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COHERENCE MEASURES AND OPTIMAL CONVERSION FOR COHERENT STATES

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We discuss a general strategy to construct coherence measures. One can build an important class of coherence measures which cover the relative entropy measure for pure states, the l_1-norm measure for pure states and the α-entropy measure. The optimal conversion of coherent states under incoherent operations is presented which sheds some light on the coherence of a single copy of a pure state.

Keywords: Coherence measure, Incoherent operation, Optimal conversion

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1 Introduction

Superposition is a critical property of quantum system resulting in quantum coherence and quantum entanglement. Quantum coherence and also entanglement provide the important resource for quantum information processing, for example, Deutschs algorithm, Shors algorithm, teleportation, superdense coding and quantum cryptography [1].

As with any such resource, there arises naturally the question of how it can be quantified and manipulated. Attempts have been made to find meaningful measures of entanglement [2, 3, 4, 5, 6, 7], and also to uncover the fundamental laws of its behavior under local quantum operations and classical communication (LOCC) [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Recently, it has attracted much attention to quantify the amount of quantum coherence. In [13], the researchers established a quantitative theory of coherence as a resource following the approach that has been established for entanglement in [6]. They introduced a rigorous framework for quantification of coherence and proposed several measures of coherence, which

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are based on the well-behaved metrics including the l_p-norm, relative entropy, trace norm and fidelity. Additional progress in this direction has been reported recently in [14, 15, 16, 17, 18, 19, 20, 21].

However, as far as a finite number of coherence measures are considered, the quantification of coherence is still in early stages. This work is intended to contribute to a better understanding of coherence. It presents a tool for build infinitely many coherence measures. Our recipes shows how to build all possible coherence measures for pure states (see sec. III).

By the tool of building coherence measures, we give the answer of the question: suppose there is a pure coherent state $|\psi\rangle = \sum_{i=1}^{d} \psi_i|i\rangle$ and we would like to convert it into another pure coherent state $|\phi\rangle = \sum_{i=1}^{d} \phi_i|i\rangle$ by incoherent operations. Which is the greatest probability of success in such a conversion? In [13], the authors provide a specific set of Kraus operators that allow us–with finite probability–to transform a pure state into another. There, they remarked that this protocol may not be optimal. In sec. IV, we provide a computation formula for the greatest probability and construct explicitly an incoherent operation achieving the greatest probability, i.e., the optimal protocol.

2 Preliminary

Let $\mathcal{H}$ be a finite dimensional Hilbert space with $d = \dim(\mathcal{H})$. Fixing a particular basis $\{|i\rangle\}_{i=1}^{d}$, we call all density operators (quantum states) that are diagonal in this basis incoherent, and this set of quantum states will be labelled by $\mathcal{I}$, all density operators $\rho \in \mathcal{I}$ are of the form

$$\rho = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|.$$

Incoherent operation— A quantum operation Φ is a trace-preserving completely positive linear map. By the classical Kraus representation theorem, the quantum operation Φ can be represented in an elegant form known as the operator-sum representation. That is, Φ is an operation if and only if there exist finite bounded linear operators K_n satisfying $\sum_n K_n^\dagger K_n = I$ and $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$, I is the identity operator on $\mathcal{H}$. From [13], the quantum operation Φ is incoherent if it fulfils $K_n \rho K_n^\dagger / Tr(K_n \rho K_n^\dagger) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$ and for all n. This definition guarantees that in an overall quantum operation $\rho \mapsto \sum_n K_n \rho K_n^\dagger$, even if one does not have access to individual outcomes n, no observer would conclude that coherence has been generated from an incoherent state. It is easy to see that a quantum operation is incoherent if and only if every column of K_n in the fixed basis $\{|i\rangle\}_{i=1}^{d}$ has at most one nonzero entry.

Based on Baumgratz et al.'s suggestion [13], any proper measure of coherence $\mathcal{C}$ must satisfy the following axiomatic postulates.

(C1) The coherence measure vanishes on the set of incoherent states, $\mathcal{C}(\rho) = 0$ for all $\rho \in \mathcal{I}$;

(C2a) Monotonicity under incoherent operation Φ, $\mathcal{C}(\Phi(\rho)) \leq \mathcal{C}(\rho)$,

or (C2b) Monotonicity under selective measurements on average: $\sum_n p_n \mathcal{C}(\rho_n) \leq \mathcal{C}(\rho)$, where $p_n = tr(K_n \rho K_n^\dagger)$, $\rho_n = \frac{1}{p_n} K_n \rho K_n^\dagger$, for all $\{K_n\}$ with $\sum_n K_n^\dagger K_n = I$ and

$$K_n \rho K_n^\dagger / Tr(K_n \rho K_n^\dagger) \in \mathcal{I} \text{ for all } \rho \in \mathcal{I};$$

(C3) Non-increasing under mixing of quantum states (convexity),

$$\mathcal{C}(\sum_n p_n\rho_n) \leq \sum_n p_n\mathcal{C}(\rho_n)$$

for any ensemble $\{p_n, \rho_n\}$.

We remark that conditions (C2b) and (C3) imply condition (C2a). And it has been recently shown that the coherence measure induced by the fidelity satisfies (C2a), violates (C2b) [14]. For the coherence measure induced by the trace norm, it is still not known whether it satisfies criterion (C2b).

3 Building coherence measures

The following focuses on coherent measures for pure states and extends these coherent measures over the whole set of quantum states. Our idea is originated from [7] which is devoted to entanglement monotone. Similarly, we define coherence monotone to be any magnitude satisfying conditions (C2b) and (C3). From the following Theorem 1 and Theorem 2, readers familiar with entanglement theory will see, in the case of pure states, the f considered in [7] can derive a coherence monotone. While the converse is not true. The key lies in that the entanglement monotone is local unitary invariant, but the coherence monotone is only invariant under some special unitary transformation (the permutation of a diagonal unitary).

