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RÉNYI AND TSALLIS FORMULATION OF NOISE-DISTURBANCE TRADE-OFF RELATIONS

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We address an information-theoretic approach to noise and disturbance in quantum measurements. Properties of corresponding probability distributions are characterized by means of both the Rényi and Tsallis entropies. Related information-theoretic measures of noise and disturbance are introduced. These definitions are based on the concept of conditional entropy. To motivate introduced measures, some important properties of the conditional Rényi and Tsallis entropies are discussed. There exist several formulations of entropic uncertainty relations for a pair of observables. Trade-off relations for noise and disturbance are derived on the base of known formulations of such a kind.

Keywords: conditional entropy, noise-disturbance relation, quantum instrument, error probability
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1. Introduction

The Heisenberg uncertainty principle [1] is one of the most known restrictions distinguishing the quantum world from the classical one. Scientists have made a great effort to understand and extend its scope and meaning. Basic developments in this direction are reviewed in [2, 3, 4]. Various quantitative measures can be used to describe quantum uncertainties formally [5]. In very traditional formulation [6, 7], we deal with the standard deviations of corresponding observables. Such an approach was criticized in the papers [8, 9], in which entropic formulation has been developed. The references [10, 11, 12, 13] considered the entropic principle in the case of an observer with quantum side information. An attention is attracted to the entropic formulation rather due to its connection with some topics of quantum information theory [3, 10]. On the other hand, Heisenberg's initial argument is better formulated in terms of noise and disturbance [14, 15]. Thus, we cannot measure precisely an observable without causing a disturbance to another incompatible observable.

There are more than one approaches to fit a quantitative formulation of trade-off between noise and disturbance in quantum measurements. The first universal uncertainty relation of noise-disturbance type was derived by Ozawa [15]. Other formulations have been proposed in [16, 17, 18, 19, 20, 21]. The authors of [22] reported experimental evidences for violation of so-called Heisenberg's error-disturbance uncertainty relation. For a discussion of this conclusion, see [23] and references therein. An information-theoretic approach to quantifying noise and disturbance in quantum measurements has been examined in [24, 25]. Corresponding definitions are based on the notion of conditional entropy. Formulations of such a kind are very useful due to several advances. The quantities introduced in [24] are invariant under relabelling of outcomes. The possibility of quantum or classical correcting operations is naturally taken into account. In addition, the information-theoretic noise can be related to the error probability of used decision rule.

The present work is devoted to formulating noise-disturbance relations in terms of generalized entropies. As information-theoretic measures, entropies of both the Rényi and Tsallis types are used. One of motivations to develop entropic uncertainty relations is connected with their potential applications in quantum cryptography [26, 27]. Although Rényi's entropies are rather meaningful in studies of such a kind, the role of Tsallis' ones deserves investigations as well. Another utility of uncertainty relations with a parametric dependence was illustrated in [9]. The presented measures of noise and disturbance in quantum measurements are defined with using the conditional Rényi and Tsallis entropies. The paper is organized as follows. Required material is reviewed in Section 2. First, we discuss quantum measurements and instruments. Second, basic properties of Tsallis and Rényi entropies are recalled. In particular, we consider relations between conditional entropies and error probability. Third, formulations of entropic uncertainty relations for a pair of observables are discussed. Main results are presented in Section 3. First, we introduce information-theoretic measures of noise and disturbance in terms of the conditional Tsallis and Rényi entropies. Reasons for proposed definitions are treated with the use of essential entropic properties. Using entropic uncertainty relations, we further derive noise-disturbance trade-off relations with a parametric dependence. In Section 4, we conclude the paper with a summary of results.

2. Preliminaries

In this section, preliminary material is reviewed. First, we recall the formalism of quantum operations, including quantum measurements and quantum instruments. Second, we write definitions and some properties of used entropic measures. In particular, we focus on existing relations between conditional entropies and error probability. Some formulations of entropic uncertainty relations are discussed as well.

2.1. *Quantum measurements and instruments*

Let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on d-dimensional Hilbert space $\mathcal{H}$. By $\mathcal{L}_{s.a.}(\mathcal{H})$ and $\mathcal{L}_{+}(\mathcal{H})$, we respectively denote the real space of Hermitian operators on $\mathcal{H}$ and the set of positive ones. The state of a quantum system is described by a density matrix $\boldsymbol{\rho}\in\mathcal{L}_{+}(\mathcal{H})$ normalized as $\mathrm{Tr}(\boldsymbol{\rho})=1$. A common approach to quantum measurements is based on the notion of positive operator-valued measures (POVMs). A positive operator-valued measure $\mathcal{N}=\{\mathsf{N}(y)\}$ is a set of elements $\mathsf{N}(y)\in\mathcal{L}_{+}(\mathcal{H})$ satisfying the completeness relation [28]

$$\sum\nolimits_{y}\mathsf{N}(y)=\mathbb{1}\ . \tag{1}$$

Here, the symbol $\mathbb{1}$ denotes the identity operator on $\mathcal{H}$. If the pre-measurement state is described by $\boldsymbol{\rho}$, then the probability of y-th outcome is $\mathrm{Tr}\big(\mathsf{N}(y)\,\boldsymbol{\rho}\big)$ [28]. The standard measurement of an observable is described by a projector-valued measure, when POVM elements form an orthogonal resolution of the identity. As an entropy-based approach deals with probability distributions, it does not refer to eigenvalues. Special types of POVM measurements are especially important. Informationally complete measurements are an indispensable tool in many questions [29, 30, 31]. Entropic uncertainty relations for symmetric informationally complete POVMs are derived in [32]. The informational power of such preparations and measurements is considered in [33].

A unified description of the operation of a laboratory detector is provided by the concept of quantum instruments [34]. Consider a linear map $\Phi:\ \mathcal{L}(\mathcal{H}_A)\rightarrow\mathcal{L}(\mathcal{H}_B)$. This map is positive, when $\Phi(\mathsf{A})\in\mathcal{L}_{+}(\mathcal{H}_B)$ for each $\mathsf{A}\in\mathcal{L}_{+}(\mathcal{H}_A)$ [35, 36]. To describe physical processes, linear maps must be completely positive [35, 36]. Let id_R be the identity map on $\mathcal{L}(\mathcal{H}_R)$, where the space $\mathcal{H}_R$ is assigned to a reference system. The complete positivity implies that the map $\Phi\otimes\mathrm{id}_R$ with the input space $\mathcal{H}_A\otimes\mathcal{H}_R$ is always positive irrespectively to a dimensionality of $\mathcal{H}_R$. Any completely positive map can be represented in the form [35, 36]

$$\Phi(\mathsf{A})=\sum\nolimits_{n}\mathsf{K}(n)\,\mathsf{A}\,\mathsf{K}(n)^{\dagger}\ . \tag{2}$$

Here, the Kraus operators $\mathsf{K}(n)$ map the input space $\mathcal{H}_A$ to the output space $\mathcal{H}_B$. When physical process is closed, the corresponding map preserves the trace, $\mathrm{Tr}\big(\Phi(\mathsf{A})\big)=\mathrm{Tr}(\mathsf{A})$. Trace-preserving completely positive (TPCP) maps are often called quantum channels [35, 37]. For a quantum channel, the Kraus operators satisfy

$$\sum\nolimits_{n}\mathsf{K}(n)^{\dagger}\,\mathsf{K}(n)=\mathbb{1}_A\ . \tag{3}$$

