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MAXIMALLY COHERENT STATES

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The relative entropy measure quantifying coherence, a key property of quantum system, is proposed recently. In this note, we firstly investigate structural characterization of maximally coherent states with respect to the relative entropy measure. It is shown that mixed maximally coherent states do not exist and every pure maximally coherent state has the form $U|\psi\rangle\langle\psi|U^\dagger$, $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle$, U is diagonal unitary. Based on the characterization of pure maximally coherent states, for a bipartite maximally coherent state with $d_A = d_B$, we obtain that the super-additivity equality of relative entropy measure holds if and only if the state is a product state of its reduced states. From the viewpoint of resource in quantum information, we find there exists a maximally coherent state with maximal entanglement. Originated from the behaviour of quantum correlation under the influence of quantum operations, we further classify the incoherent operations which send maximally coherent states to themselves.

Keywords: Maximally coherent state, Relative entropy measure, Incoherent operation

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1 Introduction

Being at the heart of interference phenomena, quantum coherence plays a central role in physics as it enables applications that are impossible within classical mechanics or ray optics. It provides an important resource for quantum information processing, for example, Deutsch's algorithm, Shor's algorithm, teleportation, superdense coding and quantum cryptography [1]. Maximally coherent states are especially important for such quantum information processing tasks.

Recently, it has attracted much attention to quantify the amount of quantum coherence. In [2], the researchers establish a quantitative theory of coherence as a resource following the approach that has been established for entanglement in [3]. They introduce a rigorous framework for quantification of coherence by determining defining conditions for measures of coherence and identifying classes of functionals that satisfy these conditions. The relative entropy measure and l_1-norm measure are proposed. Other potential candidates such as the measures induced by the fidelity , l_2-norm and trace norm are also discussed. It is shown that the coherence measure induced by l_2-norm is not good. Since then, a lot of further considerations about quantum coherence are stimulated [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

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It has been shown that a good definition of coherence does not only depend on the state of the system, but also depends on a fixed basis for the quantum system [2]. The particular basis (of dimension d) we choose throughout this manuscript is denoted by $\{|k\rangle\}_{k=1}^{d}$. In [2], Baumgratz etc. identify the pure state $|\psi\rangle := \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle$ as a maximally coherent state (MCS) with respect to any measure of coherence because every state can be prepared from $|\psi\rangle$ by a suitable incoherent operation. Two natural questions arise immediately. Under a given coherence measurement, whether it is the unique pure state whose coherence is maximal and whether there exists a mixed maximally coherent state? Given a coherence measure $\mathcal{C}$, we call a state ρ to be a maximally coherent state (MCS) with respect to $\mathcal{C}$ if $\mathcal{C}(\rho)$ attends the maximal value of $\mathcal{C}$.

The relative entropy measure is able to not only quantify coherence but also quantify superposition and frameness [16, 17, 18, 19, 20, 21]. In [22], the regularized relative entropy measure of a resource can be used to describe the optimal rate of converting (by asymptotically resource non-generating operations) n copies of a resource state ρ into m copies of another resource state σ. On considering the importance of the relative entropy measure, we are aimed to characterize the structure of the maximally coherent states under the relative entropy coherence measure. We obtain that mixed maximally coherent states do not exist and each pure maximally coherent state has the form $U|\psi\rangle$, where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle$, U is diagonal unitary. While it does not mean maximally coherent states with respect to any coherence measure have the form $U|\psi\rangle$. Indeed there exists a coherence measure such that maximally coherent states with respect to this measure do not have the form $U|\psi\rangle$ (see the example after Result 1).

Quantum correlation includes quantum entanglement and quantum discord. Both entanglement and discord have a common necessary condition—quantum coherence [23]. In [4], Z. Xi etc. study the relative entropy coherence for a bipartite system in a composite Hilbert space $\mathcal{H}^{AB} = \mathcal{H}^{A} \otimes \mathcal{H}^{B}$. They obtain an interesting property for the relative entropy of coherence, that is, the super-additivity,

$$\mathcal{C}_{RE}(\rho) \geq \mathcal{C}_{RE}(\rho_A) + \mathcal{C}_{RE}(\rho_B).$$

At the same time, they leave an open question that whether the equality holds if and only if $\rho = \rho_A \otimes \rho_B$. Using characterization of MCS with respect to relative entropy coherence measure, we will show that this question holds true if the two subsystems have the same dimension and ρ is a MCS. A counterexample is also given to tell us that the answer is negative if the two subsystems have different dimension. Furthermore, we obtain that there is a state with maximal coherence and maximal entanglement.

