Quantum Information and Computation, Vol. 17, No. 13&14 (2017) 1206–1220

A NOTE ON COHERING POWER AND DE-COHERING POWER

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Received March 3, 2017
Revised September 18, 2017

Cohering power and de-cohering power have recently been proposed to quantify the ability of a quantum operation to produce and erase coherence respectively. In this paper, we investigate the properties of cohering power and de-cohering power. First, we prove the equivalence between two different kinds of cohering power for any quantum operation on single qubit systems, which implies that l_1 norm of coherence is monotone under Maximally incoherent operation (MIO) and Dephasing-covariant operation (DIO) in 2-dimensional space. In higher dimensions, however, we show that the monotonicity under MIO or DIO does not hold. Besides, we compare the set of quantum operations with zero cohering power with Maximally incoherent operation (MIO) and Incoherent operation (IO). Moreover, two different types of de-cohering power are defined and we find that they are not equal in single qubit systems. Finally, we make a comparison between cohering power and de-cohering power for single qubit unitary operations and show that cohering power is always larger than de-cohering power.

Keywords: coherence, cohering power, de-cohering power

Communicated by: R Jozsa & K Mlmer

1 Introduction

Quantum resource theory [1, 2] plays an important role in the development and quantitative understanding of various physical phenomena in quantum physics and quantum information theory. A resource theory consists of two basic elements: free operations and free states. Any operation (or state) is dubbed as a resource if it falls out of the set of free operations (or the set of free states). The most significant resource theory is the resource theory of quantum entanglement defined on bipartite or multipartite systems [3], which is a basic resource for various information processing protocols including superdense coding [4] and teleportation [5]. However, for single quantum systems, quantum coherence, which is based on the superposition rule, must be thought of a peculiar feature of quantum mechanic just like entanglement in bipartite systems. Recently significant advancements in fields like thermodynamic theory [6–

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9], quantum biology [10–12], has suggested coherence to be a useful resource at the nanoscale, which leads to the development of the resource theory of coherence [13–45].

One advantage of having a resource theory for some physical quantity is the operational quantification of the relevant resources and the resource production through a quantum operation. In the resource theory of entanglement, entangling power [46] of quantum operations has been proposed to quantify the ability of quantum operations to produce entanglement. Besides, cohering power and de-cohering power of quantum operations have also been proposed to quantify the ability to produce coherence and erase coherence respectively [26]. And it has been shown that the cohering power of single qubit unitary operations is equal to de-cohering power in the skew information of coherence [24]. Two different types of cohering power have been defined on the set of incoherent states and the set of all quantum states respectively, and it has been proved that these two types of cohering power are equal for unitary operation in single qubit case [47,48]. However, whether this statement can be generalized to any quantum operation in single qubit case remains unclear. In the present work, we further investigate cohering power and de-cohering power. And we prove that these two types of cohering power are equal for any quantum operation in 2-dimensional space, which extends the result on unitary operations [47,48] to general quantum operations. Besides, as the cohering power of incoherent operations is always zero, we compare the sets of quantum operations with zero cohering power with several different free operations for coherence [25], namely, Incoherent operation (IO), Maximally incoherent operation (MIO) and Dephasing-covariant incoherent operation (DIO) [13,19,20]. As free operations cannot increase the amount of the relevant resource, the monotonicity of resource measure under free operations is crucial to the resource theory. Whether l_1 norm of coherence is monotone under MIO and DIO or not is an open problem proposed in [19,20]. In this work, we prove that l_1 norm of coherence is not monotone under MIO or DIO. Due to this statement, we demonstrate the operational gap between DIO and IO in terms of state transformation, which is also an open problem proposed in [19,20]. Furthermore, we derive the exact expression for de-cohering power of unitary operations on single qubit systems. Two different kinds of de-cohering power have also been defined on the set of maximally coherent states and the set of all quantum states respectively. We also compare these two kinds of de-cohering power but find they are not equal in single qubit systems, which is different from the cohering power. Finally, we make a comparison between the cohering power and de-cohering power and find that de-cohering power is always less than the cohering power for unitary operations on single qubit systems.

This work is organized as follows. In Sec.2, we provide the preliminary material in the resource theory of coherence. We investigate two types of cohering power are equal for any quantum operation in single qubit case. And we show that there is no monotonicity for l_1 norm of coherence under MIO or DIO in Sec.3. Besides, we derive the explicit formula for de-cohering power and compare two different types of de-cohering power in Sec.4. Moveover, we compare the cohering power and the de-cohering power in 2-dimensional space in Sec.5. Finally, we conclude in Sec.6.

2 Preliminary and notations

Free states and free operations in the resource theory of coherence (see [13] and [19, 20])– Given a fixed reference basis, say $\{|i\rangle\}$, any state which is diagonal in the reference basis is called an incoherent state. And the set of all incoherent states is denoted by $\mathcal{I}$. Then we introduce several different free operations in the resource theory of coherence from [13,19,20].

- Incoherent operation (IO). A quantum operation Φ is called an incoherent operation if there exists a set of Kraus operators $\{K_n\}$ of Φ such that $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$ for any n.
- Maximally incoherent operation (MIO). A quantum operation Φ is called a maximally incoherent operation if $\Phi(\mathcal{I}) \subset \mathcal{I}$.
- Dephasing-covariant incoherent operation (DIO). A quantum operation Φ is called a Dephasing-covariant incoherent operation if

$$[\Delta, \Phi] = 0, \tag{1}$$

where $\Delta(\rho) := \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$.

l_1 norm and relative entropy measure (see [13])–

(i) l_1 norm measure $\mathcal{C}_{l_1}$ is defined by

$$\mathcal{C}_{l_1}(\rho) := \sum_{i\neq j} |\rho_{ij}|. \tag{2}$$

(ii) Relative entropy measure $\mathcal{C}_r$ is defined by

$$\mathcal{C}_r(\rho) := \mathrm{S}(\rho^{(d)}) - \mathrm{S}(\rho), \tag{3}$$

where $\mathrm{S}(\rho) = -\mathrm{Tr}\rho\log\rho$ is the von Neumann entropy of ρ and $\rho^{(d)}$ is the diagonal state of ρ.

