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A NOTE ON COHERENCE POWER OF N-DIMENSIONAL UNITARY OPERATORS

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The coherence power of a quantum channel, that is, its maximum ability to increase the coherence of input states, is a fundamental concept within the framework of the resource theory of coherence. In this note we discuss various possible definitions of coherence power and coherence rate and their basic properties. Then we prove that the coherence power of a unitary operator acting on a qubit, computed with respect to the l_1-coherence measure, can be calculated by maximizing its coherence gain over pure incoherent states. We proceed to show that this result fails in the general case, that is, the maximal coherence gain is found when acting on a state with non-vanishing coherence in the case of the l1-coherence and dimension $N > 2$, the relative entropy of coherence and the geometric measure of coherence.

Keywords: Coherence of Operators, Coherence, Decoherence

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1 Introduction

The development of quantum information science has led to a reassessment of quantum physical properties such as non-locality or entanglement, elevating them to resources that may be exploited to achieve tasks that are impossible when these properties are not available. The quantitative theory of entanglement [1] was perhaps the first example of a theory that was formulated by taking seriously the idea that quantum properties are physical resources. The starting point was to take the view that constraints, here the restriction to local operations and classical communication, prevent certain non-local physical operations from being realizable unless resources, here entangled states, are available which may be consumed to allow us to overcome the imposed constraints. The resource theory approach therefore allows to analyse physical tasks which are hard in a specific setting (such as non-local tasks for distant parties) by naturally qualifying and quantifying the resources needed (e.g. entanglement). Recently, this perspective has uncovered interrelations between thermodynamics and entanglement theory [2, 3] and made it possible to generalize the second law in the microscopic regime [4, 5, 6, 7, 8]. Furthermore, in the resource theory of thermodynamics [6, 9, 10], results

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from the theory of reference frames [11, 12]—yet another successful application of the resource theory approach—proved useful [13, 14].

Recently, [15] formulated a resource theory for quantum coherence, which is a fundamental trait of quantum mechanics. In this work the authors defined a number of coherence measures and outlined, following the example of the theory of entanglement, various extensions that would have to be completed to explore all the aspects of the resource theory of coherence. This includes the study of the interconversion of coherent states by means of incoherent operations both, in the single copy [16, 17] and the asymptotic regime [18, 19] as well as the characterisation of incoherent operations [20, 21, 22, 23] and the analysis of coherence measures [24]. Although not addressed from the perspective of resource theories, [25, 26] have also dealt with the quantification of quantum coherence and the formal characterization of coherence-decreasing processes. The relationship between coherence and entanglement has been studied from various angles [28, 29, 23, 30, 22, 31, 32, 33, 34, 35].

Aside of these developments it was pointed out in [15] that following the example of entanglement theory [36, 37] it would be natural to develop a quantitative theory of the coherence of operations which may have applications in the study of coherence in dynamical processes including biological systems where the presence and role of coherence remains a matter of current debate [38, 39]. Indeed, first steps in this direction were taken in [40, 41] which mostly considered the coherence power of operations when acting on incoherent states. In our work we will demonstrate that while being consistent, this is too restrictive as it can be shown that the achievable coherence gain can be higher when accepting states as inputs which already possess some coherence [46]. This mirrors similar observations in the realm of entanglement theory [42, 43].

After this introduction, in section 2 of our manuscript we repeat some basic definitions concerning coherence measures which will be followed by a discussion of possible definitions of coherence properties of operations. This will be followed in section 3 by a discussion of the l1-coherence power of unitaries on qubits where we prove that the maximum increase of coherence is achieved for incoherent states. Techniques used in this proof will then be useful in Section 4, which proceeds by demonstrating by means of three simple examples that for higher dimensional systems one needs to consider states with coherence to calculate the largest gain in coherence for an evolution. We conclude with a summary and outlook.

2 Basic Definitions

In this section we provide the basic definitions of the quantities that we will be exploring in this work.