Let us consider a collection of completely positive maps $\mathcal{M}=\big\{\Phi^{(m)}\big\}$. The collection $\mathcal{M}$ is a quantum instrument, when the maps $\Phi^{(m)}$ are summarized to a trace-preserving map [24]. For all $\mathsf{A}\in\mathcal{L}(\mathcal{H}_A)$, one obeys

$$\sum\nolimits_{m}\mathrm{Tr}\big(\Phi^{(m)}(\mathsf{A})\big)=\mathrm{Tr}(\mathsf{A})\ . \tag{4}$$

If the pre-measurement state of an input system is described by density matrix $\boldsymbol{\rho}$, then the m-th outcome occurs with probability

$$p(m)=\mathrm{Tr}\big(\Phi^{(m)}(\boldsymbol{\rho})\big)\ . \tag{5}$$

In this case, the measuring apparatus will return an output system in the state described by [24]

$$\boldsymbol{\rho}'=p(m)^{-1}\,\Phi^{(m)}(\boldsymbol{\rho})\ . \tag{6}$$

It is convenient to use a trace-preserving completely positive map defined as

$$\Phi_{\mathcal{M}}(\boldsymbol{\rho}):=\sum\nolimits_{m}\Phi^{(m)}(\boldsymbol{\rho})\otimes|m\rangle\langle m|\ . \tag{7}$$

The "flag" states $|m\rangle$ of an auxiliary system are orthonormal and, herewith, perfectly distinguishable [24]. Such states are used for encoding measurements outcomes.

2.2. *Rényi and Tsallis entropies*

Together with the Shannon entropy, other entropic measures are extensively used. Among them, the Rényi and Tsallis entropic functionals are especially important [37]. Let discrete random variable X take values on the finite set Ω_X, and let $\{p(x)\}$ be its probability distribution. For $0<\alpha\neq 1$, the Rényi entropy is defined as [38]

$$R_\alpha(X):=\frac{1}{1-\alpha}\,\ln\biggl(\sum_{x\in\Omega_X} p(x)^\alpha\biggr)\,. \tag{8}$$

If the set Ω_X has cardinality $|\Omega_X|=d$, then the maximal value of (8) is equal to $\ln d$. It is reached with the uniform distribution. The entropy (8) is a non-increasing function of order α [38]. Other properties related to the parametric dependence are discussed in [39]. In the limit $\alpha\to 1$, the entropy (8) gives the Shannon entropy. For $\alpha\in(0,1)$, the entropy (8) is certainly concave [40]. Convexity properties of $R_\alpha(X)$ with orders $\alpha>1$ depend on dimensionality of probabilistic vectors [37, 41]. For instance, for every $\alpha>1$ there exist an integer $d_\star$ such that the entropy (8) is neither convex nor concave for all $d>d_\star$ [41]. The two-dimensional case is of special interest. As was explicitly shown in [41], the binary Rényi entropy is concave for $0<\alpha\leq 2$. We also recall that the Rényi entropy is Schur-concave.

Tsallis entropies also form an important family of generalized entropies. The Tsallis entropy of degree $0<\alpha\neq 1$ is defined as [42]

$$H_\alpha(X):=\frac{1}{1-\alpha}\,\biggl(\sum_{x\in\Omega_X} p(x)^\alpha-1\biggr)\,. \tag{9}$$

For brevity, we will omit in sums the symbols such as Ω_X. For $0<\alpha\neq 1$ and $\xi>0$, we will use the α-logarithm $\ln_\alpha(\xi)=\bigl(\xi^{1-\alpha}-1\bigr)/(1-\alpha)$. One can rewrite the entropy (9) as

$$H_\alpha(X)=-\sum\nolimits_x p(x)^\alpha\,\ln_\alpha p(x)=\sum\nolimits_x p(x)\,\ln_\alpha\!\left(\frac{1}{p(x)}\right). \tag{10}$$

When $|\Omega_X|=d$, the maximal value of (9) is equal to $\ln_\alpha(d)$. It is reached with the uniform distribution. In the limit $\alpha\to 1$, we also obtain the Shannon entropy $H_1(X)=-\sum_x p(x)\,\ln p(x)$. Applications of generalized entropies in quantum theory are reviewed in [37]. Entropic trade-off relations for a single quantum channel are discussed in [43, 44].

In the following, we will also use conditional entropic forms. Let Y be another random variable. The standard conditional entropy is defined as [45]

$$H_1(X|Y):=\sum\nolimits_y p(y)\,H_1(X|y)=-\sum\nolimits_x\sum\nolimits_y p(x,y)\,\ln p(x|y)\ . \tag{11}$$

Here, we use joint probabilities $p(x,y)$ and the particular functional

$$H_1(X|y)=-\sum\nolimits_x p(x|y)\,\ln p(x|y)\ , \tag{12}$$

where $p(x|y)=p(x,y)/p(y)$. Similarly to (12), we introduce the quantity

$$H_\alpha(X|y):=\frac{1}{1-\alpha}\,\Bigl(\sum\nolimits_x p(x|y)^\alpha-1\Bigr)\ . \tag{13}$$

Keeping (10) in mind, the two kinds of conditional Tsallis entropy can be considered [46, 47]. These forms are respectively defined as

$$H_\alpha(X|Y):=\sum\nolimits_y p(y)^\alpha\,H_\alpha(X|y)\ , \tag{14}$$

$$\widetilde{H}_\alpha(X|Y):=\sum\nolimits_y p(y)\,H_\alpha(X|y)\ . \tag{15}$$

For all $\alpha > 0$, the first form (14) shares the chain rule [46, 48]. In this paper, we will rather need another property. It is natural to demand that conditioning on more may only reduce the entropy. In effect, the standard conditional entropy satisfies [45]

$$H_1(X|Y,Z) \leq H_1(X|Y) \ . \tag{16}$$

For all $\alpha > 0$, the second form (15) of conditional α-entropy obeys [49]

$$\widetilde{H}_\alpha(X|Y,Z) \leq \widetilde{H}_\alpha(X|Y) \ . \tag{17}$$

The first form (14) satisfies such a property only for $\alpha \geq 1$ [49]. Since the mentioned property is of great importance in our research, we will use the second form. It should be noted that the form (15) does not share the chain rule. As the first form (14) of conditional α-entropy obeys the chain rule for all $\alpha > 0$ [46, 48], it may be more appropriate in some questions. In the present work, however, the chain rule is not used.

The Rényi case is similar to the Tsallis case in the following respect. There is no generally accepted approach to the definition of conditional Rényi entropy [50]. We will use the following one. For $0 < \alpha \neq 1$, the conditional α-entropy is put by [51, 52, 53]

$$R_\alpha(X|Y) := \sum\nolimits_y p(y)\, R_\alpha(X|y) \ , \tag{18}$$

where

$$R_\alpha(X|y) := \frac{1}{1-\alpha}\, \ln\Bigl(\sum\nolimits_x p(x|y)^\alpha\Bigr) \ . \tag{19}$$

Like (8), the conditional entropy (18) is a non-increasing function of α. Another approach for constructing conditional entropies is connected with the notion of relative entropy [13]. Then conditional entropies are defined via an optimization problem. The corresponding formulation of conditional Rényi's entropy is considered in [13], mainly in quantum setting. In the following, we will use the definition (18) due to its connection with error probability.