Cohering power– Two types of cohering power (see [26] and [47]):

$$\mathscr{C}_X(\Phi) : = \max_{\rho\in\mathcal{I}} \{\mathcal{C}_X(\Phi(\rho))\}, \tag{4}$$

$$\widehat{\mathscr{C}}_X(\Phi) : = \max_{\rho\in\mathcal{D}(\mathcal{H})} \{\mathcal{C}_X(\Phi(\rho)) - \mathcal{C}_X(\rho)\} \tag{5}$$

where X denotes a coherence measure and $\mathcal{I}$ is the set of incoherent states. To distinguish these two powers, we call $\mathscr{C}$ and $\widehat{\mathscr{C}}$ the cohering power and generalized cohering power, respectively. Obviously, $\mathscr{C}_X(\Phi) \leq \widehat{\mathscr{C}}_X(\Phi)$ for any coherence measure X.
Formula of cohering power for unitary operations (see [47])– It has been shown in [47] that the cohering power for a unitary operation $U = [U_{ij}]_{d\times d}$ can be written as

$$\mathscr{C}_{l_1}(U) = \|U\|^2_{1\to 1} - 1, \tag{6}$$

where $\|U\|_{1\to 1} = \max\left\{\sum_{i=1}^d |U_{ij}| : j = 1,\dots,d\right\}$. And

$$\mathscr{C}_r(U) = \max\{S(|U_{1i}|^2, |U_{2i}|^2, \cdots, |U_{di}|^2), i \in [d]\}, \tag{7}$$

where $S(\{p_i\}) = \sum -p_i \log p_i$.
De-cohering power (see [26])– Two types of decohering power:

$$\mathscr{D}_X(\Phi) : = \max_{\rho\in\mathcal{M}} \{\mathcal{C}_X(\rho) - \mathcal{C}_X(\Phi(\rho))\}, \tag{8}$$

$$\widehat{\mathscr{D}}_X(\Phi) : = \max_{\rho\in\mathcal{D}(\mathcal{H})} \{\mathcal{C}_X(\rho) - \mathcal{C}_X(\Phi(\rho))\}. \tag{9}$$

where X denotes a coherence measure and $\mathcal{M}$ is the set of maximally coherent states. To distinguish them, we call $\mathscr{D}$ and $\widehat{\mathscr{D}}$ the de-cohering power and generalized de-cohering power, respectively. Clearly, $\mathscr{D}_X(\Phi) \leq \widehat{\mathscr{D}}_X(\Phi)$ for any coherence measure X. Note that maximally coherent state must be pure state and can be expressed as $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_k e^{i\theta_k}|k\rangle$ [30] and it will takes the maximal value $\log d$ and $d-1$ in d-dimensional space for the coherence measures defined by relative entropy and l_1 norm, respectively. Thus, (generalized) de-cohering power is non-negative.

3 Results about cohering power

In view of the definitions, cohering power and generalized cohering power are different in essence: one is defined on the set of incoherent states and the other is defined on the set of all quantum states. As can be seen, cohering power is always less than the generalized cohering power. Moreover, it has been proved that for any unitary operation U on a single qubit system, the cohering power and the generalized cohering power coincides, that is, $\mathscr{C}_{l_1}(U) = \widehat{\mathscr{C}_{l_1}}(U)$ [47]. This means the maximal coherence produced by unitary operation over all states can be obtained by considering only the incoherent states which is a smaller set of states. Here, we generalize this statement to any quantum operation Φ on single qubit systems.

Proposition 1 *For any quantum operation Φ on a single qubit system, the cohering power and the generalized cohering power coincides, that is, $\mathscr{C}_{l_1}(\Phi) = \widehat{\mathscr{C}_{l_1}}(\Phi)$.*

Proof: For any quantum operation Φ on a single qubit system, it can be expressed by a set of Kraus operators $\{K_n\}_n$ as

$$\Phi(\cdot) = \sum_n K_n \cdot K_n^\dagger,$$

where $K_n = \begin{bmatrix} K_n^{(1,1)} & K_n^{(1,2)} \\ K_n^{(2,1)} & K_n^{(2,2)} \end{bmatrix}$ and $\sum_n K_n^\dagger K_n = \mathbb{I}$. Any qubit state ρ can be written as $\rho = \frac{\mathbb{I}}{2} + \frac{1}{2}\vec{\sigma}\cdot\vec{r}$, where $\vec{r} = (x, y, z)$ is a unit vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrices. Thus, the l_1 norm of coherence of initial state ρ and final state $\Phi(\rho)$ are specified by

$$\mathcal{C}_{l_1}(\rho) = |x + iy|,$$

and

$$\begin{aligned} \mathcal{C}_{l_1}(\Phi(\rho)) &= \Big|\sum_n [\overline{K_n^{(2,1)}} K_n^{(1,1)}(1+z) + \overline{K_n^{(2,2)}} K_n^{(1,2)}(1-z) \\ &\quad + \overline{K_n^{(2,1)}} K_n^{(1,2)}(x-iy) + \overline{K_n^{(2,2)}} K_n^{(1,1)}(x+iy)]\Big|. \end{aligned}$$

Since the cohering power is only defined on incoherent states, then cohering power of Φ can be written as

$$\mathscr{C}_{l_1}(\Phi) = 2\max\left\{\left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}\right|, \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}\right|\right\}.$$

Since

$$\begin{aligned}
& \mathcal{C}_{l_1}(\Phi(\rho)) \\
\leq \quad & \left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}(1+z)\right| + \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}(1+z)\right| \\
+ \quad & \left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,2)}\right| |x-iy| + \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,1)}\right| |x+iy| \\
\leq \quad & 2\max\left\{\left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}\right|, \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}\right|\right\} \\
+ \quad & \left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,2)}\right| |x-iy| + \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,1)}\right| |x+iy| \\
\leq \quad & 2\max\left\{\left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}\right|, \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}\right|\right\} \\
+ \quad & \sum_n \frac{\sum_{i,j=1}^2 \left|K_n^{(i,j)}\right|^2}{2} |x+iy| \\
= \quad & 2\max\left\{\left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}\right|, \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}\right|\right\} \\
+ \quad & |x+iy|,
\end{aligned}$$

then

$$\begin{aligned}
& \mathcal{C}_{l_1}(\Phi(\rho)) - \mathcal{C}_{l_1}(\rho) \\
\leq \quad & 2\max\left\{\left|\sum_n \overline{K_n^{(2,1)}} K_n^{(1,1)}\right|, \left|\sum_n \overline{K_n^{(2,2)}} K_n^{(1,2)}\right|\right\} \\
\leq \quad & \mathscr{C}_{l_1}(\Phi),
\end{aligned}$$

which implies

$$\widehat{\mathscr{C}_{l_1}}(\Phi) \leq \mathscr{C}_{l_1}(\Phi).$$

Therefore, $\widehat{\mathscr{C}_{l_1}}(\Phi) = \mathscr{C}_{l_1}(\Phi)$ for any quantum operation Φ on qubit system. □

