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# RENORMALIZATION OF QUANTUM DEFICIT AND MONOGAMY RELATION IN THE HEISENBERG XXZ MODEL

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In this study, the dynamical behavior of quantum deficit and monogamy relation in the Heisenberg XXZ model is investigated by implementing quantum renormalization group theory. The results demonstrate that the quantum deficit can be used to capture the quantum phase transitions point and show scaling behavior with the spin chain size increasing. It was also found that the critical exponent has no change when varying measure from entanglement to quantum correlation. The monogamy relation is influenced by the steps of quantum renormalization group and the ways of splitting the block states. Furthermore, the monogamy relation of generalized $W$ state also is given by means of quantum deficit.

*Keywords*: renormalization; quantum correlation; monogamy relation
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## 1 Introduction

Entanglement is regarded as the most important resource in the past years since it can be used to perform quantum teleportation, quantum key distribution, and quantum computation. Because of its crucial role in quantum information processing, many efforts have been devoted to study the properties of entangled state [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One of the meaningful results in these studies is called monogamy of entanglement, i.e., quantum entanglement cannot be freely shared among the parties [13]. The monogamy of entanglement

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plays a key role in condensed matter physics and it is also extremely significant in making quantum cryptography secure. There are many investigations [14, 15, 16] on this theme after the seminal work of Coffman et al. [13]. However, some recent results suggest that the concept of quantum correlation are more general than entanglement that contains nonlocal correlations [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. The investigation on monogamy relation using quantum discord or other quantum correlation measures implies that quantum correlation does not necessarily obey the monogamy relation [29, 30]. These investigations also have significant effects on some condensed matter physics areas, for example, quantum phase transitions (QPTs).

QPTs are one of the most important subjects in many-body physics. The ground state properties of a system will undergo an abrupt change around the critical point when the external parameter is adjusted slightly. It occurs at absolute zero where the quantum fluctuations play a dominant role [31, 32]. Mean field theory (MFT) is the traditional way to investigate QPTs. But people have found that the theoretical results based on MFT are not in agreement with the experiment because MFT ignores the effect of fluctuation. In 1971, Wilson successfully introduced the renormalization group idea into quantum statistical physics and established the renormalization group theory of critical phenomena [33]. Investigating the relations between entanglement and QPTs in solid state system is one of the important applications of quantum renormalization group (QRG) theory. Surveys regarding Ising models and Heisenberg models have been done in previous studies [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. The results have shown that the implementation of QRG theory is valuable in detecting the nonanalytic behavior of entanglement and the scaling behavior in the vicinity of critical point.

As is mentioned before, entanglement cannot capture all the quantum correlation in a system. So, it is meaningful to study the relations between quantum correlation and QPTs by means of QRG theory. Some quantum correlation measures like quantum discord, however, are somewhat difficult to calculate the analytic result. Rajagopal and Rendell [46] have proposed the quantum deficit to quantify the quantum correlation in a system. The quantum deficit uses a decohered density matrix which maintains the same information contained in the marginal states [46]. This measure is symmetric about the subsystems and it is most readily suited to analytic investigation and does not need any optimal measurement schemes. Here, we will use the quantum deficit to investigate the QPTs and the monogamy relation in the Heisenberg XXZ model by implementing QRG theory. The XXZ model is well known model, it is worthwhile to investigate the QPTs properties of such model by the quantum correlation measures combing QRG theory. Especially we want to get the effects of QRG on the monogamy relation.

The paper is organized as follows. In Section 2, we will introduce the concept of quantum deficit. In Section 3, we briefly review the Heisenberg XXZ model and give the analytical results of this model using quantum deficit. The scaling behavior of this model also is shown in this part. In Section 4, we will investigate the monogamy relation behavior of this model. In Section 5, we extend our result to the generalized $W$ state. Finally a summary is given in section 6.