Coherence, as a kind of resource, enables applications that are impossible within classical information. If an incoherent operation sends the MCSs to MCSs, we say it preserves MCSs. Naturally, does this kind of operation reduce the resource or is it a without noise process? We will show that an incoherent operation preserves MCSs if and only if it has the form $U \cdot U^{\dagger}$, U is a permutation of some diagonal unitary.

The structure of this paper is as follows. Section II recalls the axiomatic postulates for measures of coherence, the concepts of the relative entropy measure and incoherent operations in [2]. In section III, we focus on the structural characterization of maximally coherent states. We apply this characterization to bipartite system to answer the question on super-additivity equality in section IV. The section V is devoted to the incoherent operations preserving maximally coherent states. The paper is ended with the conclusion in section VI.

2 Preliminary

Let $\mathcal{H}$ be a finite dimensional Hilbert space with $d = \dim(\mathcal{H})$. Fixing a basis $\{|k\rangle\}_{k=1}^{d}$, we call all density operators (quantum states) that are diagonal in this basis incoherent, and this set of quantum states will be labelled by $\mathcal{I}$, all density operators $\rho \in \mathcal{I}$ are of the form

$$\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k|.$$

Quantum operations are specified by a finite set of Kraus operators $\{K_n\}$ with $\sum_n K_n^\dagger K_n = I$, I is the identity operator on $\mathcal{H}$. From [2], quantum operations are incoherent if they fulfil $K_n \rho K_n^\dagger / Tr(K_n \rho K_n^\dagger) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$ and for all n. This definition guarantees that in an overall quantum operation $\rho \mapsto \sum_n K_n \rho K_n^\dagger$, even if one does not have access to individual outcomes n, no observer would conclude that coherence has been generated from an incoherent state. Incoherent operations are of particular importance for the decoherence mechanisms of single qubit [24, 25]. As a special case, the unitary incoherent operation has the form $\rho \mapsto U\rho U^\dagger$, here U is a permutation of a diagonal unitary.

Based on Baumgratz et al.'s suggestion [2], any proper measure of coherence $\mathcal{C}$ must satisfy the following axiomatic postulates.

(i) The coherence vanishes on the set of incoherent states (faithful criterion), $\mathcal{C}(\rho) = 0$ for all $\rho \in \mathcal{I}$;

(ii) Monotonicity under incoherent operation Φ, $\mathcal{C}(\Phi(\rho)) \leq \mathcal{C}(\rho)$;

(iii) Non-increasing under mixing of quantum states (convexity),

$$\mathcal{C}(\sum_n p_n \rho_n) \leq \sum_n p_n \mathcal{C}(\rho_n)$$

for any ensemble $\{p_n, \rho_n\}$.

For any quantum state ρ on the Hilbert space $\mathcal{H}$, the measure of relative entropy coherence is defined as

$$\mathcal{C}_{RE}(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho || \sigma),$$

where $S(\rho||\sigma) = Tr(\rho \log_2 \rho - \rho \log_2 \sigma)$ is relative entropy. In particular, there is a closed form solution that makes it easy to evaluate analytical expressions [2]. For Hilbert space $\mathcal{H}$ with the fixed basis $\{|k\rangle\}_{k=1}^{d}$, we write $\rho = \sum_{k,k'} p_{k,k'} |k\rangle\langle k'|$ and denote $\rho_{diag} = \sum_k p_{kk} |k\rangle\langle k|$. By the properties of relative entropy, it is easy to obtain

$$\mathcal{C}_{RE}(\rho) = S(\rho_{diag}) - S(\rho),$$

here $S(\cdot)$ is von Neumann entropy. Some basic properties of relative entropy coherence have been given in [2].

Throughout the paper, if not specified, ρ is a maximally coherent state (MCS) means that it is with respect to $\mathcal{C}_{RE}$. As we mention in introduction, $|\psi\rangle := \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle$ is a maximally coherent state. That is, $\mathcal{C}_{RE}(|\psi\rangle\langle\psi|) = \log_2 d$ is the maximal value of $\mathcal{C}_{RE}$. The structural characterization of MCS plays a key role in section IV and V. An incoherent operation Φ preserves MCSs means that $\Phi(\rho)$ is a MCS if ρ is a MCS.