The above proposition is also an evidence that cohering power $\mathscr{C}_{l_1}$ can be used to quantify the ability of a quantum operation to generate coherence even if it is only defined on incoherent states. Besides, this result can be used to demonstrate the monotonicity of l_1 norm of coherence under DIO and MIO in single qubit system directly. However, monotonicity of l_1 norm coherence under DIO and MIO does not hold in higher dimensional space.

Proposition 2 (Non-monotonicity for l_1 norm of coherence under DIO and MIO) *In single qubit system, the l_1 norm of coherence can not increase under DIO and MIO. However, such statement is not true in N-qubit system with $N \geq 2$, that is, there exists a state $\rho_N \in D(\mathbb{C}^{\otimes N})$ and a DIO (or MIO) Φ_N such that $\mathcal{C}_{l_1}(\Phi_N(\rho_N)) > \mathcal{C}_{l_1}(\rho_N)$.*

Proof: Due to the definition of cohering power, it is easy to see that $\mathscr{C}_{l_1}(\Phi) = 0$ is equivalent to $\Phi(\mathcal{I}) \subset \mathcal{I}$, which means that such Φ is a MIO. Due to Proposition 1, we have $\widehat{\mathscr{C}_{l_1}}(\Phi) = 0$ for any MIO Φ on a single qubit system. Thus, the l_1 norm of coherence can not increase under MIO. Since $DIO \subset MIO$, then we also have the monotone of l_1 norm of coherence under DIO in single qubit case.

Next, we show there exists a DIO Φ and a state a state ρ such that $\mathcal{C}_{l_1}(\Phi(\rho)) > \mathcal{C}_{l_1}(\rho)$ in 2-qubit system. Consider the quantum operation Φ with following Kraus operators

$$M_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 & -\frac{\sqrt{6}}{3} \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix},$$

It can be easily verified such operation Φ is a DIO according to [19, 20]. Besides, let us take the state as following

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\rho_{12} = \rho_{21} > 0$. Then, through some calculation, $\mathcal{C}_{l_1}(\Phi(\rho)) = \frac{4}{\sqrt{3}}\rho_{12}$, which is lager than $\mathcal{C}_{l_1}(\rho) = 2\rho_{12}$. Furthermore, for any N qubit system with $N \geq 3$, let us take $\Phi_N = \Phi \otimes \mathbb{I}_{N-2}$ and $\rho_N = \rho \otimes \sigma_{N-2}$ where $\mathbb{I}_{N-2}$ denotes the identity operator on the remaining (N-2)-qubit system and σ_{N-2} is a state of the remaining (N-2)-qubit system with $\mathcal{C}_{l_1}(\sigma_{N-2}) > 0$. It is easily to see that such Φ_N is also a DIO. Thus,

$$\begin{aligned} & \mathcal{C}_{l_1}(\Phi_N(\rho_N)) - \mathcal{C}_{l_1}(\rho_N) \\ = \; & \mathcal{C}_{l_1}(\Phi(\rho)) \otimes \sigma_{N-2}) - \mathcal{C}_{l_1}(\rho \otimes \sigma_{N-2}) \\ = \; & [\mathcal{C}_{l_1}(\Phi(\rho)) - \mathcal{C}_{l_1}(\rho)][\mathcal{C}_{l_1}(\sigma_{N-2}) + 1] \\ > \; & \mathcal{C}_{l_1}(\Phi(\rho)) - \mathcal{C}_{l_1}(\rho) > 0, \end{aligned}$$

where the second equality comes from the multiplicity of l_1 norm of coherence, that is $\mathcal{C}_{l_1}(\tau_1 \otimes \tau_2) + 1 = [\mathcal{C}_{l_1}(\tau_1) + 1][\mathcal{C}_{l_1}(\tau_2) + 1]$ for any two states τ_1 and τ_2. Thus, the l_1 norm of coherence is not monotonous under DIO in N-qubit system with $N \geq 2$. Since DIO is a subset of MIO, it also implies that there is no monotonicity of l_1 norm coherence under MIO.□

Corollary 3 *There exists state transformation $\rho \to \sigma$ by DIO which is not possible by IO.*

Proof: Let us take the states ρ and $\Phi(\rho)$ given in the Proof of Proposition 2, then state transformation $\rho \longrightarrow \sigma = \Phi(\rho)$ is feasible by DIO, but not possible by IO, as $\mathcal{C}_{l_1}(\Phi(\rho)) > \mathcal{C}_{l_1}(\rho)$ and IO can not increase coherence of the states. □

This corollary shows the operational gap between DIO and IO in terms of state transformation which is an open problem proposed in [19, 20]. Besides, it has been shown in [13] that the distance measure, which is contracting under CPTP maps, can be used construct a potential coherence quantifier, e.g the relative entropy of coherence. Here, non-monotonicity of l_1 norm coherence under MIO implies that l_1 norm is not contracting under CPTP maps, but l_1 norm of coherence is still a proper coherence measure. Thus, the contractivity under CPTP maps may not be a necessary condition for distance measure to be a proper coherence measure.