## 2 Quantum Deficit

For any bipartite state, the quantum deficit is defined as the relative entropy of the state $\rho_{AB}$ with its classically decohered counterpart $\rho^d_{AB}$ as below [28, 30, 46]:

$$D_{AB} = S(\rho_{AB} \| \rho^d_{AB}) = Tr(\rho_{AB} \ln \rho_{AB} - Tr(\rho_{AB} \ln \rho^d_{AB}). \tag{1}$$

The quantum deficit $D_{AB}$ determines the quantum excess of correlations in the state $\rho_{AB}$, with reference to its classical counterpart $\rho^d_{AB}$. The classical state $\rho^d_{AB}$ has the same marginal states $\rho_A$, $\rho_B$ as those of $\rho_{AB}$. It is diagonal in the eigenbasis $\{|a\rangle, |b\rangle\}$ of $\rho_A$, $\rho_B$ and the expression is

$$\rho^d_{AB} = \sum_{ab} P_{ab} |a\rangle\langle a| \bigotimes |b\rangle\langle b|, \tag{2}$$

where $P_{ab} = \langle a, b|\rho_{AB}|a, b\rangle$ stand for the diagonal terms of $\rho_{AB}$, and $\sum_{ab} P_{ab} = 1$.

So, it is easy to see that $Tr(\rho_{AB} \ln \rho^d_{AB}) = \sum_{ab} P_{ab} \ln P_{ab}$, which leads to

$$D_{AB} = Tr(\rho_{AB} \ln \rho_{AB} - Tr(\rho_{AB} \ln \rho^d_{AB}) = \sum_i \lambda_i \ln \lambda_i - \sum_{ab} P_{ab} \ln P_{ab}. \tag{3}$$

where $\lambda_i$ denote the eigenvalues of the state $\rho_{AB}$.

## 3 The Model Hamiltonian and the Solutions

The Hamiltonian of the Heisenberg XXZ model on a periodic chain of $N$ sites is given by [35]

$$H = \frac{J}{4} \sum_{i=1}^{N} (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \Delta \sigma^z_i \sigma^z_{i+1}), \tag{4}$$

where $J$ is the exchange interaction, $\Delta$ is the anisotropy parameter, and $\sigma^\tau_i$ are standard Pauli matrices at site $i$.

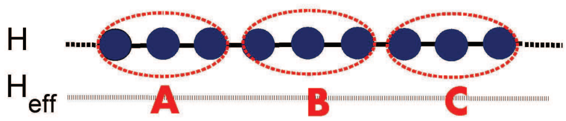


Fig. 1. A schematic description of QRG for three sites in a block.

To get a self-similar Hamiltonian after each QRG step, we can divide the spin chain into three-site blocks, see figure 1. The corresponding block Hamiltonian has two degenerate ground states as follows,

$$|\psi_0\rangle = \frac{1}{\sqrt{2+q^2}}(|110\rangle + q|101\rangle + |011\rangle), \\ |\psi_0^{'}\rangle = \frac{1}{\sqrt{2+q^2}}(|100\rangle + q|010\rangle + |001\rangle), \tag{5}$$

where

$$q = -\frac{\Delta + \sqrt{\Delta^2 + 8}}{2}. \tag{6}$$

The effective Hamiltonian of the renormalized chain is again a XXZ chain with the scaled couplings

$$H = \frac{J'}{4}\sum_{i=1}^{N/3}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta'\sigma_i^z\sigma_{i+1}^z), \tag{7}$$

here the renormalized couplings are

$$J' = J(\frac{2q}{2+q^2})^2, \\ \Delta' = \frac{\Delta q^2}{4}. \tag{8}$$

The density matrix of the ground state is defined as

$$\rho_{ABC} = |\psi_0\rangle\langle\psi_0|, \\ \rho_{ABC} = |\psi_0'\rangle\langle\psi_0'|. \tag{9}$$

Now, we first investigate the pairwise dynamical behavior of reduced density matrices $\rho_{AB}$ and $\rho_{AC}$ by tracing over site $C$ or $B$.

$$\rho_{AB} = \frac{1}{2+q^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2 & q & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

$$\rho_{AC} = \frac{1}{2+q^2}\begin{bmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

The diagonal elements of $\rho_{AB}^d$ and $\rho_{AC}^d$ are given by

$$\rho_{AB}^d = \langle\chi_i\chi_i|\rho_{AB}|\chi_i\chi_i\rangle = \frac{1}{2+q^2}diag\{1, q^2, 1, 0\}, \tag{12}$$

$$\rho_{AC}^d = \langle\chi_i\chi_i|\rho_{AC}|\chi_i\chi_i\rangle = \frac{1}{2+q^2}diag\{q^2, 1, 1, 0\}, \tag{13}$$