Incoherent states and operations – In many situations the noisy evolution of the state will average out superpositions between eigenstates of an observable (e.g. the free Hamiltonian) and it can be hard to produce and control the off-diagonal elements with respect to a given set of projectors. To model this behaviour by a resource theory, [15] calls states incoherent that are diagonal in a fixed basis. Similarly, operations are called incoherent if they cannot produce non-incoherent states from incoherent ones.

Measures of coherence of states – One result of the resource theory of coherence are well-defined quantifiers of coherence, coherence measures, which are quantities that cannot increase under the action of incoherent operations. Several such coherence measures could be identified and include the relative entropy of coherence as well as the l_1-coherence [15]. While most definitions concerning the coherence power of operations can be formulated for any choice of coherence measure, for explicit calculations it is of advantage to consider the l_1-coherence measure

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|. \tag{1}$$

On the other hand, one might want to ask how much more coherence can be distilled after the application of a map than before. To answer this, notice that it has been shown that the distillable coherence in the asymptotic limit [18] is quantified by the relative entropy of coherence [15], which therefore takes a similarly central role in coherence theory as the relative entropy of entanglement [44] in the theory of entanglement [2, 3]. It follows that the greatest increase of distillable coherence is given by the coherence power calculated with respect to the relative entropy of coherence. The latter can be expressed by

$$C_{rel.ent.}(\rho) = S(\rho_{diag}) - S(\rho), \tag{2}$$

where ρ_{diag} denotes the diagonal part of the density matrix ρ [15].
As a third example, in [27] they consider the fidelity between quantum states

$$F(\rho, \delta) = \left(\mathbf{Tr} \left[\sqrt{\rho^{1/2} \delta \rho^{1/2}} \right] \right)^2 \tag{3}$$

and prove that one can use it to define a coherence monotone, dubbed geometric measure of coherence and given by

$$C_g(\rho) = 1 - \max_{\delta \in \mathcal{I}} F(\rho, \delta) \tag{4}$$

which for qubits can be easily computed as

$$C_g(\rho) = \frac{1}{2}(1 - \sqrt{1 - 4|\rho_{01}|^2}) \tag{5}$$

where $|\rho_{01}|$ is the off-diagonal element of ρ with respect to the reference basis [27].

Coherence properties of operations – Many physical questions relate to quantum operations and time evolution rather than directly to quantum states. Hence it is of considerable interest to examine the coherence properties of quantum operations or of their generators. We want to quantify how much a map is not incoherent. We thus define the coherence power of a map by the maximal amount of coherence it can produce, i.e. the maximal *coherence gain*:

Definition 1 *The coherence power $P(\Phi)$ of a completely positive operation Φ is defined relative to the coherence measure $C(.)$ via*

$$P(\Phi) = \max_{\rho} [C(\Phi(\rho)) - C(\rho)]. \tag{6}$$

For a unitary operation the coherence power is therefore

$$P(U) = \max_{\rho} [C(U \rho U^\dagger) - C(\rho)]. \tag{7}$$

We have deliberately left unrestricted the range over which the ρ in the maximization are taken, since we want the coherence power to bound the amount of coherence the operation can produce. This is useful, as it implies for instance that the coherence power of applying two operations consequently should not exceed the sum of the single coherence powers, that is, $P(\Phi_1) + P(\Phi_2) \geq P(\Phi_1 \circ \Phi_2)$.[d] In [40] the maximization range was restricted to the set of incoherent states, i.e. the states for which $C(\rho) = 0$. While this may appear to be a natural choice, it is not immediately clear that $C(\Phi(\rho)) - C(\rho)$ may actually be larger for some ρ with $C(\rho) > 0$. Indeed, motivated by similar observations in the theory of entanglement, we consider this question and answer it in the affirmative [46] in section 4.

We now note some properties that follow immediately from the definition.