Corollary 4 *l_1 norm is not contracting under CPTP maps, that is, there exists quantum states ρ, σ and CPTP map Φ such that $\|\Phi(\rho)-\Phi(\sigma)\|_{l_1} > \|\rho-\sigma\|_{l_1}$, where $\|\rho\|_{l_1} := \sum_{i,j}|\rho_{ij}|$.*

Proof: If l_1 norm is contracting under CPTP maps, then for any quantum state ρ and any MIO Φ,

$$\begin{aligned}\mathcal{C}_{l_1}(\rho) &= \min_{\sigma\in\mathcal{I}}\|\rho-\sigma\|_{l_1}\\ &\geq \min_{\sigma\in\mathcal{I}}\|\Phi(\rho)-\Phi(\sigma)\|_{l_1}\\ &\geq \min_{\sigma\in\mathcal{I}}\|\Phi(\rho)-\sigma\|_{l_1}\\ &= \mathcal{C}_{l_1}(\Phi(\rho)),\end{aligned}$$

which contradicts with Proposition 2. □

In fact, as the cohering power $\mathcal{C}_{l_1}$ and $\mathcal{C}_r$ are both defined on the set of incoherent states $\mathcal{I}$, it is easy to see that the quantum operations with zero cohering power in l_1 norm of coherence or relative entropy of coherence is MIO, that is $MIO = \{\Phi : \mathscr{C}_{l_1}(\Phi) = 0\} = \{\Phi : \mathscr{C}_r(\Phi) = 0\}$, which means that MIO is the set of all operation that can not increase the coherence of incoherent states. We also consider the quantum operations with zero generalized cohering power as following,

$$NIO_{l_1} = \{\Phi : \widehat{\mathscr{C}_{l_1}}(\Phi) = 0\}, \tag{10}$$

$$NIO_r = \{\Phi : \widehat{\mathscr{C}_r}(\Phi) = 0\}. \tag{11}$$

Note that the set NIO_{l_1}(resp. NIO_r) is the set of all quantum operations that will not increase the coherence of all states in l_1 norm of coherence (resp. relative entropy of coherence). Due to the definition of generalized cohering power, we have $NIO_r \subset MIO$. Since relative entropy of coherence is monotone under MIO [19, 20], then $MIO \subset NIO_r$, which implies that $MIO = NIO_r$. That is, MIO is just the set of all quantum operations that will not increase the coherence of all quantum states in relative entropy measure. Moreover, we get the relationship between IO, MIO, NIO_{l_1} and NIO_r.

Corollary 5 *The relationship between IO, MIO, NIO_{l_1} and NIO_r in N-qubit system ($N \geq 2$) is*

$$IO \subsetneq NIO_{l_1} \subsetneq MIO = NIO_r \tag{12}$$

However, in single qubit system, the relationship will become

$$IO \subsetneq NIO_{l_1} = MIO = NIO_r. \tag{13}$$

Proof: Since $\mathscr{C}_{l_1}(\Phi) = \widehat{\mathscr{C}_{l_1}}(\Phi)$ in single qubit system, then $NIO_{l_1} = MIO$ due to the definition of NIO_{l_1} and the fact $MIO = \{\Phi : \mathscr{C}_{l_1}(\Phi) = 0\}$. Besides, it has been demonstrated that there exists a quantum operation on single qubit system $\Phi \in MIO$ but $\Phi \notin IO$ (see [25] and the Erratum of [20]). Thus $IO \subsetneq NIO_{l_1} = MIO = NIO_r$.

In N-qubit system ($N \geq 2$), $NIO_{l_1} \subsetneq MIO$ comes from Proposition 2. Thus, the relationship between IO, MIO and NIO_{l_1} in N-qubit system ($N \geq 2$) will become $IO \subsetneq NIO_{l_1} \subsetneq MIO = NIO_r$. □

The above proposition tells us that in single qubit system, MIO is also the set of quantum operations that will not increase the coherence of all quantum states in the l_1 norm measure, that is, NIO_{l_1} and NIO_r coincides in this case. The relationship between these sets may help us understand the role of IO and MIO in the resource theory of coherence and be complementary to the previous work [19, 20]. Besides, since the relationship between l_1 norm of coherence and relative entropy coherence has been considered in [32], we also consider the relationship between cohering power defined in l_1 norm $\mathscr{C}_{l_1}$ and that defined in relative entropy $\mathscr{C}_r$ for unitary operations.

Proposition 6 *Given a unitary operation U in d-dimensional space, we have*

$$\mathscr{C}_{l_1}(U) \geq \max\{\mathscr{C}_r(U), 2^{\mathscr{C}_r(U)} - 1\}. \tag{14}$$

Proof: Since l_1 norm coherence and relative entropy coherence in pure states has the the following relationship $\mathcal{C}_{l_1}(|\psi\rangle) \geq \max\{\mathcal{C}_r(|\psi\rangle), 2^{\mathcal{C}_r(|\psi\rangle)} - 1\}$ [32], it is easy to see the cohering power $\mathscr{C}_{l_1}(U) = \max\{\mathcal{C}_{l_1}(U|i\rangle) : i = 1, ..., d\}$ and $\mathscr{C}_r(U) = \max\{\mathcal{C}_r(U|i\rangle) : i = 1, ..., d\}$ also satisfy this relationship, that is,

$$\mathscr{C}_{l_1}(U) \geq \max\{\mathscr{C}_r(U), 2^{\mathscr{C}_r(U)} - 1\}.$$

□

However, whether the cohering power of any quantum operation Φ satisfy (14) is still a question, which is closely related to the open problem: the potential relationship between l_1 norm of coherence $\mathcal{C}_{l_1}$ and relative entropy of coherence $\mathcal{C}_r$ [32].

4 Results about de-cohering power

As mentioned before, de-cohering power and generalized de-cohering power are defined by the maximization over the set of maximally coherent states and all quantum states respectively. As both sets contain too many states, it is difficult to calculate the exact value of de-cohering power and generalized de-cohering power of a given quantum operation. Here, we consider a simple case and give the exact formula of de-cohering power and generalized de-cohering power for unitary operations in single qubit case, which makes the comparison between de-cohering power and generalized de-cohering power possible.