where $|\chi_i\rangle = |0\rangle, |1\rangle$, $i = 1, 2$ is the eigenstate of $\rho_A = \frac{1}{2+q^2}diag\{q^2+1, 1\}$ . The eigenvalues of $\rho_{AB}$ are $\frac{q^2+1}{2+q^2}$ and $\frac{1}{2+q^2}$. The eigenvalues of $\rho_{AC}$ are $\frac{q^2}{2+q^2}$ and $\frac{2}{2+q^2}$ . Therefore, we can derive the quantum deficit of $\rho_{AB}$ and $\rho_{AC}$ as

$$D_{AB} = \frac{1}{2+q^2}(q^2\ln\frac{q^2+1}{q^2} + \ln(q^2+1)), \tag{14}$$

$$D_{AC} = \frac{2}{2+q^2}\ln 2. \tag{15}$$

### *3.1 Dynamical properties of renormalized quantum deficit*

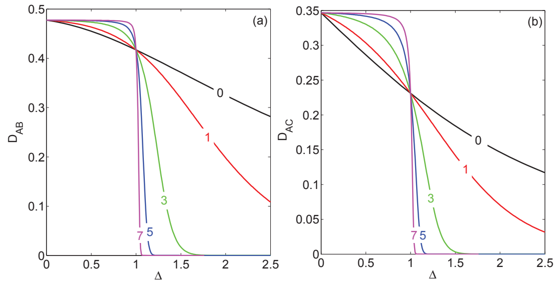


Fig. 2. The quantum deficit $D_{AB}$ (a) and $D_{AC}$ (b) of the model versus $\Delta$ at different QRG steps.

We have plotted $D_{AB}$ (left) and $D_{AC}$ (right) versus $\Delta$ in figure 2 at different QRG steps. After enough steps of renormalization, $D_{AB}$ and $D_{AC}$ will develop two saturated values, $D_{AB}$=0.4774 for $0 \leq \Delta < 1$ and 0 for $\Delta > 1$, while $D_{AC}$ =0.3466 for $0 \leq \Delta < 1$ and 0 for $\Delta > 1$ . In the region of $0 \leq \Delta < 1$ the model represents spin-fluid phase and for $\Delta > 1$ the model stands for Neel phase [35]. So, the sudden change of quantum deficit at $\Delta_c = 1$ reflect that the QPTs occur at this point. The cross point coordinates for $D_{AB}$ and $D_{AC}$ are (1, 0.417) and (1, 0.231) respectively. Moreover, the saturated value of $D_{AB}$ is larger than that of $D_{AC}$ for $0 \leq \Delta < 1$. This means that the quantum correlation of asymmetric $\rho_{AB}$ is larger than that of symmetric $\rho_{AC}$.

### *3.2 The scaling behavior*

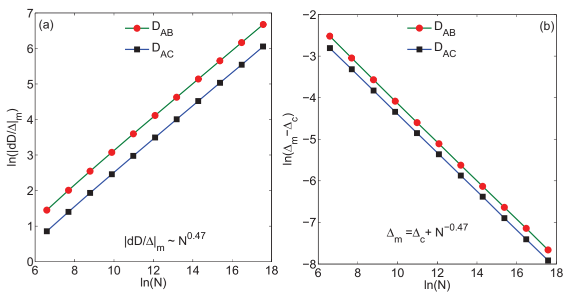


Fig. 3. The scaling behavior of $\ln |dD/d\Delta|_m$ and $\Delta_m$ in terms of system size $\ln N$.

Figure 3 depicts the scaling behavior of $|dD/d\Delta|_m$ and $\Delta_m$ with respect to $N$. In this figure, the $x$-axis stands for the length of the spin chain $N$. In figure 3(a), the $y$-axis stands for the minimum value of the first derivative of $|dD/d\Delta|_m$. In figure 3(b), the $y$-axis stands for the position of the minimum $\Delta_m$ of $|dD/d\Delta|_m$ approaches the critical point. The linear behavior exists both for $\ln |dD/d\Delta|_m$ and $\ln(\Delta_m - \Delta_c)$ with respect to $\ln N$. The exponent relations are $|dD/d\Delta|_m \sim N^{-0.47}$ and $\Delta_m \sim \Delta_c + N^{-0.47}$ respectively. Like concurrence [35], quantum deficit also can reflect the critical long-range correlation. Furthermore, we notice that the critical exponent have no change when varying the measure from concurrence to quantum deficit.