3 Maximally coherent states on $\mathcal{H}$

Result 1: ρ is a MCS if and only if $\rho = U|\psi\rangle\langle\psi|U^\dagger$, where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle$ and U is a diagonal unitary.

Proof: It is easy to see, for every diagonal unitary element U, $\rho \mapsto U\rho U^\dagger$ is an incoherent operation. From the monotonicity under incoherent operations, it follows that $U|\psi\rangle\langle\psi|U^\dagger$ is a MCS.

For if part, we firstly prove that ρ is pure. Note that the maximal value of $\mathcal{C}_{RE}$ is $\log_2 d$. For every pure state ensemble $\rho = \sum_i p_i\rho_i$. If $\mathcal{C}_{RE}(\rho) = \log_2 d$, then

$$\log_2 d = \mathcal{C}_{RE}(\rho) \leq \sum_i p_i \mathcal{C}_{RE}(\rho_i) \leq \log_2 d.$$

Thus $\mathcal{C}_{RE}(\rho) = \sum_i p_i\mathcal{C}_{RE}(\rho_i)$ and $\mathcal{C}_{RE}(\rho_i) = \log_2 d$. Let $\mathcal{C}_{RE}(\rho_i) = S(\rho_i \parallel \sigma_i)$ and $\sigma = \sum_i p_i\sigma_i$. By the jointly convex of relative entropy,

$$\log_2 d \leq S(\rho||\sigma) \leq \sum_i p_i S(\rho_i||\sigma_i) = \log_2 d.$$

This implies $S(\rho||\sigma) = \sum_i p_i S(\rho_i||\sigma_i)$. From [26, Theorem 10], it follows that $\rho_i = \rho_j$ and so ρ is a pure state.

Now, we write $\rho = |\phi\rangle\langle\phi|$ and $|\phi\rangle = \sum_{k=1}^{d}\alpha_k|k\rangle$. By the property of relative entropy,

$$\mathcal{C}_{RE}(\rho) = S(\rho_{diag}) = -\sum_{k=1}^{d}|\alpha_k|^2\log_2(|\alpha_k|^2).$$

A direct computation shows that $\mathcal{C}_{RE}(\rho) = \log_2(d)$ implies that $|\alpha_k|^2 = 1/d$. One can write $\alpha_k = \frac{1}{\sqrt{d}}e^{i\theta_k}$, then $|\phi\rangle = \sum_{k=1}^{d}\frac{1}{\sqrt{d}}e^{i\theta_k}|k\rangle$. Let $U = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_d})$, so $|\phi\rangle = U|\psi\rangle$. □

In [2], it is mentioned that if $\mathcal{D}$ is distance measure satisfying contracting under CPTP maps and jointly convex, i.e., satisfying

$$\mathcal{D}(\rho,\sigma) \geq \mathcal{D}(\Phi_{CPTP}(\rho), \Phi_{CPTP}(\sigma))$$

and

$$\mathcal{D}(\sum_n p_n\rho_n, \sum_n p_n\sigma_n) \leq \sum_n p_n\mathcal{D}(\rho_n,\sigma_n),$$

then one may define a coherence measure by

$$\mathcal{C}_{\mathcal{D}}(\rho) = \min_{\sigma\in\mathcal{I}}\mathcal{D}(\rho,\sigma).$$

From the proof of Result 1, it is easy to see that if $\mathcal{D}$ possesses the property that the equality of jointly convex holds true implies $\rho_n = \rho_m$, then the MCSs with respect to the coherence measure induced by $\mathcal{D}$ are pure. It is known that l_1-norm [2] and quantum skew divergence [27] are with such property.

Here we remark that Result 1 does not hold true for any coherence measure. The following is a counter example.