Proposition 7 *For a qubit unitary operation U, which can be expressed as (up to a phase factor) $U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ where $|a|^2 + |b|^2 = 1$, the de-cohering power in l_1 norm of coherence* *and relative entropy of coherence can be expressed as*

$$\mathscr{D}_{l_1}(U) = 1 - ||a|^2 - |b|^2| \tag{15}$$

$$\mathscr{D}_r(U) = 1 - S(\frac{1}{2} + |ab|, \frac{1}{2} - |ab|) \tag{16}$$

And the generalized de-cohering power of U *is equal to the generalized cohering power of* $U^\dagger$*, that is*

$$\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U^\dagger) \tag{17}$$

$$\widehat{\mathscr{D}}_r(U) = \widehat{\mathscr{C}}_r(U^\dagger) \tag{18}$$

Proof: In single qubit system, the maximal coherent state can be written as $|\psi\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{i\theta})^t$, where t denotes transposition. Then $U|\psi\rangle$ would be $\frac{1}{\sqrt{2}}(a + be^{i\theta}, -b^* + a^* e^{i\theta})^t$. Thus

$$\begin{aligned}\mathscr{D}_{l_1}(U) &= 1 - \min_\theta |(a + be^{i\theta})(-b^* + a^* e^{i\theta})| \\ &= 1 - ||a|^2 - |b|^2|.\end{aligned}$$

Denote $a = |a|e^{i\theta}$ and $b = |b|e^{i\theta_b}$, then

$$\begin{aligned}\mathscr{D}_r(U) &= 1 - \min_\vartheta S(\frac{1}{2} + |ab|\cos\vartheta, \frac{1}{2} - |ab|\cos\vartheta) \\ &= 1 - S(\frac{1}{2} + |ab|, \frac{1}{2} - |ab|),\end{aligned}$$

where $\vartheta = \theta + \theta_b - \theta_a$.

Besides, in view of the definition of generalized de-cohering power

$$\begin{aligned}\widehat{\mathscr{D}}_{l_1}(U) &= \max_{\rho\in\mathcal{D}(\mathcal{H})} \{\mathcal{C}_{l_1}(\rho) - \mathcal{C}_{l_1}(U\rho U^\dagger)\} \\ &= \max_{\rho\in\mathcal{D}(\mathcal{H})} \{\mathcal{C}_{l_1}(U^\dagger(U\rho U^\dagger)U) - \mathcal{C}_{l_1}(U\rho U^\dagger)\} \\ &= \max_{U\rho U^\dagger\in\mathcal{D}(\mathcal{H})} \{\mathcal{C}_{l_1}(U^\dagger(U\rho U^\dagger)U) - \mathcal{C}_{l_1}(U\rho U^\dagger)\} \\ &= \widehat{\mathscr{C}}_{l_1}(U^\dagger).\end{aligned}$$

And $\widehat{\mathscr{D}}_r(U) = \widehat{\mathscr{C}}_r(U^\dagger)$ can be obtained in a similar way. □

As can be seen from (17) and (18), the amount of coherence produced by a unitary operation U is equal to that of coherence erased by $U^\dagger$ (the reverse process of U). Besides, the exact formula of de-cohering power in single qubit system makes the comparison between de-cohering power and generalized cohering power possible. According to (15) and (16), the de-cohering power and generalized cohering power of unitary operation on single qubit system are not equal in general, which is different from the relationship between cohering power and generalized cohering power.

Proposition 8 *For any unitary operations* U *on a single qubit system,* $\mathscr{D}_{l_1}(U)$ *and* $\widehat{\mathscr{D}}_{l_1}(U)$ *are not equal in general, that is, there exist a unitary operation* U_0 *such that* $\mathscr{D}_{l_1}(U_0) < \widehat{\mathscr{D}}_{l_1}(U_0)$*.*

Proof: In single qubit system, $\mathscr{D}_{l_1}(U) = 1 - ||a|^2 - |b|^2|$ and $\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U^\dagger) = \mathscr{C}_{l_1}(U^\dagger) = 2|ab|$ where $\widehat{\mathscr{C}}_{l_1}(U^\dagger) = \mathscr{C}_{l_1}(U^\dagger)$ comes from the fact that cohering power coincides with generalized cohering power in single qubit case [47]. Thus it is easy to take an unitary U_0 such that $\mathscr{D}_{l_1}(U_0) < \widehat{\mathscr{D}}_{l_1}(U_0)$. □

Proposition 9 *For unitary operations U on single qubit system, $\mathscr{D}_r(U)$ and $\widehat{\mathscr{D}}_r(U)$ are not equal, that is, there exist a unitary operation U_0 such that $\mathscr{D}_r(U_0) < \widehat{\mathscr{D}}_r(U_0)$.*

Proof: Since the generalized de-cohering power need to take maximization over all quantum states, it is difficult to get exact value of $\widehat{\mathscr{D}}_r$. Thus, a lower bound of the generalized decohering power is expected instead of the exact value. Consider the following unitary operation and quantum state,

$$U_0 = \begin{pmatrix} 0.5645 + 0.6351i & 0.4141 + 0.3264i \\ -0.1452 + 0.5069i & -0.0868 - 0.8452i \end{pmatrix},$$

$$\rho_0 = \begin{pmatrix} 0.7063 & 0.4338 - 0.1360i \\ 0.4338 + 0.1360i & 0.2937 \end{pmatrix},$$

then $\mathscr{D}_r(U_0) \approx 0.7053$ is strictly less than $[\mathcal{C}_r(\rho_0) - \mathcal{C}_r(U_1\rho_1U_1^\dagger)] \approx 0.8327$. As $\widehat{\mathscr{D}}_r(U_0) \geq [\mathcal{C}_r(\rho) - \mathcal{C}_r(U_0\rho U_0^\dagger)]$, then we prove the result. □

In view of the definition of $\widehat{\mathscr{D}}$, $\widehat{\mathscr{D}}(\Phi) = 0$ implies that $\mathcal{C}(\rho) \leq \mathcal{C}(\Phi(\rho))$ for any quantum state, that is, quantum operation will not decrease coherence of any input state. Here, we investigate the set of quantum operations with zero generalized de-cohering power,

$$NDO_{l_1} = \{\Phi : \widehat{\mathscr{D}}_{l_1}(\Phi) = 0\}, \quad (19)$$
$$NDO_r = \{\Phi : \widehat{\mathscr{D}}_r(\Phi) = 0\}. \quad (20)$$

Note that the set NDO_{l_1}(resp. NDO_r) is the set of all quantum operations that will not decrease the coherence of any state in l_1 norm of coherence (resp. relative entropy of coherence). It is easy to give some quantum operations that belongs to NDO_{l_1} (or NDO_r), for example, take the quantum operation Φ with Kraus operators $\{K_i\}_i$, where $K_i = |\Psi\rangle\langle i|$ and $|\Psi\rangle$ is a maximally coherent state, then Φ maps any quantum state to maximally coherent $|\Psi\rangle$. It seems that there is no close relation between NDO_{l_1} (or NDO_r) and IO, MIO, as there exists coherence breaking operations [40] map any quantum state to incoherent states.