## 4 Monogamy Relation of Quantum Deficit

In general, the monogamy relation is given by [47, 48, 49, 50]

$$Q_{AB} + Q_{AC} \leq Q_{A(BC)} \tag{16}$$

for a three-partite system. Here $Q_{AB}$ is the quantum correlation of the state $\rho_{AB} = Tr_C(\rho_{ABC})$ and similarly for $Q_{AC}$. $Q_{A(BC)}$ is the quantum correlation between A and BC.

The monogamy relation of quantum deficit adopted here can be expressed as

$$\begin{array}{c} D_{AB} + D_{AC} \leq D_{A(BC)}, \\ D_{BA} + D_{BC} \leq D_{B(AC)}. \end{array} \tag{17}$$

The state $Q_{ABC}$ is monogamous with respect to quantum deficit iif the Eq. (17) is obeyed and polygamous otherwise.

In order to obtain the quantum deficit $D_{A(BC)}$, we need to evaluate the eigenbasis of $\rho_{BC} = Tr_A(\rho_{ABC})$. The results are

$$\begin{array}{c} |\psi_1\rangle = |00\rangle, \\ |\psi_2\rangle = (-q|10\rangle + |01\rangle)/\sqrt{q^2+1}, \\ |\psi_3\rangle = q(|10\rangle/q + |01\rangle)/\sqrt{q^2+1}, \\ |\psi_4\rangle = |11\rangle. \end{array} \tag{18}$$

The decohered counterpart $\rho^d_{A(BC)}$ can thus be got

$$\rho^d_{A(BC)} = \langle \chi_i, \psi_i | \rho_{ABC} | \chi_i, \psi_i \rangle = \frac{1}{q^2+2} diag(q^2+1, 1). \tag{19}$$

Since $\rho_{ABC}$ is a pure state, the result of $D_{A(BC)}$ is expressed by

$$D_{A(BC)} = -\frac{1}{2+q^2}[(q^2+1)\ln\frac{q^2+1}{q^2+2} + \ln\frac{1}{q^2+2}]. \tag{20}$$

Similarly, we can also evaluate the result of $D_{B(AC)}$ as

$$D_{B(AC)} = -\frac{1}{2(2+q^2)}[(q-1)^2\ln\frac{(q-1)^2}{2(q^2+2)} + (q+1)^2\ln\frac{(q+1)^2}{2(q^2+2)} + 2\ln\frac{1}{q^2+2}]. \tag{21}$$

Now we define the quantities

$$Diff_{A(BC)} \equiv D_{AB} + D_{AC} - D_{A(BC)}, \tag{22}$$

$$Diff_{B(AC)} \equiv D_{BA} + D_{BC} - D_{B(AC)}. \tag{23}$$

It is easy to get the analytical result of the above two expressions. Next, we mainly investigate the numerical result.

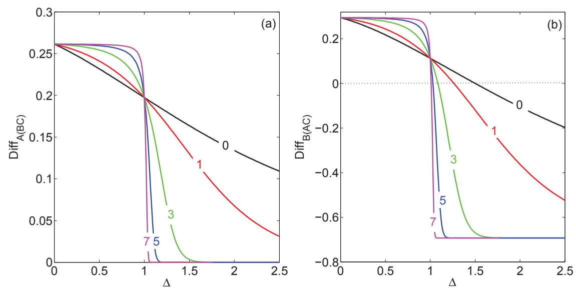

Fig. 4. The change of $Diff_{A(BC)}$ (a) and $Diff_{B(AC)}$ (b) of the model versus $\Delta$ at different QRG steps.

In figure 4, we demonstrate the evolution of $Diff_{A(BC)}$ and $Diff_{B(AC)}$ versus $\Delta$ in terms of QRG iteration steps. The $Diff_{A(BC)}$ always is positive and shows polygamous in figure 4(a). But the values of $Diff_{B(AC)}$ in figure 4(b) can be positive or negative. This indicates that the system may change from polygamous to monogamous by varying $\Delta$. The steps of QRG also have influence on such change. From this figure, we found that the monogamy relation depends on how the subsystems are partitioned and it also can be used to detect the critical point $\Delta_c = 1$.

