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CLASSIFICATION OF TRANSVERSAL GATES IN QUBIT STABILIZER CODES

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This work classifies the set of diagonal gates that can implement a single or two-qubit transversal logical gate for qubit stabilizer codes. We show that individual physical diagonal gates on the underlying qubits that compose the code are restricted to have entries of the form $e^{i\pi c/2^k}$ along their diagonal, resulting in a similarly restricted class of logical gates that can be implemented in this manner. As such, we show that all diagonal logical gates that can be implemented transversally by individual physical diagonal gates must belong to the Clifford hierarchy. Moreover, we show that for a given stabilizer code, the two-qubit diagonal transversal gates must belong to the same level of Clifford hierarchy as the single-qubit diagonal transversal gates available for the given code. We use this result to prove a conjecture about arbitrary transversal gates made by Zeng *et al.* in 2007.

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1 Introduction

Any physical realization of a quantum computing device will be subject to physical noise processes leading to potential computational errors. As such, quantum error correction is used to protect the information using multiple physical systems to encode a logical quantum state [1, 2, 3, 4]. Quantum error correction will play a central role in any fault-tolerant implementation of a quantum computer, yet it is of paramount importance that the fundamental quantum operations such as state preparation, error syndrome extraction and correction, state measurement, and state manipulation are done in a manner that does not propagate errors throughout the system [5, 6, 7, 8, 9]. In this work we focus on state manipulation, or quantum gate application. Transversal gates, that is, logical gates that are a result of the application of individual local quantum gates on qubits forming the quantum error correcting code, provide the most natural form of fault-tolerant quantum logic. Therefore, developing quantum error correcting codes that have transversal gate sets are of prime importance for

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quantum fault-tolerance. However, as first shown by Zeng *et al.* for stabilizer codes [10], and then further generalized by Eastin and Knill [11] for arbitrary quantum codes, there exists no quantum error correcting code that has a set of universal transversal gates. Additionally, Bravyi and König [12] showed that for a D-dimensional local stabilizer code with large distance only gates from the Clifford Hierarchy at level D (or lower) can be applied transversally (where we are using the convention that Pauli gates form the first level, Clifford gates the second, etc.). Their result also applies to more general local unitaries, not just transversal gates.

With these constraints in mind, there has been a push in the research community towards methods to side-step these gate restrictions for a single quantum error correcting code. Techniques that allow for the fault-tolerant application of a set of universal quantum gates involve quantum code manipulation through gate fixing [13, 14], partial transversality [15], or code conversion [16]. While these results show promise, practical techniques for implementing fault-tolerant universal gate logic without having to use techniques such as magic state distillation, with its high qubit overhead, would be useful for further improvements.

Recent results in the area of quantum gate decomposition have focused on expressing an arbitrary single-qubit quantum gate as a sequence of Hadamard (H) and V gates, where $V = \text{diag}(1+2i, 1-2i)/\sqrt{5}$ [17]. Therefore, the discovery of quantum error correcting codes that allow for the application of V in a transversal manner could potentially led to adaptation of the above mentioned techniques for universal fault-tolerant gate application without state distillation for these proposed gate decompositions.

Recently, a parallel work by Pastawski and Yoshida [18] showed many exciting results pertaining to fault-tolerant operations in topological stabilizer codes. They also proved that families of stabilizer codes with a finite loss threshold must have transversal gates in the Clifford hierarchy. Furthermore, they show that higher loss thresholds impose greater restrictions on the level in the Clifford hierarchy at which transversal gates can be implemented. Our result does not need a finite loss threshold or a family of codes to be applicable. Additionally, our result applies to transversal gates between two like codes; however, this result only applies to qubits and restricts the transversal gates to being in the Clifford hierarchy (our result does not specify the level). A parallel work has also shown that transversal logical gates for qubit stabilizer codes must belong to the Clifford hierarchy, and that the level of the Clifford hierarchy which can be implemented is bounded as a function of the number of physical qubits and code distance [19].

The main result of our paper is that for quantum qubit stabilizer codes, the only diagonal gates that can be implemented transversally using physical diagonal gates are those whose entries along the diagonal are of the form $e^{i\pi c/2^k}$, for some power of k depending on the choice of code. This result holds both for single and two-qubit gates, and moreover we show that all such gates must be contained within the Clifford hierarchy. Moreover, as Zeng, Cross, and Chuang showed [10], any transversal non-trivial single-qubit logical gate for a qubit stabilizer code must result from the application of diagonal gates along with local Clifford operations and potential swapping of qubits. Therefore, our result classifies all transversal single-qubit logical gate operations up to local Clifford equivalences and relabelling of qubits. Additionally, our result classifies all transversal diagonal single-qubit logical gates that can map one stabilizer code to another stabilizer code. It is worth noting that the Reed-Muller family of quantum codes provides a means of implement any of these diagonal transversal gates, where changing to higher order in the code family allows for the implementation of diagonal logical gates with finer angles, all of which are in the Clifford hierarchy and of the form $e^{i\pi c/2^k}$. Finally, we show that the two-qubit transversal gates that can be applied for a given quantum stabilizer code must belong to the same level of the Clifford hierarchy as the single qubit transversal logical gates that can be applied for such a code. As such, there is no increased computational logical power by considering two-qubit transversal gates and conjecture this result to generalize to multi-qubit gates.

2 Stabilizer codes and transversal logical gates

2.1 The stabilizer formalism

We begin by reviewing the stabilizer formalism [20, 21]. The Pauli matrices are defined as follows:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli group on n qubits $\mathcal{P}_n$ is generated by the above Pauli matrices on each of the n qubits. Given a set of independent commuting elements $\{G_1, \ldots, G_{n-k}\}$ from the Pauli group $\mathcal{P}_n$, the group generated by these elements modulo overall phase factors $\{1, i, -1, -i\}$, denoted $\mathcal{S} = \langle G_1, \ldots, G_{n-k}\rangle$ is the *stabilizer* of a quantum code on n qubits: $Q = \{|\psi\rangle \mid g|\psi\rangle = |\psi\rangle \; \forall \, g \in \mathcal{S}\}$. The quantum code Q corresponds to the intersection of the "+1" eigenspaces of all of the $(n-k)$ generators and has dimension size 2^k, that is, it will encode k logical qubits. Logical operators are the elements of the *normalizer* of $\mathcal{S}$, $\mathcal{N}(\mathcal{S}) = \{U \in U(2^n) \mid U\mathcal{S}U^\dagger = \mathcal{S}\}$, that are not trivially in the stabilizer $\mathcal{S}$, that is, $\mathcal{N}(\mathcal{S})\backslash S$. The *distance* of the code Q is defined as $d = \min\{\mathrm{wt}(P) \mid P \in \mathcal{P}_n, \; P \in \mathcal{N}(\mathcal{S})\backslash S\}$, where the *weight* $\mathrm{wt}(P)$ is defined as the number of non-identity elements in the Pauli operator P. A n qubit code encoding k qubits with a distance d is labelled as a $[[n, k, d]]$ stabilizer code. An error-detecting quantum stabilizer code Q is any stabilizer code whose distance $d \geq 2$. Throughout the remainder of this work, a stabilizer code will refer to an error-detecting quantum stabilizer code unless otherwise specified.

2.2 Outline of the classification of transversal gates

In this work we aim to classify the set of logical transversal gates for a stabilizer code, that is logical gates that are composed of individual unitary gates on each of the underlying qubits of the code, $U = \bigotimes_{j=1}^{n} U_j$. We begin by noting an important result from Zeng *et al.* [10] where it was shown that unitary, single-qubit logical transversal operators in qubit stabilizer codes must be of the form:

$$U = L \left(\bigotimes_{j=1}^{n} \mathrm{diag}(1, e^{i\pi\theta_j}) \right) R^\dagger P_\pi. \tag{1}$$

Here $L, R^\dagger$ are transversal Clifford operations and P_π is a coordinate permutation (a set of SWAP gates). Notice that if an $[[n, k, d]]$ stabilizer code exists which implements U transversally, then up to local Clifford equivalences, an $[[n, k, d]]$ stabilizer code exists which implements $U = \bigotimes_{j=1}^{n} \mathrm{diag}(1, e^{i\pi\theta_j})$ transversally. In this work we look at the restrictions on these diagonal, transversal gates.

We begin by considering the case of strongly transversal diagonal Z rotations, that is gates where all of the individual rotations are the same, $U = \bigotimes_{j=1}^{n} \mathrm{diag}(1, e^{i\pi\theta})$. As shown in Section 3.1, in the case of CSS codes there is a straightforward method to describe the codestates as a sum over computational basis states. This formulation of the codestates imposes a restriction on the set of angles that can implement a logical operation as the phase introduced by the transversal rotation will have to be the same for all computational basis states with non-zero overlap of a given logical state. We use this restriction to show that, for CSS codes, logical operations composed of transversal diagonal gates must be of the form $U = \bigotimes_{j=1}^{n} \mathrm{diag}(1, e^{i\pi c/2^k})$. We can then use this result to conclude that the set of logical gates that can be implemented in this manner must be similarly restricted, and moreover will belong to the Clifford hierarchy. In Section 3.2, we generalize this result to arbitrary stabilizer codes by developing a similar description for the codestates of stabilizer codes to that of CSS codes as a sum over computational basis states with the addition of complex phases. Importantly, we show that there exists a logical basis for the codespace where the sum over computational basis states for each logical qubit have non-overlapping computational basis states with those from orthogonal logical states. In Section 3.3, we further generalize the result to diagonal transversal gates where the angular rotation is not uniform across all of the qubits. We first claim (proved in Appendix A) that irrational (multiples of π) angles of the individual rotations must cancel each other out and as such cannot increase the set of gates that can be implemented transversally. In the case of rational rotation angles θ_i, we prove a *decompression lemma* which allows us to reduce the case of differing angles to the uniform θ case, and the proof therefore generalizes to the uniform case. In Section 3.4, the result is extended to comprise all mappings between stabilizer codes and as such through the single-qubit logical gate classification result of Zeng *et al.* [10], we are able to conclude that all transversal gates must belong to the Clifford hierarchy, therefore proving a conjecture in the aforementioned work.

We conclude in Section 4 by generalizing the restriction of transversal diagonal unitaries to two-qubit logical operations and show that the set of angles are similarly restricted. Moreover, we show that for a given stabilizer code any such two-qubit logical operation must reside at the same level of the Clifford hierarchy as the allowed set of single qubit transversal diagonal gates.

3 Single qubit transversal Z rotations

We begin by focusing on $Z(\theta)$ rotations, that is rotations about the Z-axis by some angle θ. For qubits, this rotation is given by a diagonal matrix

$$A = \begin{bmatrix} e^{i\pi\theta_1} & 0 \\ 0 & e^{i\pi\theta_2} \end{bmatrix} = e^{i\pi\theta_1} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi(\theta_2-\theta_1)} \end{bmatrix}. \tag{2}$$

Up to a global phase, we therefore need only consider rotations of the form

$$A = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi\theta} \end{bmatrix} \equiv Z(\theta), \tag{3}$$

where we are using the above equation as the definition of a single-qubit $Z(\theta)$ rotation of angle $\pi\theta$ (we shall assume for the remainder of this work that the angular rotations are rational multiples of π, as discussed in detail below).

In this work, we study constraints on transversal implementations of logical $Z(\theta)$ rotations. A transversal Z rotation is defined as

$$Z_T(\theta) := Z(\theta_1) \otimes Z(\theta_2) \otimes ... \otimes Z(\theta_n) \tag{4}$$

Before considering the most general form of transversal gate outlined above, we first focus on the case when all physical qubits undergo the same rotation θ, that is, we require that the logical implementation be *strongly transversal*, $Z_L(\theta') = Z(\theta)^{\otimes n}$, with n being the number of physical qubits. While each single-qubit rotation is a rotation by the same angle, we do not require that the logical Z applies the same rotation to the logical qubit.

3.1 CSS codes

A CSS code [22, 23] is a stabilizer code whose generators can be separated into two sets, the X and Z stabilizers composed only of Pauli X and Z operators, respectively. That is $\mathcal{S} = \langle G_1^X, \ldots, G_{|G_X|}^X, G_1^Z, \ldots, G_{|G_Z|}^Z \rangle$, where $|G_X|$ and $|G_Z|$ refer to the number of X and Z stabilizers. Additionally, a CSS code defines logical X and Z operators composed only of X and Z operators, again respectively. Focusing on the case of a single encoded logical qubit, the codestates can be defined as follows:

$$|0_L\rangle = \frac{1}{2^{|G_X|/2}} \prod_i (I + G_i^X)|0\rangle^{\otimes n}, \qquad |1_L\rangle = X_L|0_L\rangle, \tag{5}$$

where i runs over all X stabilizer generators G_i^X. These states are codestates as $\prod_i (I + G_i^X)$ projects onto the codespace and the state $|0_L\rangle$ must be an eigenstate of Z_L since the logical operator must consist only of Z operators. The codestates can then be expanded in terms of each of the X generators:

$$\begin{aligned} |0_L\rangle = |g_0\rangle + \sum_{i_1} |g_{i_1}\rangle + \sum_{i_1 < i_2} |g_{i_1} \oplus g_{i_2}\rangle \\ + \cdots + \sum_{i_1 < \cdots < i_{|G_X|-1}} |g_{i_1} \oplus \cdots \oplus g_{i_{|G_X|-1}}\rangle + |g_1 \oplus ... \oplus g_{|G_X|}\rangle, \end{aligned} \tag{6}$$

$$|1_L\rangle = |g_L\rangle + \sum_{i_1} |g_L \oplus g_{i_1}\rangle + \sum_{i_1 < i_2} |g_L \oplus g_{i_1} \oplus g_{i_2}\rangle + ... + |g_L \oplus g_1 \oplus ... \oplus g_{|G_X|}\rangle, \tag{7}$$

where g_i is a n-bit binary string representing the location of the X operators in the generator G_i^X, g_0 is the all-0 vector and g_L is a binary vector representing X_L. In order to classify the set of transversal Z-axis rotations that can implement a logical operation we derive a set of conditions restricting the rotation angles due to constrains arising from the form of the binary vectors g_i.