The limit $\alpha \to \infty$ gives the conditional min-entropy. For the given value y, we define

$$\hat{x}(y) := \operatorname{Arg\,max}\bigl\{p(x|y) :\, x \in \Omega_X\bigr\} \ . \tag{20}$$

It maximizes $p(x|y)$, i.e., $p(x|y) \leq p(\hat{x}|y)$ for all $x \in \Omega_X$. Note that a value (20) may be not unique. Any of such values corresponds to the standard decision in the Bayesian approach [54]. We then write

$$R_\infty(X|y) = -\ln p(\hat{x}|y) \ . \tag{21}$$

The conditional min-entropy $R_\infty(X|Y)$ is defined according to (18) and (21). The following property is related to conditioning on more. For $0 < \alpha \leq 1$, the conditional entropy (18) satisfies

$$R_\alpha(X|Y,Z) \leq R_\alpha(X|Y) \ . \tag{22}$$

This relation immediately follows from concavity of the entropy [49]. If $|\Omega_X| = 2$, then the relation (22) is valid for all $\alpha \in (0,2]$. Indeed, the binary Rényi entropy is concave for $0 < \alpha \leq 2$ [41]. Here, the proof holds irrespectively to dimensionality of any of Y and Z. The only restriction is that the variable X is two-dimensional. With arbitrary finite $|\Omega_X|$, we can use (22) only for $\alpha \in (0,1]$.

The conditional entropy (18) has interesting properties and applications in some questions [51, 52, 53]. However, this form does not share the chain rule. Conditional Rényi's entropy can be defined in a way connected with the chain rule [55, 56]. In our study, we are rather interested in properties related to conditioning on more.

2.3. *Relations between conditional entropies and error probability*

Although entropic functions are basic measures of uncertainty, the channel coding theorems are usually stated in terms of the error probability [45]. Hence, relations between entropies and the error probability are of interest. Fano's inequality provide an upper bound on the conditional entropy [57]. Known lower bounds on the conditional entropy are expressed in terms of the error of standard decision. Let variables X and Y respectively correspond to the input and the output of a communication system. We should decide on the input symbols when the output symbols are known. In the standard decision, we decide in favor of value (20) for all output values of Y. Then the error probability $\hat{p}_e$ and the probability of successful estimation $\hat{p}_s$ are written as

$$\hat{p}_e = 1 - \hat{p}_s \ , \qquad \hat{p}_s = \sum\nolimits_y p(y)\, p(\hat{x}|y) \ . \tag{23}$$

Due to the Bayesian version of the fundamental Neyman–Pearson lemma [54], no decision can have a smaller error probability than the standard decision. When there exists a decision rule with zero error probability, we inevitably have $\hat{p}_e = 0$.

As was shown in [54, 58], the standard conditional entropy (11) is bounded from below as

$$-\ln(1 - \hat{p}_e) \leq H_1(X|Y) \ . \tag{24}$$

This result was extended to some forms of generalized entropies [47]. For all $\alpha \in (0, 2]$, the conditional entropy (15) satisfies

$$\ln_\alpha\!\left(\frac{1}{1 - \hat{p}_e}\right) \leq \widetilde{H}_\alpha(X|Y) \ . \tag{25}$$

As was recently proved in [49], for $\alpha \in (0, 2]$ we also have

$$2 \ln_\alpha(2)\, \hat{p}_e \leq \widetilde{H}_\alpha(X|Y) \ . \tag{26}$$

For $\alpha > 2$, the lower bound on (15) depends also on the dimensionality $d = |\Omega_X|$. Namely, we have

$$\frac{d \ln_\alpha(d)}{d - 1}\, \hat{p}_e \leq \widetilde{H}_\alpha(X|Y) \ . \tag{27}$$

For all $\alpha \in (0, \infty)$, the conditional Rényi entropy (18) satisfies

$$-\ln(1 - \hat{p}_e) \leq R_\alpha(X|Y) \ . \tag{28}$$

In the binary case, some of the above bounds can be improved [47]. For $d = 2$, the inequality (26) remains valid for all $\alpha \in (0, \infty)$. For $d = 2$ and $\alpha \in [1, \infty)$, the conditional Rényi entropy (18) satisfies

$$2 \ln_\alpha(2)\, \hat{p}_e \leq R_\alpha(X|Y) \ . \tag{29}$$

For $d=2$ and $\alpha\in(0,1]$, we also have $(2\,\ln 2)\,\hat{p}_e\leq R_\alpha(X|Y)$ [47].

Thus, we can claim the following property. If any of the entropies (15) and (18) tends to zero, then $\hat{p}_e$ tends to zero as well. That is, vanishing of conditional entropies implies that there is a decision function with vanishing error. In general, this claim is restricted to finite dimensions. For instance, the bound (27) is applicable only when d is finite. We will now recall upper bounds related to the finite-dimensional case.

For an arbitrary decision rule $x'=g(y)$, the corresponding error probability p_e is defined similarly to (23). The well-known Fano inequality states that [59]

$$H_1(X|Y)\leq h_1(p_e)+p_e\ln(d-1)\ , \tag{30}$$

where $d=|\Omega_X|$ and the binary entropy $h_1(q)=-\,q\ln q-(1-q)\ln(1-q)$ for $q\in[0,1]$. Let us put the binary Tsallis entropy

$$h_\alpha(q):=-\,q^\alpha\ln_\alpha(q)-(1-q)^\alpha\ln_\alpha(1-q)\ . \tag{31}$$

As was proved in [49], the conditional entropy (15) satisfies

$$\widetilde{H}_\alpha(X|Y)\leq h_\alpha(p_e)+p_e^\alpha\ln_\alpha(d-1) \qquad (0<\alpha<1)\ . \tag{32}$$

$$\widetilde{H}_\alpha(X|Y)\leq h_\alpha(p_e)+p_e\ln_\alpha(d-1) \qquad (1<\alpha<\infty)\ . \tag{33}$$

When $\alpha\to1$, both the formulas (32) and (33) give the standard Fano inequality (30).

The authors of [53] derived several results concerning the conditional Rényi entropy (18). For $\alpha\geq1$, the conditional entropy $R_\alpha(X|Y)$ is bounded from above by the right-hand side of (30). Indeed, the function (19) cannot increase with growing α. For $\alpha\in(1,\infty)$, therefore, we have $R_\alpha(X|Y)\leq H_1(X|Y)$. Combining this with (30) immediately gives the claim. The upper bound (30) holds for arbitrary decision rule.