## 5 The Monogamy Relation for the Generalized $W$ States

From Eq. (5) we notice that this state is a $W$ state. Therefore, it is more meaningful if we extend our result to the generalized $W$ state. Here we will investigate the two kinds of monogamy relations with respect to quantum deficit for the generalized $W$ state. The generalized $W$ state of three-qubits has the form

$$\begin{aligned} |\psi_W\rangle &= a|110\rangle + b|101\rangle + c|011\rangle, \\ |a|^2 &+ |b|^2 + |c|^2 = 1. \end{aligned} \tag{24}$$

After some algebra, the analytical monogamy inequality for this state is

$$\begin{aligned} D_{AB} + D_{AC} &\leq D_{A(BC)}, \\ D_{BA} + D_{BC} &\leq D_{B(AC)}, \end{aligned} \tag{25}$$

where $D_{AB} = (|a|^2+|b|^2)\ln(|a|^2+|b|^2) - |a|^2\ln(|a|^2) - |b|^2\ln(|b|^2)$, $D_{AC} = (|a|^2+|c|^2)\ln(|a|^2+|c|^2) - |a|^2\ln(|a|^2) - |c|^2\ln(|c|^2)$, $D_{A(BC)} = -\frac{|b|^2(|a|-|c|)^2}{|a|^2+|b|^2}\ln\frac{|b|^2(|a|-|c|)^2}{|a|^2+|b|^2} - \frac{(|b|^2+|ac|)^2}{|a|^2+|b|^2}\ln\frac{(|b|^2+|ac|)^2}{|a|^2+|b|^2} - a^2\ln|a|^2$, $D_{BA} = D_{AB}$, $D_{BC} = (|b|^2+|c|^2)\ln(|b|^2+|c|^2) - |b|^2\ln(|b|^2) - |c|^2\ln(|c|^2)$, $D_{B(AC)} = -\frac{(|ab|-|c|^2)^2}{|a|^2+|c|^2}\ln\frac{(|ab|-|c|^2)^2}{|a|^2+|c|^2} - \frac{|c|^2(|a|+|c|)^2}{|a|^2+|c|^2}\ln\frac{|c|^2(|a|+|c|)^2}{|a|^2+|c|^2} - a^2\ln|a|^2$.

## 6 Conclusions

In conclusion, we have contributed to investigate the monogamy relation of quantum deficit by employing the QRG theory. It is found that quantum deficit will develop two different saturated values divided by the critical point, and the values depend on which two subsystems are considered. The scaling behavior is observed in the model. We also analyze the monogamy relation of this model and find that the monogamy relation depend on how to split the systems, the values of $\Delta$, and the steps of QRG. In addition, the monogamy relation is given for the generalized $W$ state via quantum deficit. We hope our result will improve the understanding about the dynamical behavior of monogamy relation.

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## References

1. R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki (2009), *Quantum entanglement*, Rev. Mod. Phys., 81, pp. 865-942.
2. R. Lo Franco and G. Compagno (2016), *Quantum entanglement of identical particles by standard information-theoretic notions*, Sci. Rep., 6, pp. 20603.
3. B. Leggio, R. Lo Franco, D. O. Soares-Pinto, P. Horodecki and G. Compagno (2015), *Distributed correlations and information flows within a hybrid multipartite quantum-classical system*, Phys. Rev. A, 92, pp. 032311.
4. L. Aolita, F. deMelo, L. Davidovich (2015), *Open-system dynamics of entanglement: a key issues review*, Rep. Prog. Phys., 78, pp. 042001.
5. A. Orieux, G. Ferranti, A. D'Arrigo, R. Lo Franco, G. Benenti, E. Paladino, G. Falci, F. Sciarrino and P. Mataloni (2015), *Experimental on-demand recovery of quantum entanglement by local operations within non-Markovian dynamics*, Sci. Rep., 5, pp. 8575.
6. T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe and J. L. OBrien (2010), *Quantum computers*, Nature, 464, pp. 45C53.
7. Z.X. Man, Y.J. Xia and R. Lo Franco (2015), *Cavity-based architecture to preserve quantum coherence and entanglement*, Sci. Rep., 5, pp. 13843.
8. J.S. Xu, K. Sun, C.F. Li, X.Y. Xu, G.C. Guo, E. Andersson, R. Lo Franco and G. Compagno (2013), *Experimental Recovery of Quantum Correlations in Absence of System-Environment Back-Action*, Nat. Commun., 4, pp. 2851.
9. A. Salles, F. de Melo, M. P. Almeida, M. Hor-Meyll, S. P. Walborn, P. H. Souto Ribeiro and L. Davidovich (2008), *Experimental investigation of the dynamics of entanglement: Sudden death, complementarity, and continuous monitoring of the environment*, Phys. Rev. A, 78, pp. 022322.
10. R. Lo Franco, A. D'Arrigo, G. Falci, G. Compagno and E. Paladino (2014), *Preserving entanglement and nonlocality in solid-state qubits by dynamical decoupling*, Phys. Rev. B, 90, pp. 054304.
11. A. D'Arrigo, R. Lo Franco, G. Benenti, E. Paladino and G. Falci (2014), *Recovering entanglement by local operations*, Ann. Phys., 350, pp. 211C224
12. A. D'Arrigo, G. Benenti, R. Lo Franco, G. Falci and E. Paladino (2014), *Hidden entanglement, system-environment information flow and non-Markovianity*, Int. J. Quantum. Inf., 12, pp.1461005.
13. V. Coffman, J. Kundu and W.K. Wootters (2000), *Distributed entanglement*, Phys. Rev. A, 61, pp. 052306.
14. J.H. Huang and S.Y. Zhu (2008), *Entanglement monogamy in a three-qubit state*, Phys. Rev. A, 78, pp. 012325.