Theorem 1 *Given a $[[n, l, d]]$ error-detecting CSS code Q (distance $d \geq 2$). A strongly transversal rotation, that is a gate of the form $Z(\theta)^{\otimes n}$, that implements a non-identity logical gate for Q is restricted to be composed of individual rotations of the form $Z(a/2^k)$, where $a, k \in \mathbb{N}$.*

Proof. Consider the action of the strongly transversal rotation $Z(\theta)^{\otimes n}$ on the codestates given in Equations 6–7. Given that $Z(\theta)^{\otimes n}$ will preserve computational basis states, other than introducing a complex phase, then in order for the transversal rotation to implement a logical operation each term in the expansion of $|0_L\rangle$ must obtain an identical phase (in this case a trivial phase of 1 since $Z(\theta)^{\otimes n}|g_0\rangle = |g_0\rangle$). Similarly, all states in the expansion of $|1_L\rangle$ must obtain an identical phase, a non-trivial complex phase in order for the logical operator to be a non-identity gate, as follows:

$$\begin{aligned}
Z(\theta)^{\otimes n}|0_L\rangle &= |g_0\rangle + Z(\theta)^{\otimes n}\left(\sum_{i_1}|g_{i_1}\rangle + \sum_{i_1<i_2}|g_{i_1}\oplus g_{i_2}\rangle + ... + |g_1\oplus ... \oplus g_{|G_X|}\rangle\right) \\
&= |g_0\rangle + \sum_{i_1} e^{i\pi\theta|g_{i_1}|}|g_{i_1}\rangle + \sum_{i_1<i_2} e^{i\pi\theta|g_{i_1}\oplus g_{i_2}|}|g_{i_1}\oplus g_{i_2}\rangle \\
&\quad + ... + e^{i\pi\theta|g_1\oplus ...\oplus g_{|G_X|}|}|g_1\oplus ... \oplus g_{|G_X|}\rangle \\
&= |g_0\rangle + \sum_{i_1}|g_{i_1}\rangle + \sum_{i_1<i_2}|g_{i_1}\oplus g_{i_2}\rangle + ... + |g_1\oplus ... \oplus g_{|G_X|}\rangle, \\
Z(\theta)^{\otimes n}|1_L\rangle &= e^{i\pi\theta|g_L|}\left(|g_L\rangle + \sum_{i}|g_L\oplus g_i\rangle + ... + |g_L\oplus g_1\oplus ... \oplus g_{|G_X|}\rangle\right),
\end{aligned}$$

where $|x|$ denotes the weight of the binary string x. The constraint that each of the states in the expansion of $|0_L\rangle$ must obtain the same phase, and similarly for $|1_L\rangle$, can be summarized by the following set of modular algebraic constraints:

$$\begin{aligned}
\theta|g_{i_1}| &= 0 \bmod 2 & \theta|g_L| &= a \bmod 2 \\
\theta|g_{i_1}\oplus g_{i_2}| &= 0 \bmod 2 & \theta|g_L\oplus g_{i_1}| &= a \bmod 2 \\
&\vdots & &\vdots \\
\theta|g_1\oplus ... \oplus g_{|G_X|}| &= 0 \bmod 2 & \theta|g_L\oplus g_1\oplus ... \oplus g_{|G_X|}| &= a \bmod 2, \\
& & \forall\ 0 < i_1 < i_2 < ... < i_{|G_X|-1} &\leq |G_X|,
\end{aligned}$$

where $\theta,\ a\in\mathbb{R}$ and $|.|\in\mathbb{N}$. The constraints on the left correspond to those for the $|0_L\rangle$ state and those on the right the $|1_L\rangle$ state.

We begin by making some observations on the above equations to rule out certain values of θ.

1. First, notice that if θ is irrational these equations can never be satisfied since $n\theta = p \equiv 0 \bmod 2 \implies \theta = \frac{2t}{n} \in \mathbb{Q}$. We can therefore restrict our attention to rational angles ($\theta = \frac{p}{q} \in \mathbb{Q}$). Without loss of generality, we can assume this fraction is irreducible and in the range $(0, 2]$. Moreover, we must have that the weight of the binary strings is a multiple of q, that is $\frac{p}{q}|\cdot| = 0 \mod q \Longrightarrow |\cdot| = 0 \mod q$.

2. We can express these constraints as conditions on overlap similarly to Bravyi and Haah [25] by noting that

$$|g_1\oplus ... \oplus g_n| = \sum_{i=1}^{n}|g_i| - 2\sum_{i<j}|g_i\wedge g_j| + ... + (-2)^{n-1}|g_1\wedge ... \wedge g_n|, \tag{8}$$

where $\wedge$ is the bitwise AND.

With these observations, we can express the constraints as:

$$\begin{aligned}
|g_{i_1}| &= 0 \bmod q \\
|g_{i_1}| + |g_{i_2}| - 2|g_{i_1} \wedge g_{i_2}| &= 0 \bmod q \\
&\vdots \\
\sum_{i_1}^{n} |g_{i_1}| - 2\sum_{i_1<i_2} |g_{i_1} \wedge g_{i_2}| + ... + (-2)^{|G_X|-1}|g_1 \wedge ... \wedge g_{|G_X|}| &= 0 \bmod q \\
\\
|g_L| &= b \bmod q \\
|g_L| + |g_{i_1}| - 2|g_L \wedge g_{i_1}| &= b \bmod q \\
&\vdots \\
\forall\ 0 < i_1 < i_2 < ... < i_{|G_X|-1} \leq |G_X|
\end{aligned}$$

We can see that these equations are not independent since the requirement that $|g_i| = 0 \bmod q$, implies

$$|g_{i_1}| + |g_{i_2}| - 2|g_{i_1} \wedge g_{i_2}| = 0 \bmod q \implies 2|g_{i_1} \wedge g_{i_2}| = 0 \bmod q. \tag{9}$$

Using this, we can express the above constraints as overlap conditions

$$\begin{aligned}
|g_{i_1}| &= 0 \bmod q, \\
2|g_{i_1} \wedge g_{i_2}| &= 0 \bmod q \\
4|g_{i_1} \wedge g_{i_2} \wedge g_{i_3}| &= 0 \bmod q \\
&\vdots \\
(2)^{|G_X|-1}|g_1 \wedge ... \wedge g_{|G_X|}| &= 0 \bmod q \\
|g_L| &\neq 0 \bmod q \\
2|g_{i_1} \wedge g_L| &= 0 \bmod q \\
4|g_{i_1} \wedge g_{i_2} \wedge g_L| &= 0 \bmod q \\
&\vdots \\
(2)^{|G_X|}|g_1 \wedge ... \wedge g_{|G_X|} \wedge g_L| &= 0 \bmod q \\
\forall\ 0 < i_1 < i_2 < ... < i_{|G_X|-1} \leq |G_X|,
\end{aligned}$$

We have also dropped the minus sign since it has no effect. For the logical operator to be nontrivial, we have assumed that $b \neq 0$. Notice that the $0 \bmod q$ conditions are independent constraints.

Observe that if q has only even prime factors (*i.e.* $q = 2^t$ for some integer t) then all higher-order overlap conditions will, at some point, become trivial. For example, if $q = 2^k$ overlap conditions will be trivial for any $k + 1$ or more rows and Reed-Muller codes exist which have any $Z(1/2^k)$ gate transversally. In fact, since the transversal gates form a group, Reed-Muller codes exist which have any $Z(c/2^k)$ (where c is an integer) gate transversally. Therefore the existence of transversal gates is already solved in the positive for that case.

In what follows we will assume that q has at least one odd prime factor and that $|g_L| \neq 0 \bmod q_o$ for at least one such q_o (we will choose this q_o). If $|g_L| = 0 \bmod q_o$ for all odd prime factors, again assuming $\theta = p/q$ then $Z(\theta)^{\otimes n}|1_L\rangle = e^{i\pi|g_L|p/q}|1_L\rangle = e^{i\pi c/2^k}|1_L\rangle$ for some positive integer k, where $c \equiv |g_L| \bmod 2^k$. In this case, the odd prime factors add nothing, and we could apply the same logical operator using $Z(1/2^k)^{\otimes n}$ instead of $Z(p/q)^{\otimes n}$, which reduces to the case we considered above. Therefore, we assume $|g_L| \neq 0 \bmod q_o$ for at least one such q_o and note that in this case all overlap conditions are nontrivial. We can write $q = q_o \cdot q_{P/o}$ where $q_{P/o}$ is the product of the other prime factors of q. Since $|g| = 0, 1 \bmod q \implies |g| = 0, 1 \bmod q_o$, we can write a weaker set of overlap conditions as

$$\begin{aligned}
|g_{i_1}| &= 0 \bmod q_o, \\
|g_{i_1} \wedge g_{i_2}| &= 0 \bmod q_o \\
|g_{i_1} \wedge g_{i_2} \wedge g_{i_3}| &= 0 \bmod q_o \\
&\vdots \\
|g_1 \wedge ... \wedge g_{|G_X|}| &= 0 \bmod q_o \\
|g_L| &\neq 0 \bmod q_o \\
|g_{i_1} \wedge g_L| &= 0 \bmod q_o \\
|g_{i_1} \wedge g_{i_2} \wedge g_L| &= 0 \bmod q_o \\
&\vdots \\
|g_1 \wedge ... \wedge g_{|G_X|} \wedge g_L| &= 0 \bmod q_o \\
\forall\ 0 < i_1 < i_2 < ... < i_{|G_X|-1} &\leq |G_X|.
\end{aligned}$$

In what follows, we will attempt to find the smallest binary matrix (in terms of number of rows) which satisfies all of the weaker overlap conditions.

Let S_X be the parity check matrix for the X stabilizers, that is S_X is a binary matrix where each row is given by independent binary strings g_i, $i \in \{1, ..., |G_X|\}$. In the statement of the Theorem we are assuming the code is an error-detecting code with distance $d \geq 2$, therefore each column of S_X must have at least one non-zero entry as this is a necessary and sufficient condition for Z error detection. We will consider the case of a single logical operator and show that no nontrivial matrix S_X with at least one X logical operator exists, such that all rows satisfy the overlap conditions derived above.[b]

Let us now try to find the smallest number of rows in S_X such that the overlap conditions are satisfied. We begin by defining the variable $w_{\vec{\mu}}$, where $\vec{\mu} = (\mu_1\ \mu_2 \dots\ \mu_{|G_X|})$ is a $|G_X|$ bit binary vector, as follows:

$$w_{\vec{\mu}} = \left|\left(\wedge_{(\mu_i=1)} g_i\right) \wedge \left(\wedge_{(\mu_i=0)} \neg g_i\right) \wedge g_L\right|, \tag{10}$$

that is $w_{\vec{\mu}}$ is the weight of the bitwise AND of g_L along with all binary vectors g_i such that $\mu_i = 1$ and $\neg g_j$ (negation of g_j) such that $\mu_j = 0$. For example if $|G_X| = 2$, that is S_X is

[b] This part of our proof uses techniques developed in [25].

composed of just two rows, then $w_{(10)} = |g_1 \wedge (\neg g_2) \wedge g_L|$, $w_{(01)} = |(\neg g_1) \wedge g_2 \wedge g_L|$, and $w_{(11)} = |g_1 \wedge g_2 \wedge g_L|$. While this definition may at first seem non-intuitive, we choose such a definition in order to consider quantities from the constraints on g_i above. For example, again in the case of $|G_X| = 2$, the constraint $|g_1 \wedge g_L| = 0 \mod q_o$ can be expressed as $w_{(11)} + w_{(10)} = 0 \bmod q_o$ by the straightforward binary string relationship $|g_1 \wedge g_L| = |g_1 \wedge g_2 \wedge g_L| + |g_1 \wedge (\neg g_2) \wedge g_L|$.