Example. Let $d = 4$ and $\Omega = \{\mathbf{x} = (x_1, x_2, x_3, x_4)^t \mid \sum_{i=1}^{4} x_i = 1 \text{ and } x_i \geq 0\}$, here $(x_1, x_2, x_3, x_4)^t$ denotes the transpose of row vector (x_1, x_2, x_3, x_4). Assume

$$f(\mathbf{x}) = \begin{cases} -\sum_{i=1}^{4} x_i \log_2 x_i, & x_4^{\downarrow} = 0 \\ \log_2 3, & x_4^{\downarrow} \neq 0 \end{cases},$$

here $x_4^{\downarrow}$ is the least element in $(x_1, x_2, x_3, x_4)^t$. By [15, Theorem 1], it is easy to check that the nonnegative function f can derive a coherence measure C_f. It is clear that both $|\psi\rangle = \sum_{k=1}^{4} \sqrt{x_k}|k\rangle, x_4^{\downarrow} \neq 0$ and $|\phi\rangle = \sum_{k=1}^{3} \sqrt{\frac{1}{3}}|k\rangle$ are maximally coherent under C_f.

4 Maximally coherent states on $H_A \otimes H_B$

Consider a bipartite system in a composite Hilbert space $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$ of $d = d_A \times d_B$ dimension, here $d_A = \dim(\mathcal{H}^A)$ and $d_B = \dim(\mathcal{H}^B)$. Let $\{|k\rangle^A\}_{k=1}^{d_A}$ and $\{|j\rangle^B\}_{j=1}^{d_B}$ be the orthogonal basis for the Hilbert space $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. Given a quantum state ρ_{AB} which could be shared between two parties, Alice and Bob, and let ρ_A and ρ_B be the reduced density operator for each party.

In [4], Xi etc. show the supper-additivity of the relative entropy coherence:

$$\mathcal{C}_{RE}(\rho_{AB}) \geq \mathcal{C}_{RE}(\rho_A) + \mathcal{C}_{RE}(\rho_B). \tag{1}$$

They leave an question that whether the equality holds if and only if $\rho = \rho_A \otimes \rho_B$. In the following, we will show that the answer is affirmative if $d_A = d_B$ and ρ_{AB} is a MSC. If $d_A \neq d_B$, then the answer is negative. This implies that, in the case of $d_A = d_B$, there is a correlation between the two subsystems, this leads to the increase of the coherence on the bipartite system.

Result 2: If $d_A = d_B$ and ρ_{AB} is a MCS, then the equality in (1) holds if and only if $\rho_{AB} = \rho_A \otimes \rho_B$.

Proof: Let $\{|i\rangle^A\}_{i=1}^{d_A}$ and $\{|j\rangle^B\}_{j=1}^{d_B}$ be the orthogonal basis for the Hilbert space H_A and H_B, respectively. Let $\rho_{AB} = |\phi\rangle\langle\phi|$ with $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d_A,d_B} e^{i\theta_{ij}} |i^A j^B\rangle$. Then

$$\rho = \frac{1}{d} \sum_{i,j,s,t} e^{i(\theta_{ij} - \theta_{st})})|i^A\rangle\langle s^A| \otimes |j^B\rangle\langle t^B|, \tag{2}$$

$$\rho_A = \frac{1}{d} \sum_{i,s} (\sum_{j} e^{i(\theta_{ij} - \theta_{sj})})|i^A\rangle\langle s^A|$$

and

$$\rho_B = \frac{1}{d} \sum_{j,t} (\sum_{i} e^{i(\theta_{ij} - \theta_{it})})|j^B\rangle\langle t^B|.$$

Note that $\mathcal{C}_{RE}(\rho_{AB}) = \mathcal{C}_{RE}(\rho_A) + \mathcal{C}_{RE}(\rho_B) \Leftrightarrow \rho_A, \rho_B$ are MCSs $\Leftrightarrow |\sum_i e^{i(\theta_{ij} - \theta_{it})}| = d_A$ and $|\sum_j e^{i(\theta_{ij} - \theta_{sj})}| = d_B$. The latter equivalence follows from Result 1. By a direct computation, we have

$$\theta_{ij} - \theta_{it} = \theta_{i'j} - \theta_{i't} \text{ and } \theta_{ij} - \theta_{sj} = \theta_{ij'} - \theta_{sj'}. \tag{3}$$

On the other hand,

$$\rho_A \otimes \rho_B = \frac{1}{d}\sum_{i,j,s,t} \alpha_{ijst}|i^A\rangle\langle s^A| \otimes |j^B\rangle\langle t^B|, \tag{4}$$

here $\alpha_{ijst} = \frac{1}{d}(\sum_j e^{i(\theta_{ij}-\theta_{sj})})(\sum_i e^{i(\theta_{ij}-\theta_{it})})$. From Equations (2),(3) and (4), we finish the proof. □

What will happen if $d_A \neq d_B$? The following counterexample shows the answer is negative in this case.