Corollary 1 *The coherence power of an operation cannot increase by composing it with an incoherent operation.*

Proof:

$$\begin{aligned} P(\Phi_{inc} \circ \Phi) = \max_{\rho}[C(\Phi_{inc} \circ \Phi(\rho)) - C(\rho)] &= C(\Phi_{inc} \circ \Phi(\tilde{\rho})) - C(\tilde{\rho}) \\ &\leq C(\Phi(\tilde{\rho})) - C(\tilde{\rho}) \\ &\leq \max_{\rho}[C(\Phi(\rho)) - C(\rho)] = P(\Phi) \end{aligned}$$

where Φ_{inc} is an incoherent operation, Φ is an operation and $\tilde{\rho}$ is the state on which $P(\Phi_{inc} \circ \Phi)$ is achieved □.

Corollary 2 *The coherence power of a quantum operation is convex.*

Proof:

$$\begin{aligned} P(\sum_i \lambda_i \Phi_i) &= \max_{\rho}[C(\sum_i \lambda_i \Phi_i(\rho)) - C(\rho)] \\ &\leq \max_{\rho}[\sum_i \lambda_i C(\Phi_i(\rho)) - \sum_i \lambda_i C(\rho)] \\ &\leq \sum_i \lambda_i P(\Phi_i) \end{aligned}$$

where $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$ □.

Of interest in the context of dynamical systems are the time dependent generalizations of the above concepts. Let us consider for example a Markovian time evolution $\Phi_t(\rho)$ with generator $\mathcal{G}$, that is $\Phi_t = e^{\mathcal{G}t}$ or for the special case of a unitary operator $U_t = e^{-iHt}$. Then one may either apply directly definition 1 at a time t resulting in the coherence power for that time-step, or one may consider the amount of coherence that can maximally be generated in an infinitesimal time-step per time-step, the coherence rate by

[d]Since $P(\Phi_1 \circ \Phi_2) = \max_\rho[C(\Phi_1 \circ \Phi_2(\rho)) - C(\rho)] = \max_\rho[C(\Phi_1 \circ \Phi_2(\rho)) - C(\Phi_2(\rho)) + C(\Phi_2(\rho)) - C(\rho)] \leq P(U_1) + P(U_2)$.

Definition 2 *For a time evolution $\Phi_t = e^{\mathcal{G}t}$ we determine the coherence rate*

$$\Gamma(\mathcal{G}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \max_{\rho} [C(e^{\mathcal{G}\Delta t}\rho) - C(\rho)] \tag{8}$$

and in case of unitary evolutions $U(t) = e^{-iHt}$ we write

$$\Gamma(H) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \max_{\rho} [C(e^{-iH\Delta t}\rho e^{iH\Delta t}) - C(\rho)]. \tag{9}$$

The coherence rate thus captures the idea of how fast coherence may be produced in the course of a Markovian evolution. To get an intuition as to why this might be useful, suppose there is some environment acting on a system, forcing it to decohere. In this scenario an evolution which has a high coherence rate might still produce some coherence, while one with a small coherence rate compared with the coherence time of the system will keep the system almost incoherent throughout the whole evolution. In this case therefore, the evolution with a high coherence rate may still be quantum, notwithstanding the decohering environment. This line of thoughts suggests that the coherence rate might prove useful for quantifying quantumness in scenarios with a high temperature, such as biological systems. Furthermore, in the course of a quantum mechanical evolution the generation of coherence is inextricably linked with population transfer (as coherence between states requires population in both) and therefore a large coherence rate will imply a large population transfer rate. Linking such concepts quantitatively will first require a proper understanding of the concept of coherence rate.