Using the definition of $w_{\vec{\mu}}$ we can arrive at a contradiction to the set of overlap conditions. First consider the case of $w_{\vec{\mu}}$ where $\vec{\mu} = (11 \ \ldots 1)$ is the all-1 vector. Therefore, by the above constraints,

$$w_{(11\ldots1)} = |g_1 \wedge \ldots \wedge g_{|G_X|} \wedge g_L| = 0 \bmod q_o.$$

Now consider the overlap condition consisting of $|G_X| - 1$ binary vectors g_i, that is $|g_{i_1} \wedge \ldots \wedge g_{i_{|G_X|-1}} \wedge g_L| = 0 \bmod q_o$. Without loss of generality, consider the case of the first $|G_X| - 1$ vectors g_i. Then the following identity holds:

$$|g_1 \wedge \ldots \wedge g_{|G_X|-1} \wedge g_L| = w_{(11\ldots11)} + w_{(11\ldots10)} = 0 \bmod q_o,$$

and therefore $w_{(11\ldots10)} = 0$, where equality for the remainder of the proof is mod q_o. Thus, without loss of generality $w_{\vec{\mu}} = 0$ for all vectors $\vec{\mu}$ of weight $|G_X| - 1$. We can now recursively prove the same statement for any weight $\vec{\mu}$. Suppose $w_{\vec{\mu}} = 0$ for all $|\vec{\mu}| > k$ and consider without loss of generality the bitwise AND of the first k vectors g_i and g_L,

$$|g_1 \wedge \ldots \wedge g_k \wedge g_L| = \sum_{i=k+1}^{|G_X|} \sum_{\mu_i=0}^{1} w_{(1\ldots1\mu_{k+1}\ldots\mu_{|G_X|})} = 0 \bmod q_o.$$

However, since all of the terms in the above sum are equal to 0 by the induction hypothesis except for the term where $\mu_i = 0 \ \forall \ i > k$, we arrive at the conclusion that $w_{(\vec{1}_k\vec{0}_{|G_X|-k})=0}$, that is $w_{\vec{\mu}} = 0$ when the only the first k entries of $\vec{\mu}$ are non-zero. Since the choice of which k entries were non-zero was arbitrary, we arrive at the conclusion that $w_{\vec{\mu}} = 0 \ \forall |\vec{\mu}| \geq k$. We have therefore proven the induction statement that $w_{\vec{\mu}} = 0 \ \forall \ \vec{\mu}$. Finally, in order to arrive at the contradiction to the satisfiability of the vector constrains, note that $|g_L| = \sum_{\vec{\mu}} w_{\vec{\mu}} = 0$, which contradicts the condition for a non-trivial gate $|g_L| \neq 0 \bmod q_o$ Therefore no such q_o exists.

Finally, it is worth noting that the set of constraints are only dependant on the X type stabilizer generators and their parity check matrix S_X, and are not dependant on the number of logical operators. Therefore, additional logical qubits (and thus constraints for the different binary strings $g_{X_{L,i}}$ of the different logical $X_{L,i}$ operators would only further constrain the problem and reduce the set of angles. As such, this result holds for multiple logical qubits as well. $\square$.

Remark 1 We made the assumption that the logical X operator was composed of a set of individual X operators on a collection of qubits characterized by the bit string g_L, where $g_L(i) = 1$ if X_L performs the operation X at qubit i. However in theory, X_L could also be comprised of Z (or Y) operations as well. A particular Z (or Y) gate could introduce a phase on some of the state vectors in the expansion of the logical $|1_L\rangle$, yet these phases must be preserved by the action of $Z(\theta)^{\otimes n}$. Since these diagonal rotations will not change the form of the computational basis state, they will only introduce a phase. In that manner, the presence of Z (or Y) operations in the logical X_L gate will not change the set of algebraic conditions for the physical rotations $Z(\theta)$.

Remark 2 We made the assumption that the individual rotation on the physical qubits, $Z(\theta)$, were of the form $\text{diag}(1, e^{i\theta})$, however in full generality the diagonal gates can be of the form $\text{diag}(e^{i\varphi}, e^{i\theta})$. The resulting conditions on the transformation of the logical states $|0_L\rangle$ and $|1_L\rangle$ will have the form:

$$\begin{aligned} Z(\theta)^{\otimes n}|0_L\rangle &= Z(\theta)^{\otimes n}\left(|g_0\rangle + \sum_{i_1}|g_{i_1}\rangle + \sum_{i_1<i_2}|g_{i_1}\oplus g_{i_2}\rangle + ... + |g_{i_1}\oplus ... \oplus g_{|G_X|}\rangle\right) \\ &= e^{i\varphi n}|g_0\rangle + \sum_{i_1} e^{i\theta|g_{i_1}|+i\varphi(n-|g_i|)}|g_{i_1}\rangle + \sum_{i_1<i_2} e^{i\theta|g_{i_1}\oplus g_{i_2}|+i\varphi(n-|g_{i_1}\oplus g_{i_2}|)}|g_{i_1}\oplus g_{i_2}\rangle \\ &\quad + \cdots + e^{i\theta|g_{i_1}\oplus ... \oplus g_{|G_X|}|+i\varphi(n-|g_{i_1}\oplus ... \oplus g_{|G_X|}|)}|g_{i_1}\oplus ... \oplus g_{|G_X|}\rangle \\ &= e^{i\varphi n}\left(|g_0\rangle + \sum_{i_1}|g_{i_1}\rangle + \sum_{i_1<i_2}|g_{i_1}\oplus g_{i_2}\rangle + ... + |g_{i_1}\oplus ... \oplus g_{|G_X|}\rangle\right) \end{aligned}$$

and

$$Z(\theta)^{\otimes n}|1_L\rangle = e^{i\varphi n + i(\theta-\varphi)|g_L|}\left(|g_L\rangle + \sum_{i_1}|g_L\oplus g_{i_1}\rangle + ... + |g_L\oplus g_{i_1}\oplus ... \oplus g_{|G_X|}\rangle\right).$$

The constraints can then be shown to have the form:

$$\begin{aligned} (\theta-\varphi)|g_{i_1}| &= 0 \bmod 2 \\ 2(\theta-\varphi)|g_{i_1}\wedge g_{i_2}| &= 0 \bmod 2 \\ &\vdots \\ 2^{|G_X|-1}(\theta-\varphi)|g_{i_1}\wedge \ldots \wedge g_{|G_X|}| &= 0 \bmod 2 \\ \\ (\theta-\varphi)|g_L| &\neq 0 \bmod 2 \\ 2(\theta-\varphi)|g_L\wedge g_{i_1}| &= 0 \bmod 2 \\ &\vdots \\ 2^{|G_X|}(\theta-\varphi)|g_L\wedge g_{i_1}\wedge ... \wedge g_{|G_X|}| &= 0 \bmod 2 \\ \forall\, 0 < i_1 < i_2 < ... < i_{|G_X|-1} &\leq |G_X|, \end{aligned}$$

which are the same constrains on the difference of the phases $(\theta-\varphi)$ as the case when $\varphi = 0$. Therefore, an arbitrary global phase can be introduced on the individual rotations of the form $\text{diag}(1, e^{i\theta})$ which are allowed in the CSS construction.

It is worth noting that the restriction on the set of rotations that can be applied to the individual qubits of a CSS code will impose a restriction on the set of logical rotations that can be applied. This shows a strong connection to the Clifford hierarchy. The Clifford hierarchy is defined recursively, where the first level of the hierarchy on n qubits is defined as the Pauli operators on n qubits, denoted $\mathcal{C}_n^{(1)} = \mathcal{P}_n$. Higher levels ($k \geq 2$) of the Clifford hierarchy are then defined as follows:

$$\mathcal{C}_n^{(k)} = \{U \in U(2^n) \mid UPU^\dagger \in \mathcal{C}_n^{(k-1)} \; \forall P \in \mathcal{P}_n\},$$

that is, a unitary U in the k-th level of the Clifford hierarchy maps by conjugation the Pauli operators on n qubits to an element in the $(k-1)$-th level of the Clifford hierarchy. Namely, the second level of the Clifford hierarchy is the Clifford operators, mapping Pauli operators to Pauli operators. It is worth noting that each level of the Clifford hierarchy contains all lower levels of the Clifford hierarchy, that is $\mathcal{C}_n^{(p)} \subsetneq \mathcal{C}_n^{(q)}$, if $p < q$.

Proposition 1 *Let $A = Z(\theta)$ be a diagonal single-qubit operator. If $\theta = c/2^k$, for any integer $k \geq 0$ where θ is in its most reduced form, then $A \in \mathcal{C}_1^{(k+1)}$. Otherwise, A is not in the Clifford hierarchy, that is $A \notin C_1^{(k)}$ for all k.*

Proof. Consider the action of conjugation of the operator $A = Z(\theta)$ on the single qubit Pauli matrix X, the action on Pauli Z is trivial due to the commutation of diagonal matrices. Consider the recursive construction of the matrices A_p defined as: $A_p = A_{p-1}XA_{p-1}^\dagger$, where $A_0 = A$. Notice the following:

$$\begin{aligned}
A_1 = A_0XA_0^\dagger &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\pi\theta} \\ e^{i\pi\theta} & 0 \end{pmatrix}, \\
A_2 = A_1XA_1^\dagger &= \begin{pmatrix} 0 & e^{-i\pi\theta} \\ e^{i\pi\theta} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-i\pi\theta} \\ e^{i\pi\theta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-2i\pi\theta} \\ e^{2i\pi\theta} & 0 \end{pmatrix}, \\
&\vdots \\
A_p = A_{p-1}XA_{p-1}^\dagger &= \begin{pmatrix} 0 & e^{-2^p i\pi\theta} \\ e^{2^p i\pi\theta} & 0 \end{pmatrix}.
\end{aligned}$$

If $A \in \mathcal{C}_1^{(k+1)}$ for some $k \geq 0$, then by definition $A_1 \in \mathcal{C}_1^{(k)}$, $A_2 \in \mathcal{C}_1^{(k-1)}$, …, $A_k \in \mathcal{C}^{(1)} = \mathcal{P}_1$. However, notice by the form of A_k that $A_k = X \Leftrightarrow \theta = c/2^{k-1}$, $A_k = Y \Leftrightarrow \theta = c/2^k$, and $A_k \neq Z \; \forall \; \theta$, where the angle θ is in its most reduced form. □.

Corollary 1 *Strongly transversal logical gates $Z(\theta)^{\otimes n}$ on CSS stabilizer codes must be composed of individual rotations that are an element of the Clifford hierarchy, that is $Z(\theta) \in \mathcal{C}_1^{(k)}$, for some value of k. Moreover, the logical gate that is implemented must also be an element of Clifford hierarchy on the logically encoded subspace.*

Proof. The first statement follows from Theorem 1 and Proposition 1. The second statement follows by considering the action of the individual rotations on the logical states written out in their expansion in terms of the computational basis. Since the individual phases must be of the form $\theta = c/2^k$, any logical phase will be a multiple of such a phase. □.

3.2 Stabilizer codes

We have proven that the only possible strongly transversal operations for CSS codes must be composed of rotations of the form $Z(a/2^k)$. The proof relied upon the fact that for CSS codes it is trivial to write logical states expressed as a sum of states in the computational basis such that orthogonal logical states do not share common computational states in their sums. Namely, if $\mathcal{S}_X$ represents the stabilizer group formed by all the X stabilizer generators $\mathcal{S}_X =$

$\langle G_i^X \rangle$ of a given CSS code $\mathcal{C}_\mathcal{S}$, then the logical basis states can be expressed as follows:

$$|0_L\rangle = \frac{1}{2^{|\mathcal{S}_X|/2}} \prod_i (I + G_i^X)|0\rangle^{\otimes n} = \frac{1}{\sqrt{|\mathcal{S}_X|}} \sum_{s \in \mathcal{S}_X} |s\rangle,$$
$$X_{L_i}|0_L\rangle = \frac{1}{2^{|\mathcal{S}_X|/2}} \prod_i (I + G_i^X) X_{L_i}|0\rangle^{\otimes n} = \frac{1}{\sqrt{|\mathcal{S}_X|}} \sum_{s \in \mathcal{S}_X} |s \otimes g_{X_{L_i}}\rangle,$$

where X_{L_i} is the logical X operators composed of individual X operators. Importantly, for two orthogonal states $X_{L_i}|0_L\rangle$ and $X_{L_j}|0_L\rangle$, any two elements of the sum will be orthogonal, that is $\langle s \otimes g_{X_{L_i}} | s' \otimes g_{X_{L_j}} \rangle = 0, \ \forall \ s, \ s' \in \mathcal{S}_X$. This fact is important for showing the restriction of the transversal Z rotations as it implies that any transversal phase rotation must introduce the same phase on all elements of the sum in order to remain in the codespace. Therefore, in order to impose the same argument for restricting the phase in the case of stabilizer codes, we must find a basis for a general stabilizer code that has the same properties with respect to non-overlapping computational basis states when written out as a sum over computational basis states. The following Lemma 1 and Corollary 2 will show such a construction.

Lemma 1 *Given a $[[n, k, d]]$ stabilizer code $\mathcal{C}_\mathcal{S}$, one can always find a set of 2^k logical basis states of the following form:*

$$|\psi_m\rangle = \sum_l i^{a_{m,l}} |m_l\rangle, \tag{11}$$

where $a_{m,l}$ is an integer and m_l is an n-bit binary string such that two different logical basis states cannot share any elements in the computation basis expansion. That is, given $|\psi_p\rangle$, $|\psi_q\rangle$ such that $p \neq q$ then $\langle p_s | q_t \rangle = 0 \ \forall \ s, t$.