Assume $d_A = 2$ and $d_B = 3$. Let

$$|\phi\rangle = \frac{1}{\sqrt{6}}(|1\rangle + e^{i\theta}|2\rangle + e^{2i\theta}|3\rangle + e^{3i\theta}|4\rangle + e^{4i\theta}|5\rangle + e^{5i\theta}|6\rangle),$$

$\theta \in (0, 2\pi)$. Clearly, $\rho = |\phi\rangle\langle\phi|$ is a MCS. By an elementary computation,

$$\rho = \frac{1}{6}\begin{pmatrix} 1 & e^{-i\theta} & e^{-2i\theta} & e^{-3i\theta} & e^{-4i\theta} & e^{-5i\theta} \\ e^{i\theta} & 1 & e^{-i\theta} & e^{-2i\theta} & e^{-3i\theta} & e^{-4i\theta} \\ e^{2i\theta} & e^{i\theta} & 1 & e^{-i\theta} & e^{-2i\theta} & e^{-3i\theta} \\ e^{3i\theta} & e^{2i\theta} & e^{i\theta} & 1 & e^{-i\theta} & e^{-2i\theta} \\ e^{4i\theta} & e^{3i\theta} & e^{2i\theta} & e^{i\theta} & 1 & e^{-i\theta} \\ e^{5i\theta} & e^{4i\theta} & e^{3i\theta} & e^{2i\theta} & e^{i\theta} & 1 \end{pmatrix}.$$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & e^{-3i\theta} \\ e^{3i\theta} & 1 \end{pmatrix},$$

$$\rho_B = \frac{1}{3}\begin{pmatrix} 1 & e^{-i\theta} & e^{-2i\theta} \\ e^{i\theta} & 1 & e^{-i\theta} \\ e^{2i\theta} & e^{i\theta} & 1 \end{pmatrix}.$$

It is evident that both ρ_A and ρ_B are MCSs and $\mathcal{C}_{RE}(\rho_{AB}) = \mathcal{C}_{RE}(\rho_A) + \mathcal{C}_{RE}(\rho_B)$, however $\rho \neq \rho_A \otimes \rho_B$.

It is wellknown that both coherence and entanglement are considered as resource in quantum information. Whether is there a state which is not only maximally coherent but also maximally entangle? We will discuss this important question at the end of this section.

Result 3: There is a MCS ρ which is maximal entanglement.

Proof: Let $\rho = |\phi\rangle\langle\phi|$ with

$$|\phi\rangle = \frac{1}{\sqrt{d}}\sum_{i,j=1}^{d_A,d_B} e^{i\theta_{ij}}|i^A j^B\rangle.$$

Then $\rho_A = \frac{1}{d}\sum_{i,s}(\sum_j e^{i(\theta_{ij}-\theta_{sj})})|i^A\rangle\langle s^A|$. Recall that ρ is maximally entangled if and only if $\rho_A = \frac{I}{d_A}$. Therefore

$$\sum_j e^{i(\theta_{ij}-\theta_{sj})} = 0 \text{ for every pair } i \neq s \tag{5}$$

implies that ρ is a maximally entangle state. Note that The equation (5) has a solution. In order to understand the solution, we list an example in the case of $d_A = d_B = 3$. $\theta_{11} = \theta_{12} = \theta_{13} = 0$, $\theta_{21} = 0$, $\theta_{22} = -\frac{2\pi}{3}$, $\theta_{23} = -\frac{4\pi}{3}$, $\theta_{31} = 0$, $\theta_{32} = -\frac{4\pi}{3}$, and $\theta_{33} = -\frac{2\pi}{3}$. □

5 Incoherent operations preserving MCS

It is an interesting area to study the behavior of quantum correlation under the influence of quantum operations [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. For example, local operations that cannot create QD is investigated in [34, 37, 39], local operations that preserve the state with vanished MIN is characterized in [38] and local operations that preserve the maximally entangled states is explored in [40]. The goal of this chapter is to discuss when an incoherent operation preserves MCSs. Here is our main result in this section.