Upper bounds on the conditional Rényi entropy of order $\alpha\in(0,1)$ can be written in terms of the error probability $\hat{p}_e$ of the standard decision [49]. They are based on one of the results of [41]. The conditional Rényi entropy of order $\alpha\in(0,1)$ obeys [49]

$$R_\alpha(X|Y)\leq\frac{1}{1-\alpha}\,\ln\Big((1-\hat{p}_e)^\alpha+(d-1)^{1-\alpha}\hat{p}_e^{\,\alpha}\Big)\,. \tag{34}$$

Recall that vanishing of conditional entropies implies that there is a decision function with zero error probability. On the other hand, the above bounds of Fano's type imply that conditional entropies should vanish for $\hat{p}_e\to0$. These results are essential in motivating measures of information-theoretic noise. Note that bounds of the Fano type involve dimensionality d. We refrain from discussing relations between conditional entropies and error probability in the countably-infinite case (see [60] and references therein).

2.4. *General entropic uncertainty relations for finite-level systems*

Formulating noise-disturbance relations, we will use uncertainty relations derived in [61, 62]. For any $\mathsf{A}\in\mathcal{L}(\mathcal{H})$, we define $|\mathsf{A}|\in\mathcal{L}_+(\mathcal{H})$ to be the positive square root of $\mathsf{A}^\dagger\mathsf{A}$. The singular values $\sigma_j(\mathsf{A})$ are then introduced as eigenvalues of $|\mathsf{A}|$ [63]. In terms of the singular values, one defines the Schatten norms widely used in quantum information theory [63]. We will further use the spectral norm $\|\mathsf{A}\|_\infty=\max\big\{\sigma_j(\mathsf{A}):\,1\leq j\leq d\big\}$.

Let us consider d-dimensional observables $\mathsf{X},\mathsf{Z}\in\mathcal{L}_{s.a.}(\mathcal{H})$ with the spectral decompositions

$$\mathsf{X}=\sum_{x\in\mathrm{spec}(\mathsf{X})} x\,\mathsf{\Lambda}(x)\ , \tag{35}$$

$$\mathsf{Z}=\sum_{z\in\mathrm{spec}(\mathsf{Z})} z\,\mathsf{\Gamma}(z)\ . \tag{36}$$

Here, the sets $\{\mathsf{\Lambda}(x)\}$ and $\{\mathsf{\Gamma}(z)\}$ are corresponding orthogonal resolutions of the identity. For non-degenerate observables, we have $\mathsf{\Lambda}(x)=|x\rangle\langle x|$ and $\mathsf{\Gamma}(z)=|z\rangle\langle z|$. In this case, the well-known Maassen–Uffink uncertainty relation [9] is expressed in terms of the quantity $c:=\max\bigl|\langle x|z\rangle\bigr|$. Inspired by the results of [10], formulations in terms of quantum conditional entropies were studied. Such uncertainty relations follow from a few simple properties [11] including monotonicity of relative entropies under the action of TPCP maps. For a wide range of parameter values, this important fact has been proved for the so-called "sandwiched" Rényi entropy. This collection of new relative entropies of Rényi's type was introduced and motivated in [12]. An application of such entropies to studying noise-disturbance trade-off relations may be a theme of separate investigation.

When the range of summation is clear from the context, we will omit symbols like $\mathrm{spec}(\mathsf{X})$ and $\mathrm{spec}(\mathsf{Z})$. The authors of [62] have addressed a problem of finding c-optimal bounds on the sum of corresponding entropies. As a measure of uncertainty in quantum measurements, one uses generalized entropies of the (h,ϕ)-form examined in the papers [64, 65]. We will consider a particular case of this entropic family. Namely, for any $\alpha>0$ we define

$$E_{\alpha}^{f}(X):=\frac{1}{1-\alpha}\,f\Bigl(\sum\nolimits_{x}p(x)^{\alpha}\Bigr)\ . \tag{37}$$

Here, the function $\xi\mapsto f(\xi)$ should be continuous and strictly increasing with $f(1)=0$. This choice obeys the conditions required in [62] and is completely sufficient for our purposes. Indeed, the Rényi entropy (8) and the Tsallis entropy (9) are respectively obtained from (37) with particular choices

$$f^{(R)}(\xi):=\ln\xi\ , \qquad f^{(T)}(\xi):=\xi-1\ . \tag{38}$$

We avoid considering entropies of more general kind, since our constructions will involve conditional entropies.

Measuring the observable X in the pre-measurement state $\boldsymbol{\rho}$, the outcome x occurs with the probability $\mathrm{Tr}(\mathsf{\Lambda}(x)\,\boldsymbol{\rho})$. Substituting this distribution into (37), we obtain the quantity

$$\mathrm{E}_{\alpha}^{f}(\mathsf{X};\boldsymbol{\rho})=\frac{1}{1-\alpha}\,f\Bigl(\sum\nolimits_{x}\bigl[\mathrm{Tr}\bigl(\mathsf{\Lambda}(x)\,\boldsymbol{\rho}\bigr)\bigr]^{\alpha}\Bigr)\ . \tag{39}$$

This quantity characterizes an amount of uncertainty in performed quantum measurement. In the case of POVM $\mathcal{N}=\{\mathsf{N}(y)\}$, the entropy $\mathrm{E}_{\alpha}^{f}(\mathcal{N};\boldsymbol{\rho})$ is given similarly to (39), but with the probabilities $\mathrm{Tr}\bigl(\mathsf{N}(y)\,\boldsymbol{\rho}\bigr)$.

To two observables $\mathsf{X},\mathsf{Z}\in\mathcal{L}_{s.a.}(\mathcal{H})$, we assign the characteristic

$$c:=\max\Bigl\{\|\mathsf{\Lambda}(x)\,\mathsf{\Gamma}(z)\|_{\infty}:\ x\in\mathrm{spec}(\mathsf{X}),\ z\in\mathrm{spec}(\mathsf{Z})\Bigr\}\,, \tag{40}$$

and related parameter $\eta := \arccos c$. Concerning (40), the following fact should be noticed. It is easy to see that $\|\mathsf{A}^{\dagger}\|_{\infty} = \|\mathsf{A}\|_{\infty}$ for any A. Since both the projectors $\Lambda(x)$ and $\Gamma(z)$ are Hermitian, we then get

$$\|\Lambda(x)\,\Gamma(z)\|_{\infty} = \|\Gamma(z)\,\Lambda(x)\|_{\infty} \ . \tag{41}$$

For non-degenerate observables, the characteristic (40) is reduced to the maximal overlap between eigenstates of X and Z, i.e., to $\max\bigl|\langle x|z\rangle\bigr|$. As follows from the unitarity, the latter ranges between $d^{-1/2}$ and 1. Introducing the parametric sum

$$\mathcal{S}_{\alpha}(\theta) := \left\lfloor \frac{1}{\cos^2\theta} \right\rfloor (\cos^2\theta)^{\alpha} + \left(1 - \left\lfloor \frac{1}{\cos^2\theta} \right\rfloor \cos^2\theta \right)^{\alpha} , \tag{42}$$

for all $\alpha,\beta \geq 0$ we define the quantity

$$\overline{\mathcal{B}}_{\alpha,\beta;f}(c) := \min_{\theta\in[0,\eta]} \left(\frac{f\bigl(\mathcal{S}_{\alpha}(\theta)\bigr)}{1-\alpha} + \frac{f\bigl(\mathcal{S}_{\beta}(\eta-\theta)\bigr)}{1-\beta} \right) . \tag{43}$$

