\left((1-q)(p_0|\psi\rangle\langle\psi| + p_1(U \otimes I)|\psi\rangle\langle\psi|(U^\dagger \otimes I)) + q\frac{I}{d^2}\right) \\ &\quad - \eta(1-q - q/d^2) - (d^2 - 1)\eta(q/d^2).\end{aligned}\quad (28)$$

Notice that this expression of  $\chi$  is formally identical to that in Eq. (21) upon the substitution  $d \rightarrow d^2$  (this is analogous to what happens in entanglement-assisted communication [20, 21]). We hence obtain

$$C_{EA}^1 \geq \eta\left(\frac{q}{d^2} + \frac{(1-q)(1+|\alpha|)}{2}\right) + \eta\left(\frac{q}{d^2} + \frac{(1-q)(1-|\alpha|)}{2}\right) - \eta(1-q + q/d^2) - \eta(q/d^2),\quad (29)$$

where  $|\alpha|^2 = |\langle\psi|U \otimes I|\psi\rangle|^2$  and  $|\psi\rangle$  is a maximally entangled state.

For the sake of simplicity, let us now consider the case  $d = 2$  and consider a system of eigenvectors of  $U \otimes I$ , denoted as  $\{|jk\rangle\}$ , where  $|jk\rangle$  is an eigenvector with eigenvalue  $e^{i\theta_j}$ . The maximally entangled state  $\psi$  can be represented, without loss of generality, as  $|\psi\rangle = \sum_{j=0,1} 2^{-1/2}|jj\rangle$ , which implies  $\alpha = \frac{1}{2} \sum_{j=0,1} e^{i\theta_j}$  and in turn yields  $|\alpha| = |\cos(\delta\theta/2)|$ . Substituting this value for  $\alpha$  in Eq. (29) with  $d = 2$  we obtain

$$C_{EA}^1 \geq \eta\left(\frac{q}{4} + (1-q)\cos^2(\delta\theta/4)\right) + \eta\left(\frac{q}{4} + (1-q)\sin^2(\delta\theta/4)\right) - \eta(1-3q/4) - \eta(q/4).\quad (30)$$

Comparing with Eq. (25) we find that, unlike the noiseless setting, the assistance of entanglement is beneficial (that is,  $C_{EA}^1 > C^1$ ) in the presence of noise (see Figure 5).

## 6 Zero-error capacity

QR is closely related to the problem of quantum channel discrimination [4, 5]. The particular case of noiseless QR is in close relation with the problem of discriminating between two unitary transformations. According to [22], two unitaries,  $U_0$  and  $U_1$ , can always be perfectly discriminated, if enough copies of them are provided and using a suitable input state and, possibly, a collective measurement. To relate in a formal way this feature of the problem of unitary discrimination with QR, we need to consider the notion of zero-error QR capacity.

**Definition 3 (Zero-error quantum reading protocol)** *A  $(M, n)$  zero-error QR protocol for a memory cell  $\Phi = \{\phi_x\}_{x \in \mathcal{X}}$  is defined by an encoding map  $\mathcal{E}$  from  $i = 1, \dots, M$  to  $\mathcal{X}^n$ , a transmitter state  $\rho^n$ , and a measurement with POVM elements  $\{\Lambda(j)\}_{j \in \mathcal{J}}$ , such that the probability of error in decoding is zero. The rate of the QR protocol is  $R = \frac{1}{n} \log M$ .*

From the definition of zero-error QR protocol it follows that of zero-error QR capacity as the maximum zero-error QR rate. Coming back to the results of [22], let us consider the case

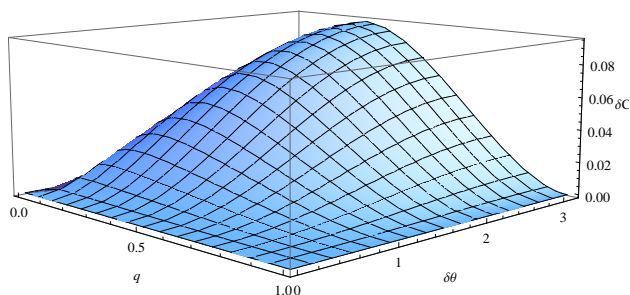


Fig. 5. The plot shows a lower bound on the gain  $\delta C = C_{EA}^1 - C^1$  for the case of the qubit binary noisy memory cell described in Section 5, where  $q$  is the depolarization probability, and  $\delta\theta$  is the relative angle of rotation. The plot shows the difference between the expressions in Eq. (30) and Eq. (25) vs  $\delta\theta$  and  $q$ . A positive value of  $C_{EA}^1 - C^1$  implies that the assistance of entanglement enhances the reading capacity.

of two-dimensional unitaries (extension to the higher dimensions is straightforward). Given that the spectrum of the unitary  $U = U_0^\dagger U_1$  is  $e^{i\theta_0}$ ,  $e^{i\theta_1}$ , [22] proved that the unitaries  $U_0^{\otimes n}$  and  $U_1^{\otimes n}$  are perfectly distinguishable if  $n \geq \pi/\delta\theta$  (with  $\delta\theta = |\theta_0 - \theta_1|$ ). This result implies that the zero-error QR capacity of the binary unitary memory cell is

$$C_0 = \frac{1}{\lceil \pi/\delta\theta \rceil}. \quad (31)$$

The zero-error capacity is plotted in Figure 4.