15. M. Koashi and A. Winter (2004), *Monogamy of quantum entanglement and other correlations*, Phys. Rev. A, 69, pp. 022309.
16. Y.C. Ou and H. Fan (2007), *Monogamy inequality in terms of negativity for three-qubit states*, Phys. Rev. A, 75, pp. 062308.
17. H. Ollivier and W.H. Zurek (2001), *Quantum discord: a measure of the quantumness of correlations*, Phys. Rev. Lett., 88, pp. 017901.
18. K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral (2012), *The classical-quantum boundary for correlations: Discord and related measures*, Rev. Mod. Phys., 84, pp. 1655.
19. B. Aaronson, R. Lo Franco and G. Adesso (2013), *Comparative investigation of the freezing phenomena for quantum correlations under nondissipative decoherence*, Phys. Rev. A, 88, pp. 012120.
20. B. Aaronson, R. Lo Franco, G. Compagno and G. Adesso (2013), *Hierarchy and dynamics of trace distance correlations*, New J. Phys., 15, pp. 093022.
21. M. Cianciaruso, T. R. Bromley, W. Roga, R. Lo Franco and G. Adesso (2015), *Universal freezing of quantum correlations within the geometric approach*, Sci. Rep., 5, pp. 10177.
22. I. A. Silva, A. M. Souza, T. R. Bromley, M. Cianciaruso, R. S. Sarthour, I. S. Oliveira, R. Lo Franco, E. R. deAzevedo, D. O. Soares-Pinto and G. Adesso (2016), *Observation of time-invariant coherence in a room temperature quantum simulator*, arXiv:1511.01971.
23. B. Bellomo, G. Compagno, R. Lo Franco, A. Ridolfo and S. Savasta (2011), *Dynamics and extraction of quantum discord in a multipartite open system*, Int. J. Quantum. Inf., 9, pp. 1665.
24. B. Bellomo, R. Lo Franco and G. Compagno (2012), *Dynamics of geometric and entropic quantifiers of correlations in open quantum systems*, Phys. Rev. A, 86, pp. 012312.
25. A. Datta, A. Shaji and C.M. Caves (2008), *Quantum discord and the power of one qubit*, Phys. Rev. Lett., 100, pp. 050502.
26. M. Qin, Y.B. Li and F.P. Wu (2014), *Relations between quantum correlations, purity and teleportation fidelity for the two-qubit Heisenberg XYZ system*, Quantum Inf. Process., 13, pp. 1573-1582.
27. S. Luo (2008), *Quantum discord for two-qubit systems*, Phys. Rev. A, 77, pp. 042303.
28. A. R. U. Devi and A.K. Rajagopal (2008), *Generalized information theoretic measure to discern the quantumness of correlations*, Phys. Rev. Lett., 100, pp. 140502.
29. R. Prabhu, A.K. Pati, A. Sen(De) and U. Sen (2012), *Conditions for monogamy of quantum correlations: Greenberger-Horne-Zeilinger versus W states*, Phys. Rev. A, 85, pp. 040102.
30. Sudha, A.R.U. Devi and A.K. Rajagopal (2012), *Monogamy of quantum correlations in three-qubit pure states*, Phys. Rev. A, 85, pp. 012103.
31. F.W. Ma, S.X Liu and X.M. Kong (2011), *Entanglement and quantum phase transition in the one-dimensional anisotropic XY model*, Phys. Rev. A, 83, pp. 062309.
32. C.H. Fan, H. N. Xiong, Y. X. Huang and Z. Sun (2013), *Quantum discord and quantum phase transition in spin-1/2 frustrated Heisenberg chain*, Quantum Inf. Comput., 13, pp. 0452-0468.
33. K.G. Wilson (1975), *The renorriialization group: Critical phenoraiena and the Konclo problem*, Rev. Mod. Phys. 47, pp. 773-840.
34. M. Kargarian, R. Jafari and A. Langari (2007), *Renormalization of concurrence: The application of the quantum renormalization group to quantum-information systems*, Phys. Rev. A, 76, pp. 060304.
35. M. Kargarian, R. Jafari and A. Langari (2008), *Renormalization of entanglement in the anisotropic Heisenberg (XXZ) model*, Phys. Rev. A, 77, pp. 032346.
36. Y. C. Li and H. Q. Lin (2011), *Thermal quantum and classical correlations and entanglement in the $XY$ spin model with three-spin interaction*, Phys. Rev. A, 83, pp. 052323.
37. R. Jafari, M. Kargarian, A. Langari and M. Siahatgar (2008), *Phase diagram and entanglement of the Ising model with Dzyaloshinskii-Moriya interaction*, Phys. Rev. B, 78, pp. 214414.
38. X.K. Song, T. Wu, S. Xu, J. He and L. Ye (2014), *Renormalization of quantum discord and Bell nonlocality in the XXZ model with Dzyaloshinskii-Moriya interaction*, Ann. Phys., 349, pp. 220-231.
39. A. Langari, F. Pollmann and M. Siahatgar (2013), *Ground-state fidelity of the spin-1 Heisenberg chain with single ion anisotropy: quantum renormalization group and exact diagonalization*