For all $\alpha,\beta \geq 0$ and two finite-dimensional observables, the corresponding generalized entropies satisfy the state-independent lower bound [62]

$$\mathrm{E}_{\alpha}^{f}(\mathsf{X};\boldsymbol{\rho}) + \mathrm{E}_{\beta}^{f}(\mathsf{Z};\boldsymbol{\rho}) \geq \overline{\mathcal{B}}_{\alpha,\beta;f}(c) \tag{44}$$

This generalized-entropy uncertainty relation for two observables has been proved recently in [62]. Note that our notation slightly differs from the notation of [62] in minor respects. Substituting the functions (38), we obtain the lower bounds for both the Tsallis and Rényi formulations

$$\overline{\mathcal{B}}_{\alpha,\beta}^{(T)}(c) := \min_{\theta\in[0,\eta]} \left(\frac{\mathcal{S}_{\alpha}(\theta)-1}{1-\alpha} + \frac{\mathcal{S}_{\beta}(\eta-\theta)-1}{1-\beta} \right) , \tag{45}$$

$$\overline{\mathcal{B}}_{\alpha,\beta}^{(R)}(c) := \min_{\theta\in[0,\eta]} \left(\frac{\ln\mathcal{S}_{\alpha}(\theta)}{1-\alpha} + \frac{\ln\mathcal{S}_{\beta}(\eta-\theta)}{1-\beta} \right) , \tag{46}$$

where $\eta = \arccos c$. In the next, we will use these bounds in obtaining both the Rényi and Tsallis formulations of noise-disturbance relations. It should be noted that the authors of [62] derived their uncertainty relations also for the case of two POVMs. However, a treatment becomes much more complicated. In particular, it depends on the maximal spectral norm among elements of a single POVM. On the other hand, the results (45) and (46) for projective measurements are sufficient for our aims.

We will also use entropic uncertainty relations of the Maassen–Uffink type. This approach was developed in deriving uncertainty relations in terms of Rényi [66] and Tsallis entropies [67]. Using Riesz's theorem leads to a specific condition imposed on entropic parameters. Developing this approach in some physical cases of specific interest is considered in [68, 69, 70]. The corresponding Tsallis entropies satisfy [67]

$$\mathrm{H}_{\alpha}(\mathsf{X};\boldsymbol{\rho}) + \mathrm{H}_{\beta}(\mathsf{Z};\boldsymbol{\rho}) \geq \ln_{\mu}\bigl(c^{-2}\bigr) \ , \tag{47}$$

where $1/\alpha + 1/\beta = 2$ and $\mu = \max\{\alpha,\beta\}$. Under the same condition on α and β, the corresponding Rényi entropies satisfy [67]

$$\mathrm{R}_{\alpha}(\mathsf{X};\boldsymbol{\rho}) + \mathrm{R}_{\beta}(\mathsf{Z};\boldsymbol{\rho}) \geq -2\ln c \, . \tag{48}$$

As was motivated in [62], the bounds (45) and (46) are not always c-optimal. In some cases, bounds of the Maassen–Uffink type are stronger. Thus, we will also derive noise-disturbance relations with the use of (47) and (48). The considered bounds are formulated in terms of only one quantity (40). Another approach to obtaining entropic bounds is dealing with more matrix elements of the form $\langle x|z\rangle$. This important topic has been studied in recent works [71, 72, 73]. Bounds of such a kind are not used in the following.

3. Main results

In this section, we formulate noise-disturbance relations with the use of generalized entropies. First, we use the conditional entropies (15) and (18) to quantify information-theoretic noise and disturbance in quantum measurements. The introduced measures are a natural extension of the quantities proposed in [24]. Second, we derive nontrivial lower bounds on the sum of introduced measures of information-theoretic noise and disturbance.

3.1. *Information-theoretic noise and disturbance*

Let X and Z be observables of a studied quantum system A with d-dimensional state space. It is assumed to be subjected to a measuring apparatus $\mathcal{M}$. We consider the following two variants of correlation experiments performed with $\mathcal{M}$ [24]. In the first experiment, some source produces eigenstates of X at random. For non-degenerate X, it should produce each eigenstate $|x\rangle$ with the probability $1/d$. According to (35), the integer $d_x := \mathrm{Tr}\big(\mathsf{\Lambda}(x)\big)$ gives degeneracy of the eigenvalue x. Therefore, it should be taken at random with the probability d_x/d [24]. The corresponding eigenstate is written as $\mathsf{\Lambda}(x)/d_x$. We feed each of the eigenstates of X into the apparatus $\mathcal{M}$ and ask for correlations of the observed outcomes m with the eigenvalues of X. The first experiment focuses on the average performance of the apparatus in discriminating between possible values of X. Only the classical outcomes are used for guessing in the first experiment [24].

In the second experiment, another source produces eigenstates of Z at random. Due to (36), each eigenvalue z is associated with the density matrix $\mathsf{\Gamma}(z)/d_z$, where $d_z := \mathrm{Tr}\big(\mathsf{\Gamma}(z)\big)$. The corresponding probability is given by d_z/d including $1/d$ for non-degenerate Z. The eigenstates of Z are fed through the apparatus $\mathcal{M}$. Then the task is to guess the input eigenvalue z. Contrary to the first test, we allow an arbitrary operation Ψ acting on both the classical outcome m and the actual quantum output of the apparatus. This operation is aimed to reverse a disturbance generated by $\mathcal{M}$ during the act of measurement. Thus, the notion of disturbance is related to the irreversible character of quantum measurements [24]. The disturbance is zero, whenever the input of the apparatus can be recast perfectly after the correction stage. A significance of unavoidable disturbance was emphasized in [24].

The pre-measurement state $\boldsymbol{\rho}$ will lead to statistics with probabilitis $p(x) = \mathrm{Tr}\big(\mathsf{\Lambda}(x)\,\boldsymbol{\rho}\big)$. Measuring by some instrument $\mathcal{M}$ results in outcomes m with corresponding probabilities (5). We wish to estimate quantitatively, whether the apparatus $\mathcal{M}$ measures X accurately. As the actual measurement outcome is kept, we try to guess which eigenstate has been input. The guessed value x' is represented as a function $g(m)$ of the measurement outcome. The "maximum *a posteriori* estimator" always gives $\hat{x}$ defined similarly to (20). Of course, an optimization over guessing functions can be taken into account.

When the pre-measurement state is taken to be completely mixed state $\boldsymbol{\rho}_* = \mathbb{1}_A/d$, we deal with the probability distribution $p(x)=d_x/d$. For non-degenerate observables, the input random variable X will be uniformly distributed. In effect, there are no general reasons to prefer one value of x to another. Then different outcomes x will equally contribute to an information-theoretic measure of noise. In the case of degeneracy, equal weights of the outcomes are rescaled appropriately. Due to Bayes' rule, the joint probability distribution of random variables is written as

$$p(m,x)=p(x)\,p(m|x)=\mathrm{Tr}\big(\mathsf{\Lambda}(x)\,\boldsymbol{\rho}_*\big)\,p(m|x)\ . \tag{49}$$

The conditional probability $p(m|x)$ is obtained by substituting the density matrix $\mathsf{\Lambda}(x)/d_x$ into the right-hand side of (5). The joint distribution (49) describes a common statistics of the input variable X and the output variable M. Hence, we can obtain conditional probabilities $p(x|m)=p(m,x)/p(m)$. The idea is that a contribution of the given m into a measure of noise should depend on corresponding conditional probabilities $p(x|m)$. The following property is physically natural for each fixed $m_\star$. The closer distribution $p(x|m_\star)$ to uniform, the larger its contribution to a measure of noise.