Proof. Let the stabilizer generators of $\mathcal{S}$ be given by $\langle G_i \rangle_{i=1}^{n-k}$, and the logical Pauli operators be given by $X_{L,j}$, $Z_{L,j}$ for $1 \leq j \leq k$, satisfying $X_{L,j} Z_{L,l} = (-1)^{\delta_{jl}} Z_{L,l} X_{L,j}$. There must exist at least one computational basis state that has non-zero overlap with the stabilizer codespace $\mathcal{C}_\mathcal{S}$. Without loss of generality, we assume that $|0\rangle^{\otimes n}$ is such a state. Then, the following state is a codestate of $\mathcal{C}_\mathcal{S}$,

$$|\phi\rangle = \frac{1}{2^{(n-k)/2}} \prod_{i=1}^{n-k} (I + G_i)|0\rangle^{\otimes n} = \frac{1}{2^{(n-k)/2}} \sum_i S_i |0\rangle^{\otimes n}$$
$$= \frac{1}{2^{(n-k)/2}} \sum_i |s_i\rangle,$$

where $\{S_i\}$ are the set of all stabilizers generated by $\langle G_i \rangle_{i=1}^{n-k}$ and we have defined the state $|s_i\rangle = S_i|0\rangle^{\otimes n}$. Consider the action of two anti-commuting logical Pauli operators $X_{L,1}$, $Z_{L,1}$, on the state $|\phi\rangle$. We know that $|\phi\rangle$ cannot be an eigenstate of both operators, as no state can be a joint eigenstate of two anti-commuting operators. Therefore, we can consider the following two cases: either $|\phi\rangle$ is an eigenstate of one of the operators, or $|\phi\rangle$ is not an eigenstate of either operator. We shall consider the case of the former first.

Without loss of generality, assume that $Z_{L,1}|\phi\rangle = |\phi\rangle = |\psi_1\rangle$ and $X_{L,1}|\phi\rangle = |\psi_2\rangle \neq \alpha|\psi_1\rangle$ (where α is a global phase). Consider the action of $X_{L,1}|\phi\rangle$:

$$\begin{aligned} X_{L,1}|\phi\rangle &= \frac{1}{2^{(n-k)/2}} \prod_{i=1}^{n-k} (I + G_i) X_{L,1}|0\rangle^{\otimes n} \\ &= \frac{1}{2^{(n-k)/2}} \sum_i S_i |g_{X_{L,1}}\rangle, \end{aligned}$$

if $|g_{X_{L,1}}\rangle = |s_j\rangle$ for some j then after the action of the sum over stabilizer operators, the final state $X_{L,1}|\phi\rangle = \alpha|\phi\rangle$ would be a contradiction. Therefore, the state $|g_{X_{L,1}}\rangle$ must be a computational basis state that is not present in the expansion of $|\phi\rangle$, and moreover, each element of the state $|\psi_2\rangle$ must have zero overlap with the state $|\psi_1\rangle$,

$$|\psi_2\rangle = X_{L,1}|\phi\rangle = \frac{1}{2^{(n-k)/2}} \sum_i |g_{X_{L,1}} \oplus s_i\rangle.$$

Therefore, two states of the form of Equation 11 have been constructed. Consider now the action of the next pair of anti-commuting logical Paulis $X_{L,2}$, $Z_{L,2}$ on the state $|\psi_1\rangle$. Again, since $|\psi_1\rangle$ cannot be a joint eigenstate of both operators, without loss of generality, assume $X_{L,2}|\psi_1\rangle = |\psi_3\rangle \neq \alpha|\psi_1\rangle$. Moreover, it must be that $|\psi_3\rangle \neq \alpha|\psi_2\rangle$ or else the following would be true: $X_{L,1}X_{L,2}|\psi_1\rangle = \alpha X_{L,1}|\psi_2\rangle = \alpha|\psi_1\rangle$, which would imply that $|\psi_1\rangle$ is an eigenstate of two anti-commuting operators, $Z_{L,1}$ and $X_{L,1}X_{L,2}$, which results in a contradiction. Therefore, we can express the state $|\psi_3\rangle$ as follows:

$$\begin{aligned} |\psi_3\rangle = X_{L,2}|\psi_1\rangle &= \frac{1}{2^{(n-k)/2}} \prod_{i=1}^{n-k} (I + G_i) X_{L,2}|0\rangle^{\otimes n} \\ &= \frac{1}{2^{(n-k)/2}} \sum_i |g_{X_{L,2}} \oplus s_i\rangle, \end{aligned}$$

where each state in the computational basis expansion must have zero overlap with the states $|\psi_1\rangle$ and $|\psi_2\rangle$. Finally. consider the action of the same anti-commuting pair on the state $|\psi_2\rangle$. As will be shown below, it does not matter which we choose, and thus, without loss of generality, we assume it to be the operator $Z_{L,2}$. First note that if $Z_{L,2}|\psi_2\rangle = \alpha|\psi_1\rangle$, then $|\psi_1\rangle$ would be the joint eigenstate of two anti-commuting Paulis, $Z_{L,1}$ and $X_{L,1}Z_{L,2}$, which is a contradiction. Moreover, if $Z_{L,2}|\psi_2\rangle = \alpha|\psi_3\rangle$ then again $|\psi_1\rangle$ would be the joint eigenstate of two anti-commuting Paulis, $Z_{L,1}$ and $X_{L,1}X_{L,2}Z_{L,2}$. Therefore, the state $Z_{L,2}|\psi_2\rangle = |\psi_4\rangle$ must have zero overlap with the previous established states and can be expressed as follows:

$$\begin{aligned} |\psi_4\rangle = Z_{L,2}|\psi_2\rangle &= \frac{1}{2^{(n-k)/2}} \prod_{i=1}^{n-k} (I + G_i) Z_{L,2} X_{L,1}|0\rangle^{\otimes n} \\ &= \frac{1}{2^{(n-k)/2}} \sum_i |g_{X_{L,1}} \oplus g_{Z_{L,2}} \oplus s_i\rangle. \end{aligned}$$

Notice the form of $|\psi_3\rangle$ and $|\psi_4\rangle$. By taking the previous states $|\psi_1\rangle$ and $|\psi_2\rangle$ and a pair of non-commuting logical Paulis, for each state in the previous level, we can construct a new state by applying the logical Pauli for which it is not an eigenstate. One can continue the same constructive process for preparing states of the form of Equation 11 by taking the m-th pair of anti-commuting logical operators and the 2^{m-1} previous constructed states, thereby creating another 2^{m-1} set of orthogonal states, following similar constraints as laid out above. Applying this to all pairs of logical Pauli gates for the given code, 2^k basis states for the codespace can be constructed.

In the case when the state $|\phi\rangle$ is not an eigenstate of either of the first two logical Pauli gates, $Z_{L,1}$ and $X_{L,1}$, the following modifications have to be made. Let $|\psi_1\rangle = Z_{L,1}|\phi\rangle$ and $|\psi_2\rangle = X_{L,1}|\phi\rangle$. If $|\psi_1\rangle = |\psi_2\rangle$ then by redefining the logical Pauli $\tilde{Z_{L,1}} = X_{L,1}Z_{L,1}$ and $|\tilde{\psi_1}\rangle = |\phi\rangle$, we recover the original case where $\tilde{Z_{L,1}}|\tilde{\psi_1}\rangle = |\tilde{\psi_1}\rangle = |\phi\rangle$ and $|\psi_2\rangle = X_{L,1}|\phi\rangle$. Therefore, the final case to consider is where $|\psi_1\rangle \neq \alpha|\psi_2\rangle$. In this case, they must not have overlapping states in the computational basis, and their expansion can be written as follows:

$$|\psi_1\rangle = Z_{L,1}|\phi\rangle = \frac{1}{2^{(n-k)/2}} \sum_i |g_{Z_{L,1}} \oplus s_i\rangle,$$
$$|\psi_2\rangle = X_{L,1}|\phi\rangle = \frac{1}{2^{(n-k)/2}} \sum_i |g_{X_{L,1}} \oplus s_i\rangle.$$

Again, as in the previous case, consider the action of the pair of logical Pauli gates $X_{L,2}$ and $Z_{L,2}$ on the state $|\psi_1\rangle$. Without loss of generality, assume that $|\psi_1\rangle$ is not an eigenstate of $X_{L,2}$. Unlike the previous case, it is now possible that $X_{L,2}|\psi_1\rangle = |\psi_2\rangle$. However, if this holds, then redefining $\tilde{Z_{L,1}} = X_{L,1}Z_{L,1}X_{L,2}$ and $|\tilde{\psi_1}\rangle = |\phi\rangle$ we recover the original case with $\tilde{Z_{L,1}}|\tilde{\psi_1}\rangle = |\tilde{\psi_1}\rangle$ and $X_{L,2}|\tilde{\psi_1}\rangle = |\psi_3\rangle$, as well as all redefined operators satisfying the appropriate commutation relations. Otherwise, we can conclude that $X_{L,2}|\psi_1\rangle = |\psi_3\rangle$ and must be orthogonal to the two previous states as well as have the following form:

$$|\psi_3\rangle = X_{L,2}|\psi_1\rangle = \frac{1}{2^{(n-k)/2}} \sum_i |g_{Z_{L,1}} \oplus g_{X_{L,2}} \oplus s_i\rangle.$$

Therefore, continuing in the same manner as in the previous case, we can construct the set of 2^k logical basis states of the form given by Equation 11. □.

Corollary 2 *Suppose $\mathcal{C}_\mathcal{S}$ is an n-qubit stabilizer containing k logical qubits. Given 2^k states $|\varphi_m\rangle \in \mathcal{C}_\mathcal{S}$ whose expansion in terms of the computational basis states are all non-overlapping, then these states must be of the form*

$$|\varphi_m\rangle = \sum_l i^{a_{m,l}} |m_l\rangle.$$

Proof. Since all 2^k states are elements of $\mathcal{C}_\mathcal{S}$, they must be convex combinations of any basis chosen for $\mathcal{C}_\mathcal{S}$. Choose the basis given by the states from Lemma 1. Then, if any of the $|\varphi_m\rangle$ were a convex combination of states from such a basis, there must be at least one overlapping state relative to the individual states in its computational basis state expansion. Otherwise the dimension of the logical Hilbert space would be too small to fit all of these logical states. □.

We know by Lemma 1, that the computational basis state expansion of $|1_L\rangle$ will be a sum of states such that each state differs from those in the representation of $|0_L\rangle$. Moreover, the gate $Z(\theta)^{\otimes n}$ will preserve all of these basis states, potentially introducing relative phases between the elements of the sum, however by Corollary 2 the resulting states must also form a basis for the stabilizer code, and in particular for the case of an automorphism the states must form the same logical basis. We can now proceed to proving the set of transversal Z rotations for stabilizer codes in a very similar manner to Theorem 1.

Proposition 2 *A nontrivial qubit stabilizer codes (distance $d \geq 2$) can only have strongly transversal Z rotations which are of the form $Z(a/2^k)$.*

Proof. We can represent a general string of Pauli matrices as a binary string using the associations: $\{I \to 00, X \to 10, Y \to 11, Z \to 01\}$. We will write an n-qubit Pauli string as a $2n$-bit string $f = (g|h)$. Here we have separated the string into the two substrings of n-bits (an X (g) and Z (h) substring). We can express the expansion of $|0_L\rangle$ and $|1_L\rangle$ in terms of binary strings of the X operators as follows (without loss of generality assume that $|0\rangle^{\otimes n}$ is in the stabilizer codespace):

$$\begin{aligned}
|0_L\rangle &= (I + Z_L)\prod_i (I + G_i)|0\rangle^{\otimes n} \\
&= |g_0\rangle + (i)^{a_{Z_L}}|g_{Z_L}\rangle + \sum_{i_1}((i)^{a_{i_1}}|g_{i_1}\rangle + (i)^{a_{Z_L \oplus i_1}}|g_{Z_L} \oplus g_{i_1}\rangle) \\
&\quad + \sum_{i_1 < i_2}((i)^{a_{i_1 \oplus i_2}}|g_{i_1} \oplus g_{i_2}\rangle + (i)^{a_{Z_L \oplus i_1 \oplus i_2}}|g_{Z_L} \oplus g_{i_1} \oplus g_{i_2}\rangle) \\
&\quad + \ldots + ((i)^{a_{i_1 \oplus \cdots \oplus i_{|G|}}}|g_{i_1} \oplus \cdots \oplus g_{i_{|G|}}\rangle + (i)^{a_{Z_L \oplus i_1 \oplus \cdots \oplus i_{|G|}}}|g_{Z_L} \oplus g_{i_1} \oplus \cdots \oplus g_{i_{|G|}}\rangle)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
|1_L\rangle &= X_L|0_L\rangle \\
&= (i)^{a_{X_L}}|g_{X_L}\rangle + (i)^{a_{X_L \oplus Z_L}}|g_{X_L} \oplus g_{Z_L}\rangle \\
&\quad + \sum_{i_1}((i)^{a_{X_L \oplus i_1}}|g_{X_L} \oplus g_{i_1}\rangle + (i)^{a_{X_L \oplus Z_L \oplus i_1}}|g_{X_L} \oplus g_{Z_L} \oplus g_{i_1}\rangle) + \\
&\quad + \ldots + ((i)^{a_{X_L \oplus i_1 \oplus \cdots \oplus i_{|G|}}}|g_{X_L} \oplus g_{i_1} \oplus \cdots \oplus g_{i_{|G|}}\rangle \\
&\qquad + (i)^{a_{X_L \oplus Z_L \oplus i_1 \oplus \cdots \oplus i_{|G|}}}|g_{X_L} \oplus g_{Z_L} \oplus g_{i_1} \oplus \cdots \oplus g_{i_{|G|}}\rangle)
\end{aligned} \tag{13}$$

Here g_0 is the "all-zeros" string, $g_i,\ g_{X_L},\ g_{Z_L}$ is a binary string corresponding to the location of the X Pauli operators in the respective operators $G_i,\ X_L,\ Z_L$, and $\oplus$ is bitwise XOR. We have added the additional projector $(I + Z_L)$ in order to project to the eigenspace of Z_L (which may contain X operators). There are relative phases between the different states as the Y and Z Pauli operators from the stabilizers G_i may introduce phases, these phases are characterized by the term $(i)^{a_{\vec{\mu}}}$, where $a_{\vec{\mu}}$ is an integer setting the phase for the particular ket $|\vec{\mu}\rangle$. Importantly, the computational basis states in the sum are all a function of the bit strings g_i which encode the location of X (or Y) operators in the stabilizer G_i. If $|0\rangle^{\otimes n}$ is not in the codespace, another computational basis state can be chosen and the proof would follow identically.