5 Comparison between cohering power and decohering power

It has been proved that the cohering power of qubit unitary operations is equal to de-cohering power in the skew information coherence [26]. Here, we consider the relationship between cohering power and de-cohering power for the unitary operations defined by l_1 norm and relative entropy respectively.

Proposition 10 *For any unitary operation U on a single qubit system, the cohering power is always larger than de-cohering power in l_1 norm, that is $\mathscr{C}_{l_1}(U) \geq \mathscr{D}_{l_1}(U)$. However, this relationship does not hold for unitary operations in higher-dimensional space.*

Proof: Since U can be written as $U = e^{i\varphi}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ with $|a|^2+|b|^2=1$, the cohering power of U is $\mathscr{C}_{l_1}(U) = 2|ab|$. And by the definition of the de-cohering power, we have

$$\mathscr{D}_{l_1}(U) = 1 - ||a|^2 - |b|^2| \leq 2|ab| = \mathscr{C}_{l_1}(U) \tag{21}$$

Take U on d-dimensional system with $d \geq 3$ as following

$$U = \frac{\sqrt{2}}{2}(|1\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 2| - |2\rangle\langle 1|) + \sum_{k>2}^{d} |k\rangle\langle k|,$$

then $\mathscr{C}_{l_1}(U) = 1$ and for maximally coherent state $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_k e^{i\theta_k}|k\rangle$, $U|\psi\rangle = \frac{1}{\sqrt{2d}}(e^{i\theta_1} + e^{i\theta_2})|1\rangle + \frac{1}{\sqrt{2d}}(e^{i\theta_1} - e^{i\theta_2})|2\rangle + \frac{1}{\sqrt{d}}\sum_{k>2}^{d} e^{i\theta_k}|k\rangle$, which implies that

$$\begin{aligned} \mathscr{D}_{l_1}(U) &= d - 1 - \min_{|\psi\rangle\in\mathcal{M}} \mathcal{C}_{l_1}(U|\psi\rangle) \\ &= (2-\sqrt{2})(2 - \frac{2-\sqrt{2}}{d}). \end{aligned}$$

Moreover, $\mathscr{D}_{l_1}(U)$ is larger than $(2-\sqrt{2})(2-\frac{2-\sqrt{2}}{3})$ when $d \geq 3$. It is easy to check that $(2-\sqrt{2})(2-\frac{2-\sqrt{2}}{3})$ is strictly larger than 1. Thus, we have $\mathscr{C}_{l_1}(U) < \mathscr{D}_{l_1}(U)$. □

Corollary 11 *For any unitary operation U on a single qubit system, we have the following relationship*

$$\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U) = \mathscr{C}_{l_1}(U) \geq \mathscr{D}_{l_1}(U) \tag{22}$$

Proof: To prove (22), we only need to prove $\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U)$. Since U can be written as $U = e^{i\varphi}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ with $|a|^2+|b|^2=1$, the cohering power of U is $\mathscr{C}_{l_1}(U) = 2|ab| = \mathscr{C}_{l_1}(U^\dagger)$. As we have proved that $\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U^\dagger)$ in Proposition 7 and $\widehat{\mathscr{C}}_{l_1}(U) = \mathscr{C}_{l_1}(U)$ [47], we have $\widehat{\mathscr{D}}_{l_1}(U) = \widehat{\mathscr{C}}_{l_1}(U^\dagger) = \mathscr{C}_{l_1}(U^\dagger) = \mathscr{C}_{l_1}(U)$. □

Proposition 12 *For any unitary operation U on a single qubit system, the cohering power is always larger than de-cohering power in relative entropy coherence, that is $\mathscr{C}_r(U) \geq \mathscr{D}_r(U)$.*

Proof: Since U can be written as $U = e^{i\varphi}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ with $|a|^2+|b|^2=1$, the cohering power of $\mathscr{C}_r(U) = S(|a|^2, |b|^2)$. And the de-cohering power of U is $\mathscr{D}_r(U) = 1 - S(\frac{1}{2}+|ab|, \frac{1}{2}+|ab|)$. Thus, $\mathscr{C}_r(U) \geq \mathscr{D}_r(U)$ is equivalent to $S(|a|^2,|b|^2) + S(\frac{1}{2}+|ab|, \frac{1}{2}-|ab|) \geq 1$. Due to Lemma 13 in Appendix, we get the result. □

Although we have proved that $\mathscr{C}_r(U) \geq \mathscr{D}_r(U)$ and $\widehat{\mathscr{D}}_r(U) = \widehat{\mathscr{C}}_r(U^\dagger)$, we cannot get the similar result like (22) as cohering power $\mathscr{C}_r(U)$ and $\widehat{\mathscr{C}}_r(U)$ are not equal even in single qubit case [47]. Besides, as the explicit formula for de-cohering power $\mathscr{D}_r$ in higher dimensions is still unknown even for unitary operations, the relationship between $\mathscr{D}_r$ and $\mathscr{C}_r$ remains to be identified.