Using generalized conditional entropies, we will develop the ideas of [24]. For $\alpha\in(0,1]$, we define Rényi's information-theoretic noise of the instrument $\mathcal{M}$ as

$$N_\alpha^{(R)}(\mathcal{M},\mathsf{X}):=R_\alpha(X|M)\ . \tag{50}$$

Here, $R_\alpha(X|M)$ is the conditional Rényi α-entropy calculated from the joint probability distribution $p(m,x)$. In the case $d=2$, we allow to use (50) for $\alpha\in(0,2]$. For all $\alpha>0$, we define Tsallis' information-theoretic noise as

$$N_\alpha^{(T)}(\mathcal{M},\mathsf{X}):=\widetilde{H}_\alpha(X|M)\ . \tag{51}$$

The quantities (50) and (51) are respectively Rényi's and Tsallis' versions of the information-theoretic measure introduced in [24]. The latter is obtained from (50) and (51) in the case $\alpha=1$. Note that the definitions (50) and (51) do not assume an optimization over guessing functions. This question is closely related to the restriction $\alpha\in(0,1]$ used in the Rényi case. Let $M\mapsto g(M)$ be a function of random variable M. The standard conditional entropy obeys

$$H_1\big(X\big|g(M)\big)\geq H_1(X|M)\ . \tag{52}$$

Like (16), the inequality (52) is connected with the concavity property. In a similar manner, for all $\alpha>0$ the conditional entropy (15) satisfies

$$\widetilde{H}_\alpha\big(X\big|g(M)\big)\geq \widetilde{H}_\alpha(X|M)\ . \tag{53}$$

This result can be proved similarly to (52). The case of Rényi's entropies is more complicated. Together with (22), for $\alpha\in(0,1]$ we can obtain

$$R_\alpha\big(X\big|g(M)\big)\geq R_\alpha(X|M)\ . \tag{54}$$

For orders $\alpha>1$, we cannot assume concavity of the conditional Rényi α-entropy. As mentioned in section 2.3 of [37], the Rényi α-entropy is not concave for $\alpha>\alpha_\star>1$, where $\alpha_\star$ depends on dimensionality of probabilistic vectors. Unfortunately, sufficiently precise lower bounds on $\alpha_\star$ are not known. In principle, for $\alpha > \alpha_\star$ we could rewrite (50) with an optimization over guessing functions. At the same time, the property (22) is crucial in proving information-theoretic relations for noise and disturbance. Within the Rényi formulation, we therefore focus on the range $\alpha \in (0,1]$ in a finite-dimensional case and on the range $\alpha \in (0,2]$ in the two-dimensional case. Finally, we point out a conclusion based on the formulas of Subsection 2.3. Each of the information-theoretic noise (50) and (51) vanishes, if and only if the minimal error probability tends to zero.

The above scheme seems to be more natural for non-degenerate observables, when each outcome x is taken with the probability $1/d$. The non-degenerate case is not very restrictive. Of course, physical systems often have degenerate observables. As a rule, the degeneracy is connected with symmetries of the system. However, real systems are typically subjected to some amount, even if small, of disorder. Such small imperfections will inevitably break the degeneracy. In this sense, the results for non-degenerate observables are sufficiently general.

The second question concerns an information-theoretic approach to quantifying the unavoidable disturbance. To do so, we consider the second observable Z. As mentioned above, the main difference between the first and the second correlation experiments is that, in the second one, we permit to use both the classical outcome and the output quantum system. To fit the unavoidable disturbance, we assume any possible action after the measurement process [24]. A general correction procedure is represented by a trace-preserving completely positive map Ψ. It is used for reconstruction of the initial system A from the output system B and the measurement record. The final estimation is then obtained by a standard measurement of Z performed on the result of correction stage. The information-theoretic disturbance will depend on the joint probability distribution [24]

$$p(z',z)=p(z)\,p(z'|z)=\mathrm{Tr}\big(\mathsf{\Gamma}(z)\,\boldsymbol{\rho}_*\big)\,p(z'|z)\ . \tag{55}$$

This distribution characterizes correlations between the input eigenvalue z and the final estimation z'. The related conditional probability is expressed as

$$p(z'|z)=\frac{1}{d_z}\,\mathrm{Tr}\big[\,\mathsf{\Gamma}(z')\,\Psi\circ\Phi_{\mathcal{M}}\big(\mathsf{\Gamma}(z)\big)\big]\ . \tag{56}$$

Following [24], we use the two definitions. For $\alpha \in (0,1]$, we define Rényi's information-theoretic disturbance of the instrument $\mathcal{M}$ as

$$D_\alpha^{(R)}(\mathcal{M},\mathsf{Z}):=\min_{\Psi} R_\alpha(Z|Z')\ . \tag{57}$$

Here, the minimization is taken over all possible TPCP maps Ψ. In the case $d=2$, the measure (57) will be used for $\alpha \in (0,2]$. The conditional entropy $R_\alpha(Z|Z')$ is calculated from the joint probability distribution (55). Further, we define Tsallis' information-theoretic disturbance

$$D_\alpha^{(T)}(\mathcal{M},\mathsf{Z}):=\min_{\Psi} \widetilde{H}_\alpha(Z|Z')\ , \tag{58}$$

Let us discuss briefly some reasons for the above definitions. We write (57) with the restriction $\alpha \in (0,1]$, since the property (22) will be essential in the proofs. Further, the error probability of the final estimation is written as

$$q_e=\sum\nolimits_z p(e,z)\ , \qquad p(e,z)=\sum\nolimits_{z'\neq z} p(z',z)\ . \tag{59}$$

As was shown in [24] for the non-degenerate case, the error probability q_e is immediately connected with the average fidelity of correction. For non-degenerate Z, one has

$$1 - q_e = \frac{1}{d} \sum\nolimits_z \mathrm{F}\bigl(\Psi \circ \Phi_{\mathcal{M}}(|z\rangle\langle z|), |z\rangle\langle z|\bigr) \ . \tag{60}$$

Recall that the Schatten 1-norm $\|\mathsf{A}\|_1$ is defined as the sum of all singular values $\sigma_j(\mathsf{A})$ [63]. Then the fidelity between density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ is expressed as [74, 75]

$$\mathrm{F}(\boldsymbol{\rho}, \boldsymbol{\omega}) = \bigl\|\sqrt{\boldsymbol{\rho}}\,\sqrt{\boldsymbol{\omega}}\bigr\|_1^2 \ . \tag{61}$$

When the right-hand side of (60) reaches 1, the error probability q_e is zero and each of the quantities (57) and (58) vanishes. The latter follows from the inequalities (32)–(34).