We illustrate Definition 2 by calculating the l_1-coherence rate of the channel $\Phi(\Delta t) = e^{\mathcal{G}\Delta t}$ that for small Δt acts as

$$\rho \longrightarrow S(R\rho R^\dagger),$$

where

$$R(\alpha\Delta t) = \begin{pmatrix} \cos(\alpha\Delta t) & -\sin(\alpha\Delta t) \\ \sin(\alpha\Delta t) & \cos(\alpha\Delta t) \end{pmatrix}$$

is a 2-dimensional rotation and

$$S(\rho) = \begin{pmatrix} \rho_{00} & e^{-\gamma\Delta t}\rho_{01} \\ e^{-\gamma\Delta t}\rho_{10} & \rho_{11} \end{pmatrix}$$

is a dephasing operator. Let us first consider

$$\begin{aligned} C_{l_1}(e^{\mathcal{G}\Delta t}\rho) - C_{l_1}(\rho) &= C_{l_1}(S(R\rho R^\dagger)) - C_{l_1}(\rho) \\ &= 2e^{-\gamma\Delta t}|(R\rho R^\dagger)_{01}| - 2|\rho_{01}|. \end{aligned}$$

Let us now calculate the coherence rate of this map by maximizing the previous expression over pure incoherent states:

$$\begin{aligned} \Gamma^{l_1}(\mathcal{G}) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \max_{\rho \in \mathcal{I}} [C_{l_1}(S(R\rho R^\dagger))] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2|e^{-\gamma\Delta t}\cos(\alpha\Delta t)\sin(\alpha\Delta t)| \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2|(1-\gamma\Delta t)(1-\frac{\alpha^2\Delta t^2}{2})(\alpha\Delta t)| \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2|\alpha\Delta t| = 2|\alpha|. \end{aligned}$$

The coherence rate of the considered channel $\Phi(\Delta t)$, is therefore found to be at least $2|\alpha|$, which is the same coherence rate as that of the rotation $R(\alpha\Delta t)$ (taking into account that the coherence power of a 2-dimensional unitary operator is achieved on pure incoherent states, see Section 3). Since C_{l_1} is a proper coherence monotone satisfying $C_{l_1}(\Lambda_{inc}\rho) \leq C_{l_1}(\rho)$ we also have that

$$C_{l_1}(e^{\mathcal{G}\Delta t}\rho) - C_{l_1}(\rho) \leq C_{l_1}(R\rho R^\dagger) - C_{l_1}(\rho).$$

We conclude that the coherence rate of the considered channel can be calculated on pure incoherent states.

Note that one may also pursue questions concerning the coherence cost of an operation, that is, the amount of coherence in the form of maximally coherent states that is required to achieve an operation purely from incoherent operations. Questions regarding coherence cost and distillable coherence have been addressed in [18]. We will not pursue such quantities further here.

Of interest would be also to consider the N-dimensional unitary operations that have maximal coherence power. An example of this kind of unitaries is the discrete Fourier transform:

Corollary 3 *The coherence power of the discrete N-dimensional Fourier transform, calculated with respect to l_1-coherence, is maximal and is given by:*

$$P_{l_1}(\mathcal{F}) = N - 1. \tag{10}$$

Proof:

$$\begin{aligned} P_{l_1}(\mathcal{F}) &= \max_{\rho}[\frac{1}{N}\sum_{a\neq b}|\sum_{j,j'} e^{\frac{2\pi i}{N}(ja-j'b)}\rho_{jj'}| - \sum_{a\neq b}|\rho_{ab}|] \\ &\geq \max_{\rho=|k\rangle\langle k|}[\frac{1}{N}\sum_{a\neq b}|\sum_{j,j'} e^{\frac{2\pi i}{N}(ja-j'b)}\rho_{jj'}| - \sum_{a\neq b}|\rho_{ab}|] \\ &= N-1. \end{aligned}$$

Since $P_{l_1}(U) \leq N-1$, we conclude that a discrete N-dimensional Fourier transform is an example of unitary having maximal coherence power $\square$.