## 7 Conclusions

We have studied the ultimate rates (capacities) which are achievable in the quantum reading of classical memories, showing their super-additive feature. This means that a parallel quantum access of a classical memory where the cells are read simultaneously (which can also be seen as a type of quantum pattern recognition) provides a non-trivial advantage with respect to a sequential data access, where the cells are independently read one after the other, even by the use of quantum probes.

As already pointed out in previous works [4, 5], the protocol of QR is closely related to the task of quantum channel discrimination. For example, the fact that the assistance of entanglement enhances the QR capacity for a noisy memory cell, mirrors the fact that the use of an entangled ancilla improves the discrimination between quantum channels (see e.g. [23]). At a more formal level, the analogy between QR and quantum channel discrimination can be appreciated through the notion of zero-error QR capacity, as discussed in Section 6. QR is also closely related to the task of parameter estimation, see e.g. [24] (this connection was also discussed in [25]). For example, notice that the QR capacity is super-additive in the region of small values of  $\delta\theta$  (see Figure 4). This is the regime in which discriminating between two unitaries is essentially equivalent to estimating a small variation of the value of a relative phase.

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## References

1. M. A. Nielsen and I. L. Chuang (2000), *Quantum Computation and Quantum Information*, Cambridge University Press (Cambridge).
2. F. Caruso, V. Giovannetti, C. Lupo, S. Mancini (2014), *Quantum channels and memory effects* Rev. Mod. Phys., 86, 1203.
3. C. M. Caves, P. D. Drummond (1994), *Quantum limits on bosonic communication rates*, Rev. Mod. Phys., 66, 481.
4. S. Pirandola (2011), *Quantum Reading of a Classical Digital Memory*, Phys. Rev. Lett., 106, 090504.
5. S. Pirandola, C. Lupo, V. Giovannetti, S. Mancini, S. L. Braunstein (2011), *Quantum reading capacity*, New J. Phys., 13, 113012.
6. S. Guha, Z. Dutton, R. Nair, J. Shapiro, and B. Yen (2011), *Information Capacity of Quantum Reading*, in Frontiers in Optics 2011/Laser Science XXVII, OSA Technical Digest (Optical Society of America, 2011), paper LTuF2.
7. M. Dall'Arno, A. Bisio, G. M. D'Ariano, M. Miková, M. Ježek, M. Dušek (2012), *Experimental implementation of unambiguous quantum reading*, Phys. Rev. A, 85, 012308.
8. M. Dall'Arno, A. Bisio, G. M. D'Ariano (2012), *Ideal quantum reading of optical memories*, Int. J. Quantum Inform., 10, 1241010.
9. M. M. Wilde, S. Guha, S. H. Tan, S. Lloyd (2012), *Explicit capacity-achieving receivers for optical communication and quantum reading*, Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on, Cambridge, MA, 2012, pp. 551-555.
10. J. Prabhu Tej, A. R. Usha Devi, A. K. Rajagopal (2013), *Quantum reading of digital memory with non-Gaussian entangled light*, Phys. Rev. A, 87, 052308.
11. C. Lupo, S. Pirandola, V. Giovannetti, S. Mancini (2013), *Quantum reading capacity under thermal and correlated noise*, Phys. Rev. A, 87, 062310.
12. S. Guha, J. H. Shapiro (2013), *Reading boundless error-free bits using a single photon*, Phys. Rev. A, 87, 062306.
13. M. Dall'Arno (2014), *Quantum reading for the practical man*, Int. J. Quantum Inform., 12, 1560018.
14. W. Roga, D. Buono, F. Illuminati (2015), *Device-independent quantum reading and noise-assisted quantum transmitters*, New J. Phys., 17, 013031.
15. G. Spedalieri (2015), *Cryptographic Aspects of Quantum Reading*, Entropy, 17, 2218.
16. M. B. Hastings (2009), *Superadditivity of communication capacity using entangled inputs*, Nat. Phys., 5, 255.
17. G. Smith, J. Yard (2008), *Quantum communication with zero-capacity channels*, Science, 321, 1812.
18. A. S. Holevo (1998), *The capacity of the quantum channel with general signal states*, IEEE Trans. Inf. Theory, 44, 269.
19. B. Schumacher and M. D. Westmoreland (1997), *Sending classical information via noisy quantum channels*, Phys. Rev. A, 56, 131.
20. C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal (1999), *Entanglement-Assisted Classical Capacity of Noisy Quantum Channels*, Phys. Rev. Lett. 83, 3081.
21. C. King (2003), *The capacity of the quantum depolarizing channel*, IEEE Trans. Inf. Theory 49, 221.
22. A. Acín (2001), *Statistical distinguishability between unitary operations*, Phys. Rev. Lett., 87, 177901.
23. M. F. Sacchi (2005), *Optimal discrimination of quantum operations*, Phys. Rev. A, 71, 062340.
24. M. G. A. Paris (2009), *Quantum estimation for quantum technology*, Int. J. Quant. Inf., 7, 125.

25. J. Czajkowski, M. Jarzyna, R. Demkowicz-Dobrzanski (2016), *Super-additivity in communication of classical information through quantum channels from a quantum parameter estimation perspective*, arXiv:1603.00472.