3.2. *Tsallis and Rényi formulations*

In this subsection, we will derive Tsallis and Rényi formulations of noise-disturbance trade-off relations. We begin with relations that are based on the lower bounds (45) and (46). The first result is formulated as follows.

Proposition 1 *Let $\mathcal{M}$ be a measuring apparatus, and let X and Z be two observables. For all $\alpha > 0$ and $\beta > 0$, the Tsallis information-theoretic noise and disturbance satisfy*

$$N_{\alpha}^{(T)}(\mathcal{M}, \mathsf{X}) + D_{\beta}^{(T)}(\mathcal{M}, \mathsf{Z}) \geq \overline{\mathcal{B}}_{\alpha,\beta}^{(T)}(c) \ , \tag{62}$$

where the bound (45) is calculated for the characteristic (40).

Proof. By $\mathcal{H}_A$, we mean the Hilbert space of the principal quantum system. We also introduce its reference copy C with the isomorphic space $\mathcal{H}_C$. Fixing some orthonormal bases $\{|n_A\rangle\}$ for $\mathcal{H}_A$ and $\{|n_C\rangle\}$ for $\mathcal{H}_C$, one defines a maximally entangled state

$$|\Phi_{AC}^{+}\rangle = \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |n_A\rangle \otimes |n_C\rangle \ . \tag{63}$$

For any observable $\mathsf{X}_A \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$, we then express the partial trace

$$\mathrm{Tr}_C\Bigl((\mathbb{1}_A \otimes \mathsf{X}_C)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Bigr) = \frac{1}{d}\,\mathsf{X}_A^{\mathsf{T}} \ . \tag{64}$$

Here, the operator $\mathsf{X}_A^{\mathsf{T}}$ is transpose to X_A with respect to the prescribed basis. Hence, the so-called "ricochet" property holds [24]:

$$\frac{1}{d}\,|x_A\rangle\langle x_A| = \mathrm{Tr}_C\Bigl(\bigl(\mathbb{1}_A \otimes |x_C\rangle\langle x_C|^{\mathsf{T}}\bigr)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Bigr) \ . \tag{65}$$

Following [24], we use the fact that the two correlation experiments defining noise and disturbance can be treated as a single estimation producing a pair of random variables $U = (V, V')$. In particular, we may choose V to be a copy of M, while V' is the best possible estimate Z' for Z [24]. If some POVM $\bigl\{\mathsf{\Pi}_A(u)\bigr\}$ with $u \in \Omega_U$ corresponds to the estimation of U, then the conditional probabilities are expressed as

$$p(u|x) = \frac{1}{d_x}\,\mathrm{Tr}\bigl(\mathsf{\Pi}_A(u)\,\mathsf{\Lambda}_A(x)\bigr) \ , \tag{66}$$

$$p(u|z) = \frac{1}{d_z}\,\mathrm{Tr}\bigl(\mathsf{\Pi}_A(u)\,\mathsf{\Gamma}_A(z)\bigr) \ . \tag{67}$$

The joint probabilities are obtained after multiplying (66) by $p(x) = d_x/d$ and (67) by $p(z) = d_z/d$, respectively. So, we write

$$p(u,x) = \frac{1}{d}\,\mathrm{Tr}\big(\mathsf{\Pi}_A(u)\,\mathsf{\Lambda}_A(x)\big)\ , \tag{68}$$

$$p(u,z) = \frac{1}{d}\,\mathrm{Tr}\big(\mathsf{\Pi}_A(u)\,\mathsf{\Gamma}_A(z)\big)\ . \tag{69}$$

Due to the "ricochet" property (65) and linearity of the transpose operation, the probabilities can be rewritten as

$$p(u,x) = \mathrm{Tr}\Big(\big(\mathsf{\Pi}_A(u)\otimes\mathsf{\Lambda}_C(x)^{\mathsf{T}}\big)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Big)\ , \tag{70}$$

$$p(u,z) = \mathrm{Tr}\Big(\big(\mathsf{\Pi}_A(u)\otimes\mathsf{\Gamma}_C(z)^{\mathsf{T}}\big)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Big)\ . \tag{71}$$

We now consider an ensemble of mixed states $\boldsymbol{\rho}_C(u)$ with corresponding probabilities $p(u)$. These states and probabilities are written as

$$\boldsymbol{\rho}_C(u) = p(u)^{-1}\,\mathrm{Tr}_A\Big(\big(\mathsf{\Pi}_A(u)\otimes\mathbb{1}_C\big)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Big)\ , \tag{72}$$

$$p(u) = \mathrm{Tr}\Big(\big(\mathsf{\Pi}_A(u)\otimes\mathbb{1}_C\big)|\Phi_{AC}^{+}\rangle\langle\Phi_{AC}^{+}|\Big)\ . \tag{73}$$

We easily check that the probabilities (70) and (71) can be represented as

$$p(u,x) = p(u)\,\mathrm{Tr}\big(\mathsf{\Lambda}_C(x)^{\mathsf{T}}\boldsymbol{\rho}_C(u)\big)\ , \tag{74}$$

$$p(u,z) = p(u)\,\mathrm{Tr}\big(\mathsf{\Gamma}_C(z)^{\mathsf{T}}\boldsymbol{\rho}_C(u)\big)\ . \tag{75}$$

Hence, we have $\mathrm{Tr}\big(\mathsf{\Lambda}_C(x)^{\mathsf{T}}\boldsymbol{\rho}_C(u)\big) = p(x|u)$ and $\mathrm{Tr}\big(\mathsf{\Gamma}_C(z)^{\mathsf{T}}\boldsymbol{\rho}_C(u)\big) = p(z|u)$. Let us apply the entropic uncertainty relation for the Tsallis entropies. For each value of u, one gives

$$\mathrm{H}_\alpha\big(\mathsf{X}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\big) + \mathrm{H}_\beta\big(\mathsf{Z}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\big) \geq \overline{\mathcal{B}}_{\alpha,\beta}^{(T)}(\tilde{c})\ , \tag{76}$$

where the parameter $\tilde{c}$ is defined as

$$\tilde{c} := \max\Big\{\|\mathsf{\Lambda}(x)^{\mathsf{T}}\,\mathsf{\Gamma}(z)^{\mathsf{T}}\|_\infty :\ x\in\mathrm{spec}(\mathsf{X}),\ z\in\mathrm{spec}(\mathsf{Z})\Big\}\ , \tag{77}$$

It follows from the singular value theorem and (41) that the parameter $\tilde{c}$ coincides with (40). Multiplying (76) by $p(u)$ and summing over all $u\in\Omega_U$, we obtain

$$\widetilde{H}_\alpha(X|U) + \widetilde{H}_\beta(Z|U) \geq \overline{\mathcal{B}}_{\alpha,\beta}^{(T)}(c)\ , \tag{78}$$

due to $H_\alpha(X|u) = \mathrm{H}_\alpha\big(\mathsf{X}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\big)$ and $H_\alpha(Z|u) = \mathrm{H}_\alpha\big(\mathsf{Z}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\big)$. Since the property (17) holds for all $\alpha > 0$, we have

$$N_\alpha^{(T)}(\mathcal{M},\mathsf{X}) = \widetilde{H}_\alpha(X|M) \geq \widetilde{H}_\alpha(X|M,Z') = \widetilde{H}_\alpha(X|U)\ , \tag{79}$$

$$D_\beta^{(T)}(\mathcal{M},\mathsf{Z}) = \widetilde{H}_\beta(Z|Z') \geq \widetilde{H}_\beta(Z|M,Z') = \widetilde{H}_\beta(Z|U)\ . \tag{80}$$

Combining (78) with (79) and (80) completes the proof. ■

In a similar manner, we will obtain a formulation in the Rényi case. The following statement takes place.