The effect of applying a $Z(\theta)^{\otimes n}$ rotation to the computational basis state composed of the string g will be:

$$Z(\theta)^{\otimes n}|g\rangle = e^{i\pi\theta|g|}|g\rangle. \tag{14}$$

Using the result of Lemma 1 we see that for $Z(\theta)^{\otimes n}$ to be a valid logical operator it cannot introduce relative phases between the states in the expansions of Eqs. 12 and 13, and as such must introduce the same phase to all the states in the expansion (there can be a relative phase between $|0_L\rangle$ and $|1_L\rangle$ however).

For the CSS codes, we assumed that $X_L(Z_L)$ consisted of single qubit unitaries X and I (Z and I). In this case we make no such assumptions.

$$\begin{aligned}
\theta|g_{i_1}| &= 0 \bmod 2 \\
\theta|g_{i_1} \oplus g_{i_2}| &= 0 \bmod 2 \\
&\vdots \\
\theta|g_{i_1} \oplus ... \oplus g_{i_{|G|}}| &= 0 \bmod 2 \\
\\
\theta|g_{X_L}| &\neq 0 \bmod 2 \\
\theta|g_{X_L} \oplus g_{i_1}| &\neq 0 \bmod 2 \\
&\vdots \\
\theta|g_{X_L} \oplus g_{i_1} \oplus ... \oplus g_{i_{|G|}}| &\neq 0 \bmod 2 \\
\\
\theta|g_{Z_L}| &= 0 \bmod 2 \\
&\forall 0 < i_1 < i_2 < ... < i_{|G|} \leq |G|
\end{aligned}$$

The additional requirement is from $[Z_L, Z(\theta)^{\otimes n}] = 0$. Otherwise $Z(\theta)^{\otimes n}|0_L\rangle = |0_L\rangle \neq Z(\theta)^{\otimes n} Z_L|0_L\rangle$.

These constraints are the same (actually slightly more constraining) as the constraints from the proof of Theorem 1 and the proof carries through analogously. □.

Corollary 3 *Given a nontrivial qubit stabilizer code (distance $d \geq 2$), a strongly transversal diagonal logical gate $U = \bigotimes_i Z(\theta)$ can only implement a logical diagonal gate in the Clifford hierarchy, that is $U = Z_L(c/2^k) \in \mathcal{C}_1^{(k+1)}$.*

Proof. Consider the action of the transversal rotation on the logical basis states for the the stabilizer code chosen according to Lemma 1, that is:

$$\bigotimes_{i=1}^{n} Z(\theta)|\psi_m\rangle = \sum_l i^{a_{m,l}} \bigotimes_{i=1}^{n} Z(\theta)|m_l\rangle = \sum_l i^{a_{m,l}} e^{i\pi\theta|m_l|}|m_l\rangle. \tag{15}$$

Therefore, the action of $U = \prod_i Z(\theta)$ will result in orthogonal logical basis states to remain orthogonal by the following:

$$\begin{aligned}
\langle\psi_n|U|\psi_m\rangle &= (\sum_p (-i)^{a_{n,p}} \langle n_p|)(\sum_l i^{a_{m,l}} e^{i\pi\theta|m_l|}|m_l\rangle) &(16)\\
&= \sum_{l,p} (-i)^{a_{n,p}} i^{a_{m,l}} e^{i\pi\theta|m_l|} \langle n_p|m_l\rangle &(17)\\
&= 0, &(18)
\end{aligned}$$

since by the choice of basis $\langle n_p|m_l\rangle\ \forall\ l,p$ for $m \neq n$. As such, for this choice of logical basis the logical gate is by definition diagonal. By Proposition 2, the only allowable rotations are of the form $Z(\theta) = Z(a/2^k)$ and the logical gate must introduce the same phase to all the computational basis states in the expansion of the logical states. Therefore, the logical rotation must have the following form:

$$\bigotimes_{i=1}^{n} Z(a/2^k)|\psi_m\rangle = \sum_l i^{a_{m,l}} e^{i\pi|m_l|a/2^k}|m_l\rangle \tag{19}$$

$$= e^{i\pi|m_1|a/2^k} \sum_l i^{a_{m,l}}|m_l\rangle \tag{20}$$

$$= e^{i\pi c/2^k} \sum_l i^{a_{m,l}}|m_l\rangle. \tag{21}$$

Therefore, since the phases are restricted to be of the form $e^{i\pi c/2^k}$ the logical gate must be in the $(k+1)$-th level of the Clifford hierarchy. □.

3.3 Relaxing strong transversality

We have proven a restriction on the set of diagonal strongly transversal gates for stabilizer codes, however there is no physical reason requiring all qubits in the code have the same rotation applied to them. Namely, each physical qubit could in theory undergo a different rotation. In this section, we show that this added freedom will still not allow an increased freedom in the set of diagonal rotations that one can implement.

First, notice that if the transversal operator includes the identity anywhere, it will have no effect on that qubit and therefore we can formulate the overlap conditions on a new code with that qubit removed. Unlike puncturing a code we are not actually removing the qubit from the code, it is simply not included in the overlap conditions. In what follows, we will assume this process has been implemented, and no identity operators remain. We can do this without any difficulties since our overlap conditions make no use of the commuting properties of stabilizer generators. To prove this more general case we will introduce a new tool; the *decompression lemma*:

Lemma 2 *If an $[[n,k,d]]$ code exists with a transversal $Z_T(\theta) = Z(\theta_1)\otimes Z(\theta_2)\otimes ... \otimes Z(m\theta_n)$ gate, then there exists an $[[n+m-1,k,2]]$ code with a transversal $Z_T'(\theta) = Z(\theta_1)\otimes Z(\theta_2)\otimes ... \otimes (Z(\theta_n)^{\otimes m})$ gate.*

Proof. If a code admits a transversal operation $Z_T(\theta) = Z(\theta_1)\otimes Z(\theta_2)\otimes ... \otimes Z(m\theta_n)$, this code's X stabilizer generators and logical operators clearly satisfy the overlap conditions for the transversal Z operator. Now, if we take the last column of the check matrix and repeat it m times, we have a new code which has distance two since a repeated column in the check matrix creates a weight two logical operator. It is easy to see that $Z_T'(\theta) = Z(\theta_1)\otimes Z(\theta_2)\otimes ... \otimes (Z(\theta_n)^{\otimes m})$ satisfies the same overlap conditions on the new code that Z_T satisfied for the original code, and it follows that $Z_T'(\theta)$ implements the same logical operation as $Z_T(\theta)$. Here we have not specified the Z stabilizer generators and it should be noted that in the new code obtained after applying the decompression lemma, there will be $m-1$ new Z stabilizer generators. □.

Before proceeding to the general statement for transversal gates, we present a Lemma that rules out the possibility of irrational angles contributing to the logical gate. We present the proof of the following result in Appendix 1.

Lemma 3 *Suppose Q is a $[[n,k,d]]$ error-detecting stabilizer code ($d \geq 2$) with a transversal logical gate of the form $Z_T(\theta) = \bigotimes_{i=1}^n Z(\theta_i)$ such that $\theta_i \neq p_i/q_i$ for $i \in \Gamma$, where Γ is an arbitrary set of integers from the set $[1,\ldots,n]$. Then, one can replace the irrational angles by the identity gate and obtain the same logical gate. That is, let*

$$\beta_i = \begin{cases} 0 & i \in \Gamma \\ \theta_i & i \notin \Gamma \end{cases},$$

then $\forall\ |\psi\rangle \in Q$, $\bigotimes_{i=1}^n Z(\theta_i)|\psi\rangle = \bigotimes_{i=1}^n Z(\beta_i)|\psi\rangle$.

Proposition 3 *A nontrivial stabilizer code (distance $d \geq 2$) can only have transversal Z rotations which are of the form $Z(a/2^k)$.*

In this case we have a transversal gate

$$Z_T(\theta) := Z(\theta_1) \otimes Z(\theta_2) \otimes ... \otimes Z(\theta_n). \tag{22}$$

As per Lemma 3, irrational angles must cancel and therefore cannot contribute to the logical gate. As such, we can assume that $Z(\theta_i)$ is rational. Therefore, we have a transversal gate of the form

$$Z_T(\theta) := Z(p_1/q_1) \otimes Z(p_2/q_2) \otimes ... \otimes Z(p_n/q_n). \tag{23}$$

We can find the least common denominator q of $q_1, ..., q_n$ and express this as

$$Z_T(\theta) := Z(p_1'/q) \otimes Z(p_2'/q) \otimes ... \otimes Z(p_n'/q). \tag{24}$$

We can also use that $Z(2 + p/q) = Z(p/q)$ to claim $p_i'/q_i \in [0,2)$ and also assume that $Z(p/q) \neq I$, else we could ignore this operator and the qubit it acted upon as they would not affect the overlap conditions.

Now, we repeatedly apply the decompression lemma until we have an $[[\sum_i p_i, k, 2]]$ code with a transversal gate

$$Z_T(\theta) := Z(1/q) \otimes ... \otimes Z(1/q). \tag{25}$$

We have now reduced these more general gates to strongly transversal gates, and the proof follows as before.

3.4 Classification of all single qubit logical gates

Recall that Zeng *et al.* showed that all single-qubit logical transversal gates for a stabilizer code must have the form [10]:

$$U = L\left(\bigotimes_{j=1}^{n} \mathrm{diag}(1, e^{i\pi\theta_j})\right) R^\dagger P_\pi, \tag{26}$$

where P_π is a permutation matrix of the physical qubits while R and L are transversal single-qubit Clifford operators. Given that the action of the transversal unitary is given by the composition of a transversal diagonal gate with Clifford operations, by classifying all possible mappings between stabilizer codes given by transversal diagonal operations we can classify all possible transversal gates. We summarize this statement in the following two results.

Proposition 4 *Given two nontrivial (distance $d \geq 2$) n-qubit stabilizer codes $\mathcal{C}_\mathcal{S}$ and $\mathcal{C}_\mathcal{T}$ consisting of r logical qubits, transversal Z rotations of the form $D = \otimes_{j=1}^{r} Z(\theta_j)$ which map $\mathcal{C}_\mathcal{S} \longrightarrow \mathcal{C}_\mathcal{T}$ (and possibly apply a logical unitary in the process) must be of the form $D = \otimes_{j=1}^{r} Z(a_j/2^{k_j})$.*

Proof. Let $\{|\psi_m\rangle_{m=1}^{2^r}\}$ form a logical basis set for the stabilizer code and choose $\mathcal{C}_\mathcal{S}$ to be of the form outlined in Lemma 1. Then, given a transversal application of diagonal gates, the resulting set of states must also form a basis for a stabilizer code (in this case chosen to be $\mathcal{C}_\mathcal{T}$) such that each individual basis state will have the same expansion in terms of the computational basis states. However, the transversal application of diagonal gates may result in relative phases between the states; since the states must form a basis for the stabilizer code of the type given by Lemma 1, the relative phases must be powers of i. Therefore, the transformed states read:

$$|\psi_m\rangle = \sum_l i^{a_{m,l}} |m_l\rangle \xrightarrow{D} |\varphi_m\rangle = e^{i\pi\phi_m} \sum_l i^{a_{m,l}} i^{c_{m,l}} |m_l\rangle.$$

Therefore, repeating the action of the diagonal transversal gate 4 times must return the original set of basis states (with the possible introduction of a phase).

$$|\psi_m\rangle = \sum_l i^{a_{m,l}} |m_l\rangle \xrightarrow{D^4} e^{i4\pi\phi_m} |\psi_m\rangle = e^{i4\pi\phi_m} \sum_l i^{a_{m,l}} |m_l\rangle.$$

We are now back to the original case of classifying transversal diagonal gates for logical gates returning to the same codespace, which we have already classified to be rotations of the form $D = \otimes_{j=1}^{n} Z(a_j/2^{k_j})$. Therefore, we are similarly restricted for the case of logical mappings between stabilizer codes. $\square$.