Result 4: An incoherent operation Φ preserves MCSs if and only if $\Phi(\rho) = U\rho U^\dagger$ for every quantum state ρ, here U is a permutation of a diagonal unitary.

From Result 4, every incoherent operation preserving MCSs does not reduce the resource and is noiseless. Although this result is not surprising, the proof is not trivial. Let Φ be specified by a set of Kraus operators $\{K_n\}$, the main step of our proof is to show that each $K_n = a_n \Pi_n$ after some reduction, a_n is a complex number with $\sum_n |a_n|^2 = 1$ and Π_n is a permutation of I. The reduction process is not trivial because we need to prove Φ is unital which is based on an interesting property that identity operator can be described as a sum of d MCSs.

Proof: The if part can be obtained directly from the Result 1.

Now we check the only if part. We firstly claim that I can be written as $\sum_{k=1}^d |\phi_k\rangle\langle\phi_k|$ with all of $|\phi_k\rangle$ are MCSs. Choose $|\phi_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d e^{i\alpha_{j,k}}|k\rangle$, all of $\alpha_{j,k}$ are real numbers. Denote $M = \sum_{j=1}^d |\phi_j\rangle\langle\phi_j|$, then M has the matrix form

$$\begin{pmatrix} 1 & \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,1}-\alpha_{j,2})} & \cdots & \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,1}-\alpha_{j,d})} \\ \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,2}-\alpha_{j,1})} & 1 & \cdots & \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,2}-\alpha_{j,d})} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,d}-\alpha_{j,1})} & \frac{1}{d}\sum_{j=1}^d e^{i(\alpha_{j,d}-\alpha_{j,2})} & \cdots & 1 \end{pmatrix}.$$

If $\alpha_{j,k}$ satisfy

$$\alpha_{j+1,k} - \alpha_{j+1,l} = \alpha_{j,k} - \alpha_{j,l} + \frac{2(k-l)}{d}\pi,$$

then $\sum_{j=1}^d e^{i(\alpha_{j,k}-\alpha_{j,l})} = 0$ $(j,k,l = 1,\ldots,d,\ k \neq l)$. So $M = I$. There exist solutions of these equations, for example $\alpha_{j,k} = \frac{2}{d}(k-1)(j-1)\pi$.

In the following, we show that Φ preserving MCS is unital, that is $\Phi(I) = I$. Note that Φ is incoherent, we have $\Phi(I)$ is diagonal. From the Result 1 in section III, $\Phi(|\phi_k\rangle\langle\phi_k|) = U_k|\psi\rangle\langle\psi|U_k^\dagger$, U_k is diagonal unitary. Then $(\Phi(|\phi_k\rangle\langle\phi_k|))_{diag} = \frac{I}{d}$, here $(\Phi(|\phi_k\rangle\langle\phi_k|))_{diag}$ denotes the state obtained from $\Phi(|\phi_k\rangle\langle\phi_k|)$ by deleting all off-diagonal elements. This implies that

$$\Phi(I) = \Phi(I)_{diag} = \sum_{k=1}^d (\Phi(|\phi_k\rangle\langle\phi_k|))_{diag} = I.$$

Let K_n be the Kraus operators of Φ, we obtain $\sum_n K_n K_n^\dagger = \sum_n K_n^\dagger K_n = I$. From Φ is incoherent, we also have that every column of K_n is with at most 1 nonzero entry. From Result 1, for every diagonal unitary U, there is a diagonal unitary V_U depending on U such that $\Phi(U|\psi\rangle\langle\psi|U^\dagger) = V_U|\psi\rangle\langle\psi|V_U^\dagger$. That is $|\psi\rangle\langle\psi|$ is a fixed point of $V_U^\dagger\Phi(U\cdot U^\dagger)V_U$. This implies

$$V_U^\dagger K_n U|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|V_U^\dagger K_n U.$$

So $V_U^\dagger K_n U|\psi\rangle = \lambda_{n,U}|\psi\rangle$ for some scalar $\lambda_{n,U}$ depending on U and n. We assert that $\lambda_{n,I} \neq 0$. Otherwise, K_n is singular and so there is a row of K_n in which all entries are zero. Note that $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^d |k\rangle$, therefore all $\lambda_{n,U}$ equal zero and so $K_n U|\psi| = 0$. Since I can be written as a sum of MCSs, we have $K_n = 0$. From $\lambda_{n,I} \neq 0$, there exists a nonzero element of each row of K_n. Combining this and each column of K_n is with at most one nonzero element, we get that there is one and only one nonzero entry in every row and column of K_n. Note that $V_I^\dagger \Phi V_I$ possesses the same properties as Φ, without loss of generality, we may assume $\Phi(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$. So $K_n|\psi\rangle = \lambda_{n,I}|\psi\rangle$. This implies the entries of K_n are equal. Therefore $K_n = a_n\Pi_n$, a_n is a complex number with $\sum_n |a_n|^2 = 1$ and Π_n is a permutation of I.