3 Coherence power of a 2-dimensional unitary operator

As we have already mentioned, it is a non-trivial question whether it suffices in Definition 1 to restrict ρ to incoherent states or whether the full range of possible states, including states with coherence, need to be considered. First we formulate and prove

Theorem 1 *The coherence power of a 2-dimensional unitary operation U acting on qubits and calculated with respect to the l_1-coherence is maximal for pure incoherent states*

$$P_{l_1}(U) = \max_{i=1,2}[C_{l_1}(U|i\rangle\langle i|U^\dagger)].$$

Proof: First we note that the coherence power of $R_z(\alpha)UR_z(\beta)$, where R_z denotes a diagonal (and hence incoherent) unitary, is the same as that for U.

$$\begin{aligned} P(R_z(\alpha)UR_z(\beta)) &= \max_\rho[C(R_z(\alpha)UR_z(\beta)\rho R_z^\dagger(\beta)U^\dagger R_z^\dagger(\alpha)) - C(\rho)] \\ &= \max_\rho[C(UR_z(\beta)\rho R_z^\dagger(\beta)U^\dagger) - C(\rho)] \\ &= \max_\rho[C(U\rho U^\dagger) - C(R_z^\dagger(\beta)\rho R_z(\beta)] \\ &= \max_\rho[C(U\rho U^\dagger) - C(\rho)] \\ &= P(U). \end{aligned}$$

Where the first and fifth equalities are the definition of the coherence power, the second and the fourth follow from $R_z(\alpha)$ (respectively $R_z(\beta)$) being incoherent and the third from the set of states being invariant under a change of basis. Now consider the unitary $M = R_z(\psi)UR_z(\phi)$

$$M = \begin{pmatrix} e^{i(\psi+\alpha)} & 0 \\ 0 & e^{-i(\psi-\alpha)} \end{pmatrix} \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \begin{pmatrix} e^{i(\phi+\beta)} & 0 \\ 0 & e^{-i(\phi-\beta)} \end{pmatrix} \tag{11}$$

where α and β are global phases without physical effect. We choose α and ψ such that $u_{gg}e^{i(\psi+\alpha)} \in \mathbb{R}^+$ and $u_{eg}e^{i(-\psi+\alpha)} \in \mathbb{R}$. We find

$$M = \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \tag{12}$$

with $u_{gg} \in \mathbb{R}^+$ and $u_{eg} \in \mathbb{R}$. Now choose $\phi = -\beta$ and make use of the orthonormality of the columns in a unitary

$$u_{gg}(u_{ge}e^{-2i\phi}) + u_{eg}(u_{ee}e^{-2i\phi}) = 0 \tag{13}$$

to conclude from $u_{gg}, u_{eg} \in \mathbb{R}$ that the phase of $u_{ge}e^{-2i\phi}$ and $u_{ee}e^{-2i\phi}$ is equal and can be eliminated by appropriate choice of ϕ. Therefore we can assume

$$M = \begin{pmatrix} u_{gg} & u_{ge} \\ u_{eg} & u_{ee} \end{pmatrix} \tag{14}$$

with u_{gg}, u_{eg}, u_{ge} and $u_{ee} \in \mathbb{R}$. Hence we can start by considering real U and using $\rho_{gg} = 1 - \rho_{ee}$ and $\rho_{eg} = \rho_{ge}e^{i\gamma}$ we find

$$\begin{aligned} P(U) \quad = \quad & 2\max_\rho[|u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) \\ & \qquad + \rho_{ge}(u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge})| - |\rho_{ge}|]. \end{aligned}$$

As the first two terms are real and the third term can be chosen to have any phase by virtue of the freedom of phase of ρ_{ge} we notice that the absolute value takes on its maximum value when $\rho_{ge}(u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge})$ is real and has the same sign as the sum of the first two terms. Now let us choose $\rho_{ge}(u_{ee}u_{gg}+e^{i\gamma}u_{eg}u_{ge}) \in \mathbb{R}$ and with the same sign as $u_{ee}u_{ge}+\rho_{gg}(u_{eg}u_{gg}-$

$u_{ee}u_{ge}$) (the case for opposite sign is treated analogously). Then there are two cases:
1) $u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) > 0$ which leads to

$$\begin{aligned} P(U) &= 2\max_{\rho}[(u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) \\ &\quad +|\rho_{ge}|(|u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}| - 1)]. \end{aligned}$$