Corollary 4 *Given an n-qubit stabilizer code $\mathcal{C}_\mathcal{S}$ and a transversal unitary U implementing a logical gate, then U must be of the following form:*

$$U = L \left(\bigotimes_{j=1}^{n} Z(a_j/2^{k_j}) \right) R^\dagger P_\pi, \tag{27}$$

and must implement a logical unitary in the Clifford hierarchy.

Proof. First note that by the characterization in Ref [10], a transversal unitary gate must have the form given in Equation 26. Given that P_π is a permutation unitary, it must necessarily map $\mathcal{C}_\mathcal{S}$ to another stabilizer code $\mathcal{C}_{\mathcal{S}_1}$ with the exact same code properties, namely code distance. Additionally, since $R^\dagger$ is a transversal Clifford operation, it will map any stabilizer code $\mathcal{C}_{\mathcal{S}_1}$ to another stabilizer code $\mathcal{C}_{\mathcal{S}_2}$ whose distance is preserved. The distance is preserved as suppose one is given a Pauli error E such that $\mathrm{wt}(E) < d$, where d is the distance. Then for $|\psi_2\rangle \in \mathcal{C}_{\mathcal{S}_2}$, $R(E|\psi_2\rangle) = E'|\psi_1\rangle$, where $|\psi_1\rangle$ must be in $\mathcal{C}_{\mathcal{S}_1}$ and E' is a modified Pauli error of the same weight as E since R is a transversal Clifford. Therefore since E' is correctable by the distance property of $\mathcal{C}_{\mathcal{S}_1}$, the transformed error must remain correctable in $\mathcal{C}_{\mathcal{S}_2}$. Similarly, $L^\dagger$ must map $\mathcal{C}_\mathcal{S}$ to a stabilizer code $\mathcal{C}_{\mathcal{S}_3}$ with the same distance d. Therefore, since $R^\dagger P_\pi$ maps $\mathcal{C}_\mathcal{S}$ to $\mathcal{C}_{\mathcal{S}_2}$ and L maps $\mathcal{C}_{\mathcal{S}_3}$ to $\mathcal{C}_\mathcal{S}$, in order for U to be a logical operation that preserves the codespace, the operator $D = \otimes_{j=1}^{n} Z(\theta_j)$ must be a mapping between two nontrivial stabilizer codes $\mathcal{C}_{\mathcal{S}_2}$ and $\mathcal{C}_{\mathcal{S}_3}$. As such, the operator must be constrained by the result of Proposition 4 and the resulting gate is given by the form in Equation 27. Finally, since U is a result of the application of a transversal gate belonging to the Clifford hierarchy in composition with Clifford gates, and using the fact that Clifford gates must preserve the given level of the Clifford hierarchy, the resulting gate must belong to the Clifford hierarchy by the same argument as given in Corollary 1. □.

4 Multi-block gates

In this section we consider the case of multi-qubit logical gates, where each logical qubit (or codeblock) is encoded in the same $[[n,k,d]]$ error correcting code Q. Zeng *et al.* classified the set of gates that can be transversal across these codeblocks. Namely, if U is a transversal gate on $Q^{\otimes r}$, then for each $j \in [n]$ either $U_j \in \mathcal{L}_r$ or $U_j = L_1 V L_2$, where $L_1, L_2 \in \mathcal{L}_1^{\otimes r}$ are local Clifford gates, and V keeps the linear span of the group elements of $\langle \pm Z_j^{(i)}, i \in [r]\rangle$. This work focuses on the gates V, which must be diagonal in order to preserve the span of the group of Z operators across qubits at a fixed i.

4.1 Strong transversality for two-qubit logical gates

Consider first the implementation of a logical diagonal gate in the case of two codeblocks, where the logical gate is implemented by using a strongly transversal gate. That is, consider the implementation of the diagonal two-qubit logical gate by applying a given two-qubit gate, $U = \sum_j e^{i\pi\theta_j}|j\rangle\langle j|$ transversally $U^{\otimes n}$ among the corresponding pair of qubits between the codeblocks. The desired logical gate to be implemented has the form

$$U_L = \sum_j e^{i\pi\omega_j}|j\rangle\langle j|_L,$$

where the states $\{|j\rangle_L\}_j = \{|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L\}$ are two-qubit logical states spanning the two codeblocks.

As in the single block case, the desired action of the logical gate on the logical states will impose a restriction on the form of the two-qubit physical gates that can be implemented in a strongly transversal manner. In the case of a quantum CSS code, the above logical gate description will have the following form, similar to the construction in Theorem 1:

$$\begin{aligned}
U_L|00\rangle_L &= U^{\otimes n}\prod_{i=1}^{|G_X|}(I+G_{X_i})|0\rangle^{\otimes n}\prod_{j=1}^{|G_X|}(I+G_{X_j})|0\rangle^{\otimes n} \\
&= U^{\otimes n}\Big(|g_0\rangle + \sum_{i_1}|g_{i_1}\rangle + \sum_{i_1<i_2}|g_{i_1}\oplus g_{i_2}\rangle + \ldots + |i_1\oplus\ldots\oplus i_{|G_X|}\rangle\Big) \\
&\quad \otimes \Big(|g_0\rangle + \sum_{j_1}|g_{j_1}\rangle + \sum_{j_1<j_2}|g_{j_1}\oplus g_{j_2}\rangle + \ldots + |j_1\oplus\ldots\oplus j_{|G_X|}\rangle\Big) \\
&= e^{i\pi\omega_{00}}|00\rangle_L.
\end{aligned}$$

Note that each of the $4^{|G_X|}$ states in the summation of the $|00\rangle_L$ state are computational basis states and will not change under the action of $U^{\otimes n}$ except for the possible addition of a phase. Therefore, in order to remain a codeword, all states in the expansion must have the same phase.

Without loss of generality, one can assume that the phase $\theta_{00} = 0$ (this maps to a global phase freedom in the logical gate U_L). Consider now the phases introduced on all $2^{|G_X|}$ states in the expansion of the first qubit, along with the state $|g_0\rangle$ in the expansion of the second qubit. For clarity, we will list the state in the expansion, along with the corresponding condition imposed on its phase.

$$
\begin{aligned}
&|g_0\rangle|g_0\rangle: && n\theta_{00} = \omega_0 = 0 \mod 2\\
&|g_{i_1}\rangle|g_0\rangle: && |g_{i_1}|\theta_{10} + (n - |g_{i_1}|)\theta_{00} = |g_{i_1}|\theta_{10} = 0 \mod 2\\
&|g_{i_1} \oplus g_{i_2}\rangle|g_0\rangle: && |g_{i_1} \oplus g_{i_2}|\theta_{10} = 0 \mod 2\\
&\vdots && \\
&|g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}\rangle|g_0\rangle: && |g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}|\theta_{10} = 0 \mod 2
\end{aligned}
$$

These conditions are equivalent to the conditions derived on a single codeblock and therefore θ_{10} is restricted to be an integer multiple of $1/2^c$ (when paired with the appropriate restrictions on the logical phase ω_{10} as given below). In a very similar manner, a set of constraints can be obtained for the angle θ_{01} due to the symmetry of the two codes:

$$
\begin{aligned}
&|g_0\rangle|g_{j_1}\rangle: && |g_{j_1}|\theta_{01} + (n - |g_{j_1}|)\theta_{00} = |g_{i_1}|\theta_{01} = 0 \mod 2\\
&|g_0\rangle|g_{j_1} \oplus g_{j_2}\rangle: && |g_{j_1} \oplus g_{j_2}|\theta_{01} = 0 \mod 2\\
&\vdots && \\
&|g_0\rangle|g_{j_1} \oplus \ldots \oplus g_{j_{|G_X|}}\rangle: && |g_{j_1} \oplus \ldots \oplus g_{j_{|G_X|}}|\theta_{01} = 0 \mod 2
\end{aligned}
$$

Therefore, the phase angle θ_{01} will also be restricted to be an integer multiple of $1/2^c$.

In order to obtain a restriction on the phase angle θ_{11}, higher order state vectors must be considered in both expansions of the logical $|0\rangle_L$ states. Consider the full summation over states in the expansion of the first block, along with state vectors in the second block of the form $|g_{j_1}\rangle$.

$$
\begin{aligned}
&|g_{i_1}\rangle|g_{j_1}\rangle: && |g_{i_1} \wedge g_{j_1}|(\theta_{11} - \theta_{01} - \theta_{10}) + |g_{i_1}|\theta_{10} + |g_{j_1}|\theta_{01} = 0 \mod 2\\
&|g_{i_1} \oplus g_{i_2}\rangle|g_{j_1}\rangle: && |(g_{i_1} \oplus g_{i_2}) \wedge g_{j_1}|(\theta_{11} - \theta_{01} - \theta_{10})\\
& && + |g_{i_1} \oplus g_{i_2}|\theta_{10} + |g_{j_1}|\theta_{01} = 0 \mod 2\\
&\vdots && \\
&|g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}\rangle|g_{j_1}\rangle: && |(g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}) \wedge g_{j_1}|(\theta_{11} - \theta_{01} - \theta_{10})\\
& && + |g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}|\theta_{10} + |g_{j_1}|\theta_{01} = 0 \mod 2
\end{aligned}
$$

First notice that every term other than the first term in each of the conditions will be equal to zero (mod 2), as a result of the set of conditions imposed on the phase angles θ_{01} and θ_{10}. Therefore, what remains are conditions on the phase difference $\theta'_{11} = (\theta_{11} - \theta_{01} - \theta_{10})$, which is equivalent to a condition on θ_{11}. Moreover, consider the expansion of the direct sum as given by Equation 8:

$$
\begin{aligned}
|(g_{i_1} \oplus g_{i_2}) \wedge g_{j_1}|\theta'_{11} &= |(g_{i_1} \wedge g_{j_1}) \oplus (g_{i_2} \wedge g_{j_1})|\theta'_{11} \\
&= \Big(|g_{i_1} \wedge g_{j_1}| + |g_{i_2} \wedge g_{j_1}| - 2|g_{i_1} \wedge g_{i_2} \wedge g_{j_1}|\Big)\theta'_{11} = 0 \\
\Rightarrow 2|g_{i_1} \wedge g_{i_2} \wedge g_{j_1}|\theta'_{11} = 0,
\end{aligned}
$$

where the implication in the final line is due to $|(g_i \wedge g_{j_1})|\theta'_{11} = 0$, for all i from the first set of constraints of the state vector $|g_{i_1}\rangle|g_{j_1}\rangle$. Similarly,

$$
\begin{aligned}
|(g_{i_1} \oplus g_{i_2} \oplus g_{i_3}) \wedge g_{j_1}|\theta'_{11} &= |(g_{i_1} \wedge g_{j_1}) \oplus (g_{i_2} \wedge g_{j_1}) \oplus (g_{i_3} \wedge g_{j_1})|\theta'_{11} \\
&= \Big(|g_{i_1} \wedge g_{j_1}| + |g_{i_2} \wedge g_{j_1}| + |g_{i_3} \wedge g_{j_1}| \\
&\quad - 2|g_{i_1} \wedge g_{i_2} \wedge g_{j_1}| - 2|g_{i_1} \wedge g_{i_3} \wedge g_{j_1}| - 2|g_{i_2} \wedge g_{i_3} \wedge g_{j_1}| \\
&\quad + 4|g_{i_1} \wedge g_{i_2} \wedge g_{i_3} \wedge g_{j_1}|\Big)\theta'_{11} = 0 \\
\Rightarrow 4|g_{i_1} \wedge g_{i_2} \wedge g_{i_3} \wedge g_{j_1}|\theta'_{11} = 0.
\end{aligned}
$$

The final implication is a consequence of the above condition on $2|g_{i_1} \wedge g_{i_2} \wedge g_{j_1}|$. The same procedure will follow for all states in the expansion, and conditions on θ'_{11} can thus be modified as:

$$
\begin{aligned}
&|g_{i_1}\rangle|g_{j_1}\rangle : && |g_{i_1} \wedge g_{j_1}|\theta'_{11} = 0 && \bmod 2 \\
&|g_{i_1} \oplus g_{i_2}\rangle|g_{j_1}\rangle : && 2|g_{i_1} \wedge g_{i_2} \wedge g_{j_1}|\theta'_{11} = 0 && \bmod 2 \\
&|g_{i_1} \oplus g_{i_2} \oplus g_{i_3}\rangle|g_{j_1}\rangle : && 4|g_{i_1} \wedge g_{i_2} \wedge g_{i_3} \wedge g_{j_1}|\theta'_{11} = 0 && \bmod 2 \\
&\vdots \\
&|g_{i_1} \oplus \ldots \oplus g_{i_{|G_X|}}\rangle|g_{j_1}\rangle : && 2^{|G_X|-1}|g_{i_1} \wedge \ldots \wedge g_{i_{|G_X|}} \wedge g_{j_1}|\theta'_{11} = 0 && \bmod 2.
\end{aligned}
$$

Given that the two codebocks are encoded in the same quantum error correcting code, these conditions are a modified version of the conditions derived in the single block case, where an extra factor of 2 is present in all of the constraints. This extra factor of 2 will have a consequence on the type of logical gates that can be implemented transversally and will limit the 2-qubit gates to reside in the same level of the Clifford hierarchy as the 1-qubit gates that can be implemented for a given code.