6 Conclusion

In this work, we have investigated the cohering power and de-cohering power which are defined to quantify the ability of quantum operations to produce coherence and erase coherence respectively. It has been proved that cohering power $\mathscr{C}_{l_1}$ and generalized cohering power $\widehat{\mathscr{C}}_{l_1}$ are equal for single qubit unitary operations [47, 48]. In this work, we prove that this statement is also true for any quantum operation on single qubit systems, which implies the monotonicity of l_1 norm of coherence under MIO on single qubit systems. However, we show that l_1 norm of coherence is not monotone under DIO or MIO in higher dimensional space. Thus we give a complete answer to the open problem about the monotonicity of l_1 norm of coherence under MIO proposed in [19, 20]. And the non-monotonicity of l_1 norm coherence implies that l_1 norm is not contracting under CPTP maps. Thus contractivity under CPTP maps may not be a necessary property for norms to be coherence measures. Besides, we investigate the connections between the sets of operations with zero generalized cohering power NIO_{l_1} and NIO_r with IO and MIO: $IO \subsetneq NIO_{l_1} = MIO = NIO_r$ in single qubit case and $IO \subsetneq NIO_{l_1} \subsetneq MIO = NIO_r$ in higher dimensions; MIO is just the set of all quantum operations that will not increase the coherence of all states in relative entropy measure. Moreover, we derive the exact formula of de-cohering power of single unitary operations. By a comparison between de-cohering power and generalized de-cohering power, we have shown that they are not equal in general which is different from the coincidence between cohering power and generalized cohering power in single qubit systems. Furthermore, we compare cohering power and de-cohering power defined in l_1 norm and relative entropy, and find that cohering power is usually larger than de-cohering power for unitary operations on single qubit systems.

The results in this work present a new approach to study the free operations in the resource of coherence by cohering power and therefore, are of great value to our understanding of IO, MIO and DIO proposed in [13, 19, 20]. However, more work is needed in this context. For example, it will be useful to obtain the relationship between cohering power $\mathscr{C}_{l_1}$ and $\mathscr{C}_r$ (or de-cohering power $\mathscr{D}_{l_1}$ and $\mathscr{D}_r$) for any quantum operation. Another important question for future studies is to determine the relationship between cohering power and de-cohering power for any quantum operations on higher dimensions.

Acknowledgements

This work is supported by the Natural Science Foundations of China (Grants No.11171301 and No. 10771191) and the Doctoral Programs Foundation of the Ministry of Education of China (Grant No. J20130061).

References

1. M. Horodecki and J. Oppenheim (2013), *(QUANTUMNESS IN THE CONTEXT OF) RESOURCE THEORIES*, International Journal of Modern Physics B 27, 1345019 .
2. F. G. S. L. Brandão and G. Gour (2015), *The general structure of quantum resource theories*, Phys. Rev. Lett. 115, 070503.
3. R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki (2009), *Quantum entanglement*, Rev. Mod. Phys. 81, 865.

4. C. H. Bennett and S. J. Wiesner (1992),*Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states*, Phys. Rev. Lett. 69, 2881.
5. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters (1993), *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*. Phys. Rev. Lett. 70, 1895.
6. M. Lostaglio, D. Jennings, and T. Rudolph (2015), *Description of quantum coherence in thermodynamic processes requires constraints beyond free energy*, Nat. Commun. 6, 6383 .
7. M. Lostaglio, D. Jennings, and T. Rudolph (2015), *Thermodynamic resource theories, non-commutativity and maximum entropy principles*, quant-ph/1511.04420.
8. F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S.Wehner (2015), *The second laws of quantum thermodynamics*, Proc. Natl. Acad. Sci. U.S.A. 112, 3275.
9. V. Narasimhachar and G. Gour (2015), *Low-temperature thermodynamics with quantum coherence*, Nat. Commun. 6, 7689.
10. M. B. Plenio and S. F. Huelga (2008), *Dephasing-assisted transport: quantum networks and biomolecules*, New J. Phys. 10, 113019.
11. S. Lloyd (2011), *Quantum coherence in biological systems*, J. Phys.: Conf. Ser. 302, 012037.
12. F. Levi and F. Mintert (2014), *A quantitative theory of coherent delocalization*, New J. Phys. 16, 033007.
13. T. Baumgratz, M. Cramer, and M. B. Plenio (2014), *Quantifying Coherence*, Phys. Rev. Lett. 113, 140401.
14. B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral (2016), *Quantum processes which do not use coherence*, Phys. Rev. X 6, 041028.
15. J. Åberg (2014), *Catalytic coherence*, Phys. Rev. Lett. 113, 150402.
16. Z. Bai and S. Du (2015), *Maximally coherent states*, Quantum Inf. Comput., Vol. 15, pp. 1355-1364.
17. T. R. Bromley, M. Cianciaruso, and G. Adesso (2015), *Frozen Quantum Coherence*, Phys. Rev. Lett. 114, 210401.
18. S. Cheng and M. J. W. Hall (2015), *Complementarity relations for quantum coherence*, Phys. Rev. A 92, 042101 .
19. E. Chitambar and G. Gour (2016), *Critical Examination of Incoherent Operations and a Physically Consistent Resource Theory of Quantum Coherence*,Phys. Rev. Lett. 117, 030401 .
20. E. Chitambar and G. Gour (2016), *Comparison of incoherent operations and measures of coherence*, Phys. Rev. A 94, 052336 .
21. E. Chitambar and M.-H. Hsieh (2016), *Relating the Resource Theories of Entanglement and Quantum Coherence*, Phys. Rev. Lett. 117, 020402.
22. E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein (2016), *Assisted Distillation of Quantum Coherence*, Phys. Rev. Lett. 116, 070402.
23. S. Du, Z. Bai, and Y. Guo (2015), *Conditions for coherence transformations under incoherent operations*, Phys. Rev. A 91, 052120 .
24. D. Girolami (2014), *Observable Measure of Quantum Coherence in Finite Dimensional Systems*, Phys. Rev. Lett. 113, 170401.
25. X. Hu (2016), *Channels that do not generate coherence*, Phys. Rev. A 94, 012326.
26. A. Mani and V. Karimipour (2015), *Cohering and decohering power of quantum channels*, Phys. Rev. A 92, 032331.
27. I. Marvian and R. W. Spekkens (2014), *Modes of asymmetry: The application of harmonic analysis to symmetric quantum dynamics and quantum reference frames*, Phys. Rev. A 90, 062110.
28. D. Mondal, C. Datta, and S. Sazim (2016), *Quantum coherence sets the quantum speed limit for mixed states* , Phys. Lett. A 380, 689.
29. C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso (2016), *Robustness of Coherence: An Operational and Observable Measure of Quantum Coherence*, Phys. Rev. Lett. 116, 150502 .
30. Y. Peng, Y. Jiang, and H. Fan (2016), *Maximally coherent states and coherence-preserving operations*, Phys. Rev. A 93, 032326.