As $U \in \mathbb{R}$ we have

$$|u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}| = \left| \begin{pmatrix} u_{gg} \\ u_{ge}e^{i\gamma} \end{pmatrix} \begin{pmatrix} u_{ee} \\ u_{eg} \end{pmatrix}^{\dagger} \right|.$$

As the vectors on the right are normalized the modulus of their scalar product is bounded by 1. Therefore $2|\rho_{ge}|(|u_{ee}u_{gg} + e^{i\gamma}u_{eg}u_{ge}| - 1) \leq 0$ and takes its maximum for $\rho_{eg} = 0$.
2) $u_{ee}u_{ge} + \rho_{gg}(u_{eg}u_{gg} - u_{ee}u_{ge}) < 0$ proceeds along the same lines.
The coherence power of a 2-dimensional unitary is therefore achieved for states ρ that are incoherent. To complete the proof of the theorem we now note that by the convexity $C(\sum_n p_n\rho_n) \leq \sum_n p_n C(\rho_n)$ for any set of states $\{\rho_n\}$ and probability distribution $\{p_n\}$ we find

$$\begin{aligned} C_{l_1}(U\rho_{inc}U^{\dagger}) &= C_{l_1}(U\sum_i p_i|i\rangle\langle i|U^{\dagger}) \\ &= C_{l_1}(\sum_i p_i U|i\rangle\langle i|U^{\dagger}) \\ &\leq \sum_i p_i C_{l_1}(U|i\rangle\langle i|U^{\dagger}) \\ &\leq C_{l_1}(U|i^*\rangle\langle i^*|U^{\dagger}) \end{aligned}$$

where $|i^*\rangle\langle i^*|$ is the pure incoherent state which has the largest contribution in the sum [45]. This concludes the proof □.

From theorem 1 we easily find

Corollary 4 *The coherence power of a 2-dimensional unitary operation U, calculated with respect to the l_1-coherence, is given by*

$$P_{l_1}(U) = \max_j\{(\sum_{i=1}^{2}|U_{ij}|)^2 : j = 1, 2\} - 1. \tag{15}$$

Proof: Since in order to compute the coherence power of a 2-dimensional unitary we need to maximize the gain over pure incoherent states only, we find

$$\begin{aligned} P_{l_1}(U) &= \max_{|k\rangle\langle k|}[C_{l_1}(U|k\rangle\langle k|U^{\dagger})] : k = 1, 2 \\ &= \max_j\{(\sum_{i=1}^{2}|U_{ij}|)^2 : j = 1, 2\} - 1 \end{aligned}$$

□.

4 Coherence power of an N-dimensional unitary operator ($N > 2$)

The question then arises whether theorem 1 is valid for any coherence measure, thus implying an intrinsic property of coherence power in 2-dimensional spaces, or whether it only holds for the considered l_1-coherence. Indeed, the coherence power has been defined in this way in [40]. However, it is not self-evident that the largest coherence gain is obtained from incoherent states. Indeed, in the theory of entanglement the analogous question, i.e. whether the entanglement gain is maximized by starting on separable states, has been answered in the negative [42, 43]. In the following we show that the same observation holds for the case of coherence power for the cases of the l1-coherence, the relative entropy of coherence (both cases have been independently addressed in [41]) and the geometric measure of coherence. Through our proof of the latter, we make explicit how symmetries can be used in general to simplify the problem using the techniques shown in theorem 1.