Finally, consider the action of the strongly transversal gate on the logical states $|01\rangle_L$, $|10\rangle_L$, and $|11\rangle$ when performing a logical X_L on the appropriate qubit(s). The resulting set of conditions impose a restriction on the logical phases ω_{01}, ω_{10}, and ω_{11}. For the logical state $|01\rangle_L$ the conditions are:

$$
\begin{aligned}
&|g_0\rangle|g_L\rangle : && |g_L|\theta_{01} = \omega_{01} && \bmod 2 \\
&|g_0\rangle|g_{j_1} \oplus g_L\rangle : && |g_{j_1} \oplus g_L|\theta_{01} = \omega_{01} && \bmod 2 \\
&\vdots \\
&|g_0\rangle|g_{j_1} \oplus \ldots \oplus g_{j_{|G_X|}} \oplus g_L\rangle : && |g_{j_1} \oplus \ldots \oplus g_{j_{|G_X|}} \oplus g_L|\theta_{01} = \omega_{01} && \bmod 2.
\end{aligned}
$$

Therefore these restrictions, along with the conditions for the phase θ_{01}, will impose the restriction of the form of phases that can be applied, as shown in the single block case. In the exact same manner, restriction on the phases θ_{10} and ω_{10} are obtained. Finally, in order to obtain restrictions on the phase ω_{11}, consider the following:

$$\begin{aligned}
&|g_L\rangle|g_L\rangle : && |g_L|\theta_{11} = \omega_{11} \mod 2\\
&|g_L\rangle|g_{j_1}\oplus g_L\rangle : && |g_{j_1}\oplus g_L|\theta_{11} = \omega_{11} \mod 2\\
&\vdots && \\
&|g_L\rangle|g_{j_1}\oplus\ldots\oplus g_{j_{|G_X|}}\oplus g_L\rangle : && |g_{j_1}\oplus\ldots\oplus g_{j_{|G_X|}}\oplus g_L|\theta_{11} = \omega_{11} \mod 2,
\end{aligned}$$

there conditions are in fact exactly the same as the overlap between the g_L string in both states has to be the same since the two codes are encoded into the same codeblock. Therefore, the exact same overlap conditions are obtained for the phases θ_{11} and ω_{11}.

These set of conditions result in the following Theorem for two-qubit transversal diagonal gates.

Theorem 2 *Suppose for a given quantum error correcting code the logical gate $Z_L(1/2^k)$ can be obtained by applying a transversal $Z(1/2^k)^{\otimes n}$ on the underlying physical qubits, yet the transversal application of the logical gate $Z_L(1/2^{k+1})$ is impossible due to code constraints. Then, the two-qubit logical gate U_L that can be applied transversally is in the same level of the Clifford hierarchy as the single-qubit logical gate that transversal single qubit diagonal gates can implement, i.e. $Z_L(1/2^k)\in\mathcal{C}_1^{(k+1)}$ and $U_L\in\mathcal{C}_2^{(k+1)}$. More specifically, the set of two-qubit diagonal gates $U=\sum_j e^{i\pi\theta_j}|j\rangle\langle j|$ that can implement a logical two-qubit operation by applying such gates transversally $U^{\otimes n}$ will be restricted to the angles (up to a global phase freedom),*

$$\begin{aligned}
\theta_{00} &= 0,\\
\theta_{01} &= a/2^k,\\
\theta_{10} &= b/2^k,\\
\theta_{11} &= a/2^k + b/2^k + c/2^{k-1},
\end{aligned}$$

where a, b and c are arbitrary integers. The resulting two-qubit logical gate will have the form (up to arbitrary Clifford gates),

$$U_L = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & e^{i\pi\alpha/2^{k-1}} & 0 & 0\\ 0 & 0 & e^{i\pi\beta/2^k} & 0\\ 0 & 0 & 0 & e^{i\pi\gamma/2^k}\end{pmatrix},$$

where $\beta=0\Leftrightarrow\gamma=0$.

Proof. The first result of the proof is proved in the above section by the resulting constraints on the angles that can be implemented transversally. More specifically, since $Z_L(1/2^k)$ can be implemented transversally, we know that the generators of stabilizer of the code must satisfy,

$$\begin{aligned} |g_{i_1}| &= 0 \mod 2^k \\ 2|g_{i_1} \wedge g_{i_2}| &= 0 \mod 2^k \\ &\vdots \\ 2^{k-1}|g_{i_1} \wedge \ldots \wedge g_{i_k}| &= 0 \mod 2^k, \end{aligned}$$

for all choices of valid indices $\{i_1, \ldots i_k\}$. Moreover, since the code is constrained to not be able to implement $Z(1/2^{k+1})$ transversally, we know the following must be true for some choice of indices $\{\mu_1, \ldots, \mu_{k+1}\}$,

$$2^k|g_{\mu_1} \wedge \ldots \wedge g_{\mu_{k+1}}| \neq 0 \mod 2^{k+1}.$$

Therefore, given the resulting set of constraints on the angle difference $\theta'_{11} = \theta_{11} - \theta_{01} - \theta_{10}$,

$$\begin{aligned} |g_{i_1} \wedge g_j| &= 0 \mod q \\ 2|g_{i_1} \wedge g_{i_2} \wedge g_j| &= 0 \mod q \\ &\vdots \\ 2^{k-1}|g_{i_1} \wedge \ldots \wedge g_{i_k} \wedge g_j| &= 0 \mod q, \end{aligned}$$

where $1/q$ is the desired angular rotation. The above conditions will not be able to be satisfied for $q = 2^k$ as for the indices $\{\mu_1, \cdots, \mu_{k+1}\}$ the following would lead to a contradiction with the final condition:

$$2^k|g_{\mu_1} \wedge \ldots \wedge g_{\mu_{k+1}}| \neq 0 \mod 2^{k+1} \Longrightarrow 2^{k-1}|g_{\mu_1} \wedge \ldots \wedge g_{\mu_{k+1}}| \neq 0 \mod 2^k.$$

Conversely, we know that we can satisfy the above equations for $q = 2^{k-1}$ by using an implication from the single-qubit conditions that must be satisfied for all indices $\{i_1, \ldots i_k\}$,

$$2^l|g_{i_1} \wedge \ldots \wedge g_{i_{l+1}}| = 0 \mod 2^k \Longrightarrow 2^{l-1}|g_{i_1} \wedge \ldots \wedge g_{i_{l+1}}| = 0 \mod 2^{k-1}.$$

Therefore, combining the results for the single qubit gates on the same restrictions and the above observations we know the angle θ'_{11} is restricted to have the form $c/2^{k-1}$. Since the angles θ_{01} and θ_{10} satisfy the same restrictions as the single qubit block case, this completes the proof of the first claim of Theorem 2 regarding the allowable angles which a multi-qubit diagonal gate can implementing a logical multi-qubit gate via a transversal application of the chosen gates. In order to obtain a description of the two-qubit logical gate, we can consider the set of equations that provide the restrictions on the allowable angles in order to obtain an explicit description of the logical angle that can be applied:

$$\begin{aligned} U^{\otimes n}|00\rangle_L &= e^{i\pi n\theta_{00}}|00\rangle_L = |00\rangle, \\ U^{\otimes n}|01\rangle_L &= e^{i\pi|g_i|\theta_{01}}|01\rangle_L = e^{i\pi a|g_i|/2^k}|01\rangle_L = e^{i\pi\alpha/2^k}|01\rangle_L, \\ U^{\otimes n}|10\rangle_L &= e^{i\pi|g_i|\theta_{10}}|10\rangle_L = e^{i\pi b|g_i|/2^k}|10\rangle_L = e^{i\pi\beta/2^k}|10\rangle_L, \\ U^{\otimes n}|11\rangle_L &= e^{i\pi|g_i \wedge g_j|\theta_{11}}|11\rangle_L = e^{i\pi|g_i|(\theta_{01}+\theta_{10}+\theta'_{11})}|11\rangle_L = e^{i\pi(a+b+2c)|g_i|/2^k}|11\rangle_L \\ &= e^{i\pi(\alpha+\beta+2\eta)/2^k}|11\rangle_L. \end{aligned}$$

The above equations must hold for any choice of the weight of the individual (or pairs) of stabilizer generators and in the last equation we have chosen g_j to equal g_i as we have a freedom over which j we choose. We have also introduced the integers $\alpha = a|g_i|$, $\beta = b|g_i|$, and $\eta = c|g_i|$. Consider the angle that is applied by the logical operation to the state $|11\rangle_L$ in more detail, $(\alpha+\beta+2\eta)/2^k$. If both α and β are odd, then the overall angle will be of the form $\gamma/2^{k-1}$. If either α or β are even (or zero), but not both, then the angle will have the form $\gamma/2^k$; however this would mean that the other angle could then be expressed in its most reduced form as $\alpha'/2^{k-1}$. Finally, if both are even, it follows that all these angles can be reduced and shown to be proportional to $1/2^{k-1}$. Therefore, up to a relabelling of logical basis states (which can be achieved using either a logical X_L or $CNOT$ gate), the two-qubit logical gate can be expressed in the form

$$U_L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi\gamma/2^{k-1}} \end{pmatrix},$$

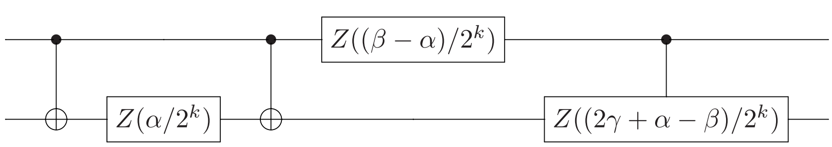
where $\alpha = 0 \Leftrightarrow \beta = 0$, such that either both phases are zero or the two phases are proportional to $1/2^k$ (in the case when only one is zero, then the gate will be a product of a single logical qubit rotation proportional to $Z(1/2^k)$ and a controlled-$Z(1/2^{k-1})$ which are both in $\mathcal{C}_2^{(k+1)}$). To prove the final statement of Theorem 2 we must show that the above gate is contained within the Clifford hierarchy at the $(k+1)$-th level of the two-qubit Clifford hierarchy, $U_L \in \mathcal{C}_2^{(k+1)}$.

We shall prove this by induction. Begin with the case $k=1$, and without loss of generality, assume $\alpha < \beta$, that both are not equal to zero, and that the angles are written in their most reduced form (if both $\alpha = \beta = 0$. Now the proof of the base case is trivial, as it becomes a controlled-Z gate which is clearly in $\mathcal{C}_2^{(2)}$, a Clifford gate). We can rewrite the logical gate as

$$U_L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi\gamma/2^{k-1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi\alpha/2^k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\pi(\beta-\alpha)/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi\gamma/2^{k-1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi\alpha/2^k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\pi(\beta-\alpha)/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi(\beta-\alpha)/2^k} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi(2\gamma+\alpha-\beta)/2^k} \end{pmatrix}.$$

The above sequence of unitaries can be expressed as the following sequence of gates:



In this case, $k=1$, and all of the single qubit gates are achieved by repeated action of the Clifford phase gate $S = \text{diag}(1,i)$. The two-qubit coupling gate is actually the application of a controlled-Z gate since $\alpha \neq 0$, $\beta \neq 0$, and both are odd; therefore, their difference is even and the gate can be expressed in the form $Z(\zeta/2)$. Since all of these gates are in $\mathcal{C}_2^{(2)}$ and the first two levels of the Clifford hierarchy form a group, the resulting composition is an element of $\mathcal{C}_2^{(2)}$.