31. M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso (2016), *Robustness of asymmetry and coherence of quantum states*, Phys. Rev. A 93, 042107.
32. S. Rana, P. Parashar, and M. Lewenstein (2016), *Trace-distance measure of coherence*, Phys. Rev. A 93, 012110.
33. L.-H. Shao, Z. Xi, H. Fan, and Y. Li (2015), *Fidelity and trace-norm distances for quantifying coherence*, Phys. Rev. A 91, 042120
34. U. Singh, M. N. Bera, H. S. Dhar, and A. K. Pati (2015), *Maximally coherent mixed states: Complementarity between maximal coherence and mixedness*, Phys. Rev. A 91, 052115.
35. U. Singh, M. N. Bera, A. Misra, and A. K. Pati (2015), *Erasing Quantum Coherence: An Operational Approach* , quant-ph/1506.08186.
36. U. Singh, L. Zhang, and A. K. Pati (2016), *Average coherence and its typicality for random pure states*, Phys. Rev. A 93, 032125.
37. U. Singh, A. K. Pati, and M. N. Bera (2016),*Uncertainty Relations for Quantum Coherence*, Mathematics 4, 47.
38. L. Zhang, U. Singh, and A. K. Pati (2017), *Average subentropy, coherence and entanglement of random mixed quantum states*, Annals of Physics 377, 125-146.
39. K. Bu, U. Singh, and J. Wu (2016),*Catalytic coherence transformations*, Phys. Rev. A 93, 042326.
40. K. Bu, Swati, U. Singh, and J. Wu (2016), *Coherence-breaking channels and coherence sudden death*, Phys. Rev. A 94, 052335
41. J. Vicente and A. Streltsov (2017), *Genuine quantum coherence*, J. Phys. A 50, 045301.
42. A. Streltsov, S. Rana, M. Nath Bera, and M. Lewenstein (2017), *Towards resource theory of coherence in distributed scenarios* , Phys. Rev. X 7, 011024.
43. Z. Xi, Y. Li, and H. Fan (2015), *Quantum coherence and correlations in quantum system*, Sci. Rep. 5, 10922.
44. Y. Yao, X. Xiao, L. Ge, and C. P. Sun (2015), *Quantum coherence in multipartite systems*, Phys. Rev. A 92, 022112.
45. A. Winter and D. Yang (2016), *Operational resource theory of coherence*, Phys. Rev. Lett. 116, 120404 .
46. P. Zanardi, C. Zalka, and L. Faoro (2000), *Entangling power of quantum evolutions*, Phys. Rev. A 62, 030301.
47. K. Bu, A. Kumar, L. Zhang, and J. Wu (2017),*Cohering power of quantum operations*, Physics Letters A, 381, 1670-1676.
48. M. García-Díaz, D. Egloff, and M. Plenio (2016), *A note on coherence power of N-dimensional unitary operators*, Quant. Inf. Comp. 16, 1282 - 1294.

Appendix A

Lemma 13 *The function* $H(x) := -x\log_2 x - (1-x)\log_2(1-x)$ *with* $x \in [0,1]$ *satisfy*

$$H(x) + H(\frac{1}{2} + \sqrt{x(1-x)}) \geq 1, \tag{A.1}$$

for any $x \in [0,1]$.

Proof: To prove this inequality is equal to prove

$$-x\ln x - (1-x)\ln(1-x) - t\ln t - (1-t)\ln(1-t) \geq \ln 2$$

with $t = \frac{1}{2} + \sqrt{x(1-x)}$. Since the symmetry of the formula, we only need to consider the the case $x \in [0.5, 1]$. As variables x, t satisfy $(x-1/2)^2 + (t-1/2)^2 = 1/2$, we change the variables x, t to $x = \frac{1+\cos\theta}{2}$ and $t = \frac{1+\sin\theta}{2}$ with $\theta \in [0, \pi/2]$. Then we prove the following the inequality:

$$\begin{aligned} f(\theta) = \quad & -(\frac{1+\cos\theta}{2})\ln(\frac{1+\cos\theta}{2}) - (\frac{1-\cos\theta}{2})\ln(\frac{1-\cos\theta}{2}) \\ & -(\frac{1-\sin\theta}{2})\ln(\frac{1-\sin\theta}{2}) - (\frac{1+\sin\theta}{2})\ln(\frac{1+\sin\theta}{2}) \\ & -\ln 2 \geq 0 \end{aligned}$$

with $\theta \in [0, \pi/2]$. Differentiate $f(\theta)$ with respect to θ, then

$$\begin{aligned} \frac{df}{d\theta} &= \frac{1}{2}\left[\sin\theta \ln\frac{1+\cos\theta}{1-\cos\theta} - \cos\theta \ln\frac{1+\sin\theta}{1-\sin\theta}\right] \\ &= \frac{1}{2\sin\theta\cos\theta}\left[\frac{1}{\cos\theta}\ln\frac{1+\cos\theta}{1-\cos\theta} - \frac{1}{\sin\theta}\ln\frac{1+\sin\theta}{1-\sin\theta}\right]. \end{aligned}$$

Consider the function $g(s) = \frac{1}{s}\ln\frac{1+s}{1-s}$ with $s \in [0, 1]$. Then $\frac{dg}{ds} = \frac{1}{s^2}[\ln(1-s) + \frac{1}{1-s} - (\ln(1+s) + \frac{1}{1+s})] > 0$, that is, $g(s)$ is a monotonous function. Thus

(1) when $\theta \in [0, \pi/4]$, then $\cos\theta \geq \sin\theta$. As the function $g(s)$ is monotonous, thus $\frac{df}{d\theta} \geq 0$.

(2) when $\theta \in [\pi/4, \pi/2]$, then $\cos\theta \leq \sin\theta$. As the function $g(s)$ is monotonous, thus $\frac{df}{d\theta} \leq 0$.

Therefore, $\min_{\theta \in [0,\pi/2]} f(\theta) = \min\{f(0), f(\pi/2)\} = 0$. □