From Result 1, for arbitrary d real numbers $\theta_1, \cdots, \theta_d$, $|\phi\rangle = \sum_k \frac{1}{\sqrt{d}} e^{i\theta_k}|k\rangle$ is a MCS. By a direct computation, $K_n|\phi\rangle = \frac{a_n}{\sqrt{d}}\sum_k e^{i\alpha_{kn}}|k\rangle$, $(\alpha_{1n}, \cdots, \alpha_{dn}) = \Pi_n(\theta_1, \cdots, \theta_d)$. Furthermore, $K_n|\phi\rangle\langle\phi|K_n^\dagger$ is the matrix

$$\frac{|a_n|^2}{d}\begin{pmatrix} 1 & e^{i(\alpha_{1n}-\alpha_{2n})} & \cdots & e^{i(\alpha_{1n}-\alpha_{dn})} \\ e^{i(\alpha_{2n}-\alpha_{1n})} & 1 & \cdots & e^{i(\alpha_{2n}-\alpha_{dn})} \\ \cdots & \cdots & \cdots & \cdots \\ e^{i(\alpha_{dn}-\alpha_{1n})} & e^{i(\alpha_{dn}-\alpha_{2n})} & \cdots & 1 \end{pmatrix}.$$

And $\Phi(|\phi\rangle\langle\phi|)$ equals

$$\frac{1}{d}\begin{pmatrix} \sum_n |a_n|^2 & \sum_n |a_n|^2 e^{i(\alpha_{1n}-\alpha_{2n})} & \cdots & \sum_n |a_n|^2 e^{i(\alpha_{1n}-\alpha_{dn})} \\ \sum_n |a_n|^2 e^{i(\alpha_{2n}-\alpha_{1n})} & \sum_n |a_n|^2 & \cdots & \sum_n |a_n|^2 e^{i(\alpha_{2n}-\alpha_{dn})} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_n |a_n|^2 e^{i(\alpha_{dn}-\alpha_{1n})} & \sum_n |a_n|^2 e^{i(\alpha_{dn}-\alpha_{2n})} & \cdots & \sum_n |a_n|^2 \end{pmatrix}.$$

By our assumption, it is a MCS. So

$$|\sum_n |a_n|^2 e^{i(\alpha_{jn}-\alpha_{kn})}| = 1$$

for $j, k = 1, 2, \cdots, d$. The arbitrariness of α_{jn} and α_{kn} implies $n = 1$. Therefore Φ has the desired form. □

6 Conclusion

In this paper, we firstly investigate the maximally coherent states with respect to the relative entropy measure of coherence. We find that there does not exist a mixed maximally coherent state and each pure maximally coherent states have the form $U|\psi\rangle$, where U is a diagonal unitary and $|\psi\rangle := \frac{1}{\sqrt{d}}\sum_{k=1}^d |k\rangle$. Applying this structural characterization of maximally coherent states to bipartite system, we answer the question left in [4] whether $\mathcal{C}_{RE}(\rho_{AB}) = \mathcal{C}_{RE}(\rho_A) + \mathcal{C}_{RE}(\rho_B)$ if and only $\rho = \rho_A \otimes \rho_B$. It is shown that the answer is affirmative if $d_A = d_B$ and ρ_{AB} is a MSC. If $d_A \neq d_B$, then the answer is negative. From the viewpoint of resource of quantum information, we show that there exists a state which is not only maximally coherent but also maximally entangled. By using the form of pure maximally coherent states, we obtain the structural characterization of incoherent operations sending maximally coherent states into maximally coherent states. That is, an incoherent operation Φ preserves MCSs if and only if $\Phi(\rho) = U\rho U^\dagger$ for every quantum state ρ, here U is a permutation of a diagonal unitary.

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