Assume the claim holds for $k-1$; we will now show that it holds for k. By definition, if $U_L \in \mathcal{C}_2^{(k)}$ it must map any two-qubit Pauli to an element in $\mathcal{C}_2^{(k-1)}$ when conjugating by U_L. We need only consider the action of U_L on the Pauli X elements, as the action on Pauli-Z is trivial since diagonal gates commute. Consider the following:

$$U_L(X \otimes I)U_L^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi\gamma/2^{k-1}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\pi\alpha/2^k} & 0 & 0 \\ 0 & 0 & e^{-i\pi\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{-i\pi\gamma/2^{k-1}} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & e^{-i\pi\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{i\pi(\alpha-2\gamma)/2^k} \\ e^{i\pi\beta/2^k} & 0 & 0 & 0 \\ 0 & e^{-i\pi(\alpha-2\gamma)/2^k} & 0 & 0 \end{pmatrix} = A.$$

Through the action of CNOT gates, we can map the above operator to the following:

$$e^{-i\pi\beta/2^k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi(\alpha+\beta-2\gamma)/2^k} & 0 & 0 \\ 0 & 0 & e^{i\pi 2\beta/2^k} & 0 \\ 0 & 0 & 0 & e^{-i\pi(\alpha-\beta-2\gamma)/2^k} \end{pmatrix}.$$

Note that the left or right action of any Clifford gate will not change the level of an element in the Clifford hierarchy, as proven in Prop. 3 in Ref [26]. Therefore we can show that the above gate is in $\mathcal{C}_2^{(k-1)}$ which is equivalent to showing that $A \in \mathcal{C}_2^{(k-1)}$. We know we can write the integers α and β as $\alpha = 2k_\alpha + 1$ and $2k_\beta + 1$. Consider the following angular expressions:

$$\frac{\alpha+\beta-2\gamma}{2^k} = \frac{2(k_\alpha+k_\beta)-2\gamma+2}{2^k} = \frac{(k_\alpha+k_\beta)-(\gamma-1)}{2^{k-1}},$$
$$\frac{\alpha-\beta-2\gamma}{2^k} = \frac{2(k_\alpha-k_\beta)-2\gamma}{2^k} = \frac{(k_\alpha-k_\beta)-\gamma}{2^{k-1}}.$$

Since $(k_\alpha + k_\beta)$ is even if and only if $(k_\alpha - k_\beta)$ is even, one of the numerators in the final expression will be even, and as such, one of the above angles will necessarily be of the form $1/2^{k-2}$. Therefore, up to a logical Clifford operation (which preserves the level of the Clifford hierarchy), the gate A will have the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi\alpha'/2^{k-1}} & 0 & 0 \\ 0 & 0 & e^{i\pi\beta'/2^{k-1}} & 0 \\ 0 & 0 & 0 & e^{i\pi\gamma'/2^{k-2}} \end{pmatrix},$$

which by the induction hypothesis is an element of $\mathcal{C}_2^{(k-1)}$. Finally, we must show the same property for the following mapping:

$$U_L(I\otimes X)U_L^\dagger = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{i\pi\alpha/2^k} & 0 & 0\\ 0 & 0 & e^{i\pi\beta/2^k} & 0\\ 0 & 0 & 0 & e^{i\pi\gamma/2^{k-1}}\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{-i\pi\alpha/2^k} & 0 & 0\\ 0 & 0 & e^{-i\pi\beta/2^k} & 0\\ 0 & 0 & 0 & e^{-i\pi\gamma/2^{k-1}}\end{pmatrix}$$

$$= \begin{pmatrix}0 & e^{-i\pi\alpha/2^k} & 0 & 0\\ e^{i\pi\alpha/2^k} & 0 & 0 & 0\\ 0 & 0 & 0 & e^{i\pi(\beta-2\gamma)/2^k}\\ 0 & 0 & e^{-i\pi(\beta-2\gamma)/2^k} & 0\end{pmatrix} = B.$$

Up to logical Clifford operations, the gate B has the following form:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{i\pi 2\alpha/2^k} & 0 & 0\\ 0 & 0 & e^{i\pi(\alpha\beta-2\gamma)/2^k} & 0\\ 0 & 0 & 0 & e^{-i\pi(\beta-\alpha-2\gamma)/2^k}\end{pmatrix}.$$

We can see that this has the same form as the case above when the roles of α and β are exchanged; therefore $B \in \mathcal{C}_2^{(k-1)}$ by the induction hypothesis. Furthermore, since $U_L(Z\otimes I)U_L^\dagger = Z\otimes I$ and $U_L(I\otimes Z)U_L^\dagger = I\otimes Z$, we conclude that $U_L \in \mathcal{C}_2^{(k)}$, thus proving the induction hypothesis correct. □.

It is fairly straightforward to note that the equivalent of Proposition 3 will also apply in the two-qubit case. That is, the gate restrictions will also apply to general transversal operations and not just to those that are strongly transversal by using the *Decompression Lemma*.

5 Conclusion

Zeng *et al.* classified the set of single-qubit logical transversal gates [10], showing that they must result from the application of single-qubit diagonal gates in addition to possible local Clifford operations and permutations (SWAP gates). In this work we have characterized the set of individual diagonal gates that can result in the application of a non-trivial logical gate, concluding that all of the entries must be of the form $e^{i\pi c/2^k}$. This severely limits the set of logical gates that can be implemented in a transversal manner for qubit stabilizer codes. It also provides an important result for fault-tolerant quantum computing, as it rules out the possibility of finding transversal implementations for important gates in certain decomposition algorithms, such as the V gate. It also places restrictions on new fault-tolerance schemes which thus far have used a combination of codes to achieve fault-tolerant quantum computation.

Additionally, we have extended our analysis to two-qubit logical gates through the use of two-qubit physical diagonal gates, showing that a very similar restriction holds. In fact, in both the single and two-qubit case, the logical gates that can be implemented by transversal diagonal gate application must belong to the Clifford hierarchy, and moreover, both the single and two-qubit gates that can be implemented for a given code must reside at the same level of the hierarchy. We conjecture that this is true for all multi-qubit gates.

Open questions for future research would be to classify the set of physical diagonal gates that can implement a non-trivial logical gate for qudit systems. Additionally, it would be interesting to consider the set of logical gates that can be generated by coupling two codeblocks corresponding to different quantum error correcting codes, and determine if the same logical gate restrictions apply. Classifying the set of transversal gates for other types of codes is another interesting direction for future research which could provide insight into ways to circumvent the gate restrictions introduced in this work.

Additionally, many magic state distillation schemes use CSS codes to distill purer magic states. These schemes use stabilizer codes with strongly transversal gates directly related to the magic state which the scheme distills. Our results suggest that magic state distillation based on these methods, can only distill gates in the Clifford hierarchy. However, it is worth noting that the 5-to-1 distillation scheme proposed in Ref. [27] is not based on such a method and does allow for the distillation of a state that can implement $Z(\pi/6)$.

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Appendix A

Proof of rationality

In this section, we show that by applying a set of transversal rotations of angles that are irrational multiples of π, that such rotations cannot increase the set of logical gates one can apply and are logically equivalent to the identity.

Proof. [Lemma 3] We allow each $Z(\theta_i)$ to be a Z rotation about any angle, not just a rational angle. Without loss of generality we can assume each θ_i is in the range $(-1, +1)$ and $\theta_i \neq 0$ (since we can just use a new code with that qubit removed). Now we have a transversal gate of the form:

$$Z_L(\theta') := Z(\theta_1) \otimes Z(\theta_2) \otimes ... \otimes Z(\theta_n). \tag{A.1}$$

We will also assume that at least one of the angles is irrational as we have already solved the rational case. Constraints from $Z_L(\theta')|0_L\rangle = |0_L\rangle$ restrict as follows:

$$\begin{aligned} \vec{\theta} \cdot g_{i_1}^T &= 0 \\ \vec{\theta} \cdot (g_{i_1} \oplus g_{i_2})^T &= 0 \\ &\vdots \\ \vec{\theta} \cdot (g_{i_1} \oplus ... \oplus g_{i_{|G_X|}})^T &= 0, \end{aligned}$$

while constraints from $Z_L(\theta')|1_L\rangle = e^{i\pi\theta}|1_L\rangle$ provide the following:

$$\begin{aligned} \vec{\theta} \cdot g_{X_L}^T &= \theta \\ \vec{\theta} \cdot (g_{X_L} \oplus g_{i_1})^T &= \theta \\ &\vdots \\ \vec{\theta} \cdot (g_{X_L} \oplus g_{i_1} \oplus ... \oplus g_{i_{|G_X|}})^T &= \theta \\ &\forall 0 < i_1 < i_2 < ... < i_{|G_X|} \leq |G_X|. \end{aligned}$$

Here the equality is taken over the Real numbers if at least one term in the sum $\vec{\theta} \cdot g_i^T$ is irrational, otherwise the equality is modulo some integer as before.

Some observations:

1. If $\theta_i = \frac{p}{q}\theta_j$, then if $\theta_i + \frac{p}{q}\theta_j = 0 \implies \frac{\theta_i}{q}(q+p) = 0 \implies \theta'(q+p) = 0$. Here $\theta' = \theta_i/q$. We can use the decompression lemma to create a new code where Z_L applies $Z(\theta')$ to $p+q$ qubits.

2. If $\theta_i \neq \frac{p}{q}\theta_j$, then $\theta_i + \theta_j = 0$ iff $\theta_i = 0$ and $\theta_j = 0$. Notice that θ_i and θ_j could be two irrational numbers which are not proportional or an irrational and a rational number (which by definition are not proportional).

3. We can use these observations to reorder the qubits in the code (and possibly apply the decompression lemma to create a new code) to write $\vec{\theta}$ as $Z(1/q) \otimes ... \otimes Z(1/q) \otimes Z(\theta_1) \otimes ... \otimes Z(\theta_1) \otimes Z(\theta_2)$.... Here the $Z(1/q)$ are from the rational part of Z_L (with q a common denomonator) and $Z(\theta_i)$ are the irrational part of Z_L. We have used the decompression lemma to express proportional irrational angles as the same θ_i. Each different i corresponds to irrational angles which are not proportional.

4. Using the second observation we see that the rational angles and each set of proportional irrational angles must individually satisfy the above constraints. We have already discussed the allowable solutions given rational angles. In what follows we will show that no nontrivial solutions exist given irrational angles.

For each θ_i we will have constraints from $Z_L(\theta_i)|0_{L|\theta_i}\rangle = |0_{L|\theta_i}\rangle$ such that,

$$\vec{\theta_i} \cdot g_{i_1|\theta_i}^T = 0$$
$$\vec{\theta_i} \cdot (g_{i_1|\theta_i} \oplus g_{i_2|\theta_i})^T = 0$$
$$\vdots$$
$$\vec{\theta_i} \cdot (g_{i_1|\theta_i} \oplus ... \oplus g_{i_{|G_X|}|\theta_i})^T = 0,$$

while constraints from $Z_L(\theta_i)|1_{L|\theta_i}\rangle = e^{i\pi\theta}|1_{L|\theta_i}\rangle$ provide the following,

$$\vec{\theta_i} \cdot g_{X_L|\theta_i}^T = \theta$$
$$\vec{\theta_i} \cdot (g_{X_L|\theta_i} \oplus g_{i_1|\theta_i})^T = \theta$$
$$\vdots$$
$$\vec{\theta_i} \cdot (g_{X_L|\theta_i} \oplus g_{i_1|\theta_i} \oplus ... \oplus g_{i_{|G_X|}|\theta_i})^T = \theta$$
$$\forall 0 < i_1 < i_2 < ... < i_{|G_X|} \leq |G_X|.$$

Here $|0_{L|\theta_i}\rangle$ refers to the restriction to qubits which $\vec{\theta_i}$ acts nontrivially upon. Note that it is possible that $\theta = 0$ for some set of proportional irrational angles. As long as $\theta \neq 0$ for some set of proportional irrational angles, then the irrational part of $\vec{\theta}$ has contributed nontrivially $Z_L(\theta')$. We will only consider the case when $\theta \neq 0$ as the other case is trivial (equivalent to applying the identity).

Now, we will try to find a set of rows of H_X and X_L which satisfy all these conditions. For the underlying code to be nontrivial we require that H_X has no zero columns. We assume that $a \neq 0$, otherwise the transversal operator is trivial ($Z_L(\theta') = I$).

If there is only one row h_1 then it must be all ones and

$$\vec{\theta} \cdot g_1^T = 0$$
$$\vec{\theta} \cdot g_{X_L}^T \neq 0$$
$$\vec{\theta} \cdot (g_1 \wedge g_{X_L})^T = 0,$$

but $\vec{\theta} \cdot (g_1 \wedge g_{X_L})^T = \vec{\theta} \cdot g_{X_L}^T = 0$ and we have a contradiction.

If H_X is nontrivial and has two rows, the columns of H_X are one of three types:

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{A.2}$$

We will refer to the combination of all columns of type a, b, c, by the matrix A, B, C, respectively.

If we have a logical operator X_L, then

$$\vec{\theta} \cdot (g_1 \wedge g_{X_L})^T = \theta(\Delta w_A + \Delta w_C) = 0,$$
$$\vec{\theta} \cdot (g_2 \wedge g_{X_L})^T = \theta(\Delta w_B + \Delta w_C) = 0,$$
$$\vec{\theta} \cdot (g_1 \wedge g_2 \wedge g_{X_L})^T = \theta(\Delta w_C) = 0,$$
$$\vec{\theta} \cdot X_L = \theta(\Delta w_A + \Delta w_B + \Delta w_C) \neq 0.$$

Here, $\Delta w_A = w_A^+ - w_A^-$ and $w_A^+(w_A^-)$ is the overlap of A and X_L which has support on $H_X^+(H_X^-)$. Since $\theta \neq 0$ the first three constraints imply that $\Delta w_A, \Delta w_B, \Delta w_C = 0$ which imply $|X_L| = 0$ and hence a contradiction.

As we can see the proof proceeds in the same manner as in Sec. 3.1 with w_i replaced by Δw_i. We reach the same contradiction given any set of proportional irrational angles and have, therefore, proven that transversal gates with single qubit rotations by irrational angles have no effect and are equivalent to applying the identity. $\square$.