Corollary 5 *The coherence power of a 2-dimensional unitary operator calculated with respect to the geometric measure of coherence may be larger than the maximal coherence gain computed on incoherent states only.*

Proof:
As we have shown in the proof of theorem 1, we can assume our unitary to be real:

$$U_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{16}$$

Let us parametrize the state over which we are maximizing as $\rho = U_\alpha|0\rangle\langle 0|U_\alpha^\dagger$, where w.l.o.g. U_α is a real unitary as well (as in the proof of theorem 1). Then the expression we are interested in maximizing is given by

$$G_\alpha(U_\phi) = C_g(U_\phi U_\alpha|0\rangle\langle 0|U_\alpha^\dagger U_\phi^\dagger) - C_g(U_\alpha|0\rangle\langle 0|U_\alpha^\dagger). \tag{17}$$

Its first derivative with respect to α is plotted in figure 1 for a fixed $\phi = 0.0001$. From there it can be concluded, since no critical point is found at $\alpha = 0$, that there may exist non-incoherent states $\tilde{\rho} = U_{\tilde{\alpha}}|0\rangle\langle 0|U_{\tilde{\alpha}}^\dagger$, where $\alpha \neq 0$, which give rise to a larger increase in coherence than incoherent states when undergoing the action of a unitary operator. In fact, we have that the maximum coherence gain is achieved for $\alpha = 0.7852$:

$$G_{0.7852}(U_{0.0001}) = 9.99 \cdot 10^{-5} > G_0(U_{0.0001}) = 1 \cdot 10^{-8}.$$

Proposition 1 *For $N > 2$, the coherence power of an N-dimensional unitary operator can be larger than the maximal coherence gain computed on incoherent states only, for both the l1-coherence and the relative entropy of coherence.*

Proof: We consider the coherence power as quantified relative to the l_1-coherence and the relative entropy of coherence [15].

l_1-coherence power – Let us consider a 3-dimensional rotation by $\theta = \frac{\pi}{4}$ around the x axis:

$$R_x\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{18}$$

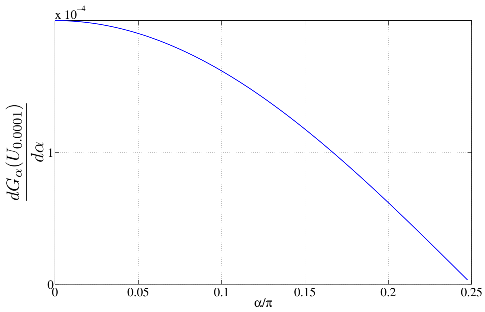

Fig. 1. First derivative of the coherence gain of the unitary $U_{0.0001}$ calculated with respect to the geometric measure of coherence.

According to corollary 4, its maximum coherence gain calculated over pure incoherent states is found to be:

$$\max_j \{ (\sum_{i=1}^{3} |R_x\left(\frac{\pi}{4}\right)_{ij}|)^2 : j = 1, 2, 3\} - 1 = 1. \tag{19}$$

It is easy to find examples of coherent states that provide a larger coherence gain for this particular rotation. The state $|\psi\rangle = c_1|1\rangle + c_3|3\rangle$ where $c_1 = 0.3$ and $c_3 = \sqrt{1-0.3^2}$, for instance, provides a coherence gain of 1.1471:

$$\begin{aligned} G_{|\psi\rangle\langle\psi|}\left(R_x\left(\frac{\pi}{4}\right)\right) &= C_{l_1}\begin{pmatrix} {c_1}^2 & -\frac{c_1\,c_3}{\sqrt{2}} & \frac{c_1\,c_3}{\sqrt{2}} \\ -\frac{c_1\,c_3}{\sqrt{2}} & \frac{{c_3}^2}{2} & -\frac{{c_3}^2}{2} \\ \frac{c_1\,c_3}{\sqrt{2}} & -\frac{{c_3}^2}{2} & \frac{{c_3}^2}{2} \end{pmatrix} - C_{l_1}\begin{pmatrix} {c_1}^2 & 0 & c_1\,c_3 \\ 0 & 0 & 0 \\ c_1\,c_3 & 0 & {c_3}^2 \end{pmatrix} \\ &= (2\sqrt{2}-2)c_1c_3 + c_3^2 \\ &= 1.1471 > 1. \end{aligned}$$