Proposition 2 *Let $\mathcal{M}$ be a measuring apparatus, and let X and Z be two observables. When the orders α and β are both in the interval $(0,1]$, the Rényi information-theoretic noise and disturbance satisfy*

$$N_{\alpha}^{(R)}(\mathcal{M},\mathsf{X})+D_{\beta}^{(R)}(\mathcal{M},\mathsf{Z})\geq\overline{\mathcal{B}}_{\alpha,\beta}^{(R)}(c)\ , \tag{81}$$

where the bound (46) is calculated for the characteristic (40). In the case $\dim(\mathcal{H}_A)=2$, the trade-off relation (81) holds for $\alpha,\beta\in(0,2]$.

Proof. Repeating the argumentation between (63)–(77), we merely replace (76) with the relation

$$\mathrm{R}_{\alpha}\bigl(\mathsf{X}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)+\mathrm{R}_{\beta}\bigl(\mathsf{Z}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)\geq\overline{\mathcal{B}}_{\alpha,\beta}^{(R)}(\tilde{c})\ , \tag{82}$$

which holds for all $\alpha>0$ and $\beta>0$. Note that we have $\mathrm{R}_{\alpha}\bigl(\mathsf{X}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)=R_{\alpha}(X|u)$ and $\mathrm{R}_{\beta}\bigl(\mathsf{Z}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)=R_{\beta}(Z|u)$. Multiplying (82) by $p(u)$ and summing over all $u\in\Omega_U$, we obtain

$$R_{\alpha}(X|U)+R_{\beta}(Z|U)\geq\overline{\mathcal{B}}_{\alpha,\beta}^{(R)}(c)\ . \tag{83}$$

Similarly to (79) and (80), we write the following relations. When both the orders α and β lie in the range $(0,1]$, the property (22) leads to

$$N_{\alpha}^{(R)}(\mathcal{M},\mathsf{X})=R_{\alpha}(X|M)\geq R_{\alpha}(X|M,Z')=R_{\alpha}(X|U)\ , \tag{84}$$

$$D_{\beta}^{(R)}(\mathcal{M},\mathsf{Z})=R_{\beta}(Z|Z')\geq R_{\beta}(Z|M,Z')=R_{\beta}(Z|U)\ . \tag{85}$$

If $d=2$, these relations holds for $\alpha,\beta\in(0,2]$. Combining (83) with (84) and (85) completes the proof. ■

Propositions 1 and 2 are respectively the Tsallis and Rényi formulations of relations for noise and disturbance. In a certain sense, they are an extension of the noise-disturbance relation given in [24]. In our notation, the information-theoretic relation of the paper [24] is written as

$$N_1(\mathcal{M},\mathsf{X})+D_1(\mathcal{M},\mathsf{Z})\geq-2\ln c\ . \tag{86}$$

The authors of [24] defined the information-theoretic noise and disturbance in terms of the standard conditional entropy. So, we left out superscripts in the formula (86). Each of the definitions (50) and (51) leads to the standard information-theoretic noise in the limit $\alpha\to1$. In the same limit, both the definitions (57) and (58) gives the standard information-theoretic disturbance of [24]. The bounds (45) and (46) are not always c-optimal in general. Moreover, for $\alpha=\beta=1$ these bounds do not coincide with the Maassen–Uffink bound. Thus, the relations (62) and (81) do not lead to (86) in the case $\alpha=\beta=1$. We shall now derive such a direct extension. It is based on the entropic bound (47).

Proposition 3 *Let $\mathcal{M}$ be a measuring apparatus, and let X and Z be two observables. If $\alpha>0$ and $\beta>0$ obey $1/\alpha+1/\beta=2$, then*

$$N_{\alpha}^{(T)}(\mathcal{M},\mathsf{X})+D_{\beta}^{(T)}(\mathcal{M},\mathsf{Z})\geq\ln_{\mu}\bigl(c^{-2}\bigr)\ , \tag{87}$$

where $\mu=\max\{\alpha,\beta\}$ and the characteristic c is defined by (40).

Proof. The argumentation can be followed like the proof of Proposition 1. For each u, combining (74) and (75) with (47) finally gives

$$\mathrm{H}_{\alpha}\bigl(\mathsf{X}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)+\mathrm{H}_{\beta}\bigl(\mathsf{Z}_C^{\mathsf{T}};\boldsymbol{\rho}_C(u)\bigr)\geq\ln_{\mu}\bigl(c^{-2}\bigr)\ , \tag{88}$$

where $\mu = \max\{\alpha, \beta\}$ and $1/\alpha + 1/\beta = 2$. Then we complete the argumentation similarly to the proof of Proposition 1. ■

As a particular case of (87), we have the noise-disturbance relation (86) derived in [24]. Thus, our result (87) is an immediate extension of (86). A final comment concerns possible Rényi's formulation based on (48). Here, the concavity and related properties are crucial. If the dimensionality is not prescribed, the property (22) can be accepted only for $\alpha \in (0, 1]$. Combining the latter with $1/\alpha + 1/\beta = 2$ gives $\alpha = \beta = 1$. With (48), therefore, we could reach no more than (86). In the two-dimensional case, we can get a little extension. Here, non-trivial observables are certainly non-degenerate. For $d = 2$, we have

$$N_{\alpha}^{(R)}(\mathcal{M}, \mathsf{X}) + D_{\beta}^{(R)}(\mathcal{M}, \mathsf{Z}) \geq -2 \ln c \ , \tag{89}$$

where $1/\alpha + 1/\beta = 2$ and $\alpha, \beta \in (0, 2]$. A search for tightest bounds remains open in general. Novel uncertainty relations would lead to new trade-off relations for noise and disturbance.

4. Conclusions

We have obtained trade-off relations for noise and disturbance in terms of the Rényi and Tsallis information-theoretic measures. Our work is a further development of the approach originally proposed in [24]. As was shown in several cases, the use of generalized entropies may give new possibilities in analyzing statistical data. The presented information-theoretic measures of noise and disturbance are based on the conditional Rényi and Tsallis entropies. Introduced measures were motivated with the use of important properties of the conditional entropies. In particular, relations between the conditional entropies and the error probability were essential. We utilized several formulations of entropic uncertainty relations for a pair of observables. These formulations lead to trade-off relations for introduced measures of noise and disturbance. The scope of obtained results also depends on concavity properties of the considered entropies. In this regard, the Rényi formulation turns out to be somewhat restricted. In the noise-disturbance relations (62) and (81), the entropic parameters do not satisfy any constraint. We only specify an interval, in which the parameters should range. When the entropic parameters obey a certain constraint, we can use entropic bounds of the Maassen–Uffink type. Hence, we have obtained the noise-disturbance relations (87) and (89).

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