*approaches*, J. Phys.: Condens. Matter, 25, pp. 406002.
40. F.W. Ma, S.X. Liu and X.M. Kong (2011), *Quantum entanglement and quantum phase transition in the XY model with staggered Dzyaloshinskii-Moriya interaction*, Phys. Rev. A, 84, pp. 042302.
41. A. Langari and A.T. Rezakhani (2012), *Quantum renormalization group for ground-state fidelity*, New J. Phys., 14, pp. 053014.
42. X.K. Song, T. Wu and L. Ye (2013), *Negativity and quantum phase transition in the anisotropic XXZ model*Eur. Phys. J. D, 67, pp. 96.
43. X.K. Song, T. Wu and L. Ye (2013), *The monogamy relation and quantum phase transition in one-dimensional anisotropic XXZ model*, Quantum Inf. Process., 12, pp. 3305-3317.
44. L. Qiu, G. Tang, X.Q. Yang and A.M. Wang (2014), *Relating tripartite quantum discord with multisite entanglement and their performance in the one-dimensional anisotropic XXZ model*, EPL, 105, pp. 30005.
45. Q. Zheng, Y. Yao and X.W. Xu (2015), *Probing BerezinskiiKosterlitzThouless phase transition of spin-Half XXZ chain by quantum fisher information*, Commun. Theor. Phys., 63, pp. 279-284.
46. A.K. Rajagopal and R.W. Rendell (2002), *Separability and correlations in composite states based on entropy methods*, Phys. Rev. A, 66, pp. 022104.
47. G.L. Giorgi (2011), *Monogamy properties of quantum and classical correlations*, Phys. Rev. A, 84, pp. 054301.
48. H.C. Braga, C. C. Rulli, T.R. de Oliveira and M.S. Sarandy (2012), *Monogamy of quantum discord by multipartite correlations*, Phys. Rev. A, 86, pp. 062106.
49. R. Prabhu, A.K. Pati, A. Sen(De) and U. Sen (2012), *Relating monogamy of quantum correlations and multisite entanglement*, Phys. Rev. A, 86, pp. 052337.
50. S.Y. Liu, Y.R. Zhang, L.M. Zhao, W.L. Yang and H. Fan (2014), *General monogamy property of global quantum discord and the application*, Ann. Phys., 348, pp. 256-269.