Relative entropy of coherence power – Assuming that the coherence power could be calculated by maximization of the gain over incoherent states, and the observation that by convexity of the relative entropy of coherence we can then restrict maximization to pure incoherent states, we find for the coherence power of an N-dimensional unitary with respect to the relative entropy of coherence, that it can be calculated by

$$P_{rel.ent.}(U) = \max_i \{ -\sum_{j=1}^{N} |U_{ij}|^2 \log(|U_{ij}|^2) : i = 1, ..., N \}, \tag{20}$$

since

$$\begin{aligned} P_{rel.ent.}(U) &= \max_{|i\rangle\langle i|} [C_{rel.ent}(U|i\rangle\langle i|U^\dagger) - C_{rel.ent.}(|i\rangle\langle i|) : i = 1, ..., N] \\ &= \max_{|i\rangle\langle i|} [S((U|i\rangle\langle i|U^\dagger)_{\text{diag}}) - S(U|i\rangle\langle i|U^\dagger) : i = 1, ..., N] \\ &= \max_{|i\rangle\langle i|} [S((U|i\rangle\langle i|U^\dagger)_{\text{diag}}) : i = 1, ..., N] \\ &= \max_i [-\sum_{j=1}^{N} |U_{ij}|^2 \log(|U_{ij}|^2) : i = 1, ..., N]. \end{aligned}$$

Let us now consider a 3-dimensional rotation of $\theta = \frac{\pi}{8}$ around the x axis:

$$R_x\left(\frac{\pi}{8}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ 0 & \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{pmatrix}. \tag{21}$$

Maximization of the coherence gain of this rotation over incoherent states results in

$$\max_i\{-\sum_{j=1}^{3} |R_x\left(\frac{\pi}{8}\right)_{ij}|^2 \log(|R_x\left(\frac{\pi}{8}\right)_{ij}|^2) : i = 1, 2, 3\} = 0.41650.$$

However we have found a number of coherent states that provide an even larger gain, such as the state $|\phi\rangle = q_2|2\rangle + q_3|3\rangle$ where $q_2 = \sqrt{1 - 0.12533^2}$ and $q_3 = 0.12533$:

$$G_{|\phi\rangle\langle\phi|}\left(R_x\left(\frac{\pi}{8}\right)\right) = 0.47648 > 0.41650. \tag{22}$$

The maximum gain of these two rotations, with respect to their corresponding coherence measure, is not achieved on pure incoherent states. Adapting the unitaries by direct addition of the identity one gets counterexamples in higher dimensions by the exact same arguments. Therefore the most natural definition of the coherence power is by maximization over *all* states □.

5 Conclusion

In this note we have discussed several possible definitions of coherence power and shown some basic properties of it. We have also proved that the coherence power of a 2-dimensional unitary operator with respect to the l_1-norm can be calculated by maximizing its coherence gain over pure incoherent states only. Giving an explicit counterexample, we could show that this result cannot be generalized for dimensions higher than $N = 2$ [46]. We also show that the result does not hold in the case of the geometric measure of coherence.

Hence, analogously to the result of entanglement theory, where it was observed that entangled states typically admit the largest gain in entanglement, we found that some initial coherence in the input state can be required for an optimal coherence gain to be attained. This result shows that it is not sufficient to maximize the coherence gain over incoherent states. It seems therefore an interesting question if one can restrict the optimization in higher dimension to a smaller subset or one needs to run it over the whole state space even for unitary evolutions. For non-unitary evolutions, while it seems challenging to try to find a generic simplification, one still might use the symmetries present in coherence theory to simplify the optimization for a given evolution, similarly as we used them here in the case of qubits and unitary evolution for proving theorem 1 or in the explicit examples in sections 2 and 4.

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45. Note that this has also been observed in [40], where the coherence power of an arbitrary quantum channel calculated for any coherence measure is proved to be maximum for pure states, provided that the maximization is performed over incoherent states only.
46. After completion of the first version of this work we became aware of [41] which independently found that the coherence power is generally achieved only by maximization over the full state space and also has a different proof for theorem 1, proposition 1 and corollary 4.