Let $\Omega = \{\mathbf{x} = (x_1, x_2, \cdots, x_d)^t \mid \sum_{i=1}^d x_i = 1 \text{ and } x_i \geq 0\}$, here $(x_1, x_2, \cdots, x_d)^t$ denotes the transpose of row vector $(x_1, x_2, \cdots, x_d)$. And let π be an arbitrary permutation of $\{1, 2, \cdots, d\}$, P_π be the permutation matrix corresponding to π which is obtained by permuting the rows of a $d \times d$ identity matrix according to π. Given any nonnegative function $f : \Omega \mapsto \mathcal{R}^+$ such that it is

- $$f(P_\pi(1, 0, \cdots, 0)^t) = 0, \tag{1}$$

  for every permutation π,

- invariant under any permutation transformation P_π, i.e.

  $$f(P_\pi\mathbf{x}) = f(\mathbf{x}) \text{ for every } \mathbf{x} \in \Omega, \tag{2}$$

- concave, i.e.

  $$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \tag{3}$$

  for any $\lambda \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \Omega$.

A coherence measure can be derived by defining it for pure states (normalized vectors $|\psi\rangle = (\psi_1, \psi_2, \cdots, \psi_d)^t$ in the fixed basis $\{|i\rangle\}_{i=1}^d$) as

$$C_f(|\psi\rangle\langle\psi|) = f((|\psi_1|^2, |\psi_2|^2, \cdots, |\psi_d|^2)^t), \tag{4}$$

and by extending it over the whole set of density matrices as

$$C_f(\rho) = \min_{p_j,\rho_j} \sum_j p_j C_f(\rho_j), \tag{5}$$

where the minimization is to be performed over all the pure-state ensembles of ρ, i.e., $\rho = \sum_j p_j \rho_j$.

Theorem 1. Any function C_f satisfying (1)-(5) is a coherence measure, i.e.,

$$\text{Eqs. } (1)-(5) \Rightarrow C1, C2b, C3. \tag{6}$$

Proof: For any $\rho \in \mathcal{I}$ and $\rho = \sum_i p_i |i\rangle\langle i|$. From the definition of C_f and Eq.(1), it follows that $C_f(\rho) \le \sum_i p_i C_f(|i\rangle\langle i|) = 0$.

To verify C2b, we assume firstly that ρ is a pure state $|\psi\rangle\langle\psi|$. Here, it is needed to use the technique to deal with the acting of incoherent operations on pure states developed in the proof of only if part of Theorem 1 [20]. Considering that there is some subtle difference and the completeness of the proof, we write it in detail. This technique allows us to consider only the three dimensional case, other cases can be treated similarly. Suppose $|\psi\rangle = (\psi_1, \psi_2, \psi_3)^t$ and there is an incoherent operation Φ with Kraus operators K_n. Since, for arbitrary permutation matrix P_π, $\{K_n P_\pi\}$ can define an incoherent operation, we assume

$$|\psi_1| \ge |\psi_2| \ge |\psi_3| \tag{7}$$

without loss of generality. Denote $p_n = \|K_n|\psi\rangle\|^2$ and $\rho_n = \frac{1}{p_n} K_n |\psi\rangle\langle\psi| K_n^\dagger$. Let $k_j^{(n)} (j = 1,2,3)$ be the nonzero element of K_n at $j-th$ column (if there is no nonzero element in $j-th$ column, then $k_j^{(n)} = 0$). Suppose $k_j^{(n)}$ locates $f_n(j)-th$ row. Here, $f_n(j)$ is a function that maps $\{2,3\}$ to $\{1,2,3\}$ with the property that $1 \le f_n(j) \le j$. Let $\delta_{s,t} = \begin{cases} 1, & s = t \\ 0, & s \ne t \end{cases}$. Then there is a permutation π_n such that

$$K_n = P_{\pi_n} \begin{pmatrix} k_1^{(n)} & \delta_{1,f_n(2)} k_2^{(n)} & \delta_{1,f_n(3)} k_3^{(n)} \\ 0 & \delta_{2,f_n(2)} k_2^{(n)} & \delta_{2,f_n(3)} k_3^{(n)} \\ 0 & 0 & \delta_{3,f_n(3)} k_3^{(n)} \end{pmatrix}. \tag{8}$$

From $\sum_n K_n^\dagger K_n = I$, we get that

$$\begin{cases} \sum_n |k_j^{(n)}|^2 = 1, (j = 1,2,3), \\ \sum_n \overline{k_1^{(n)}} \delta_{1,f_n(2)} k_2^{(n)} = 0, \\ \sum_n \overline{k_1^{(n)}} \delta_{1,f_n(3)} k_3^{(n)} = 0, \\ \sum_n (\delta_{1,f_n(2)} \delta_{1,f_n(3)} + \delta_{2,f_n(2)} \delta_{2,f_n(3)}) \overline{k_2^{(n)}} k_3^{(n)} = 0. \end{cases} \tag{9}$$

For $|\psi\rangle = (\psi_1, \psi_2, \psi_3)^t$, by a direct computation, one can get

$$K_n |\psi\rangle = P_{\pi_n} \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \\ \phi_3^{(n)} \end{pmatrix}, \tag{10}$$

here

$$\begin{cases} \phi_1^{(n)} = k_1^{(n)} \psi_1 + \delta_{1,f_n(2)} k_2^{(n)} \psi_2 + \delta_{1,f_n(3)} k_3^{(n)} \psi_3, \\ \phi_2^{(n)} = \delta_{2,f_n(2)} k_2^{(n)} \psi_2 + \delta_{2,f_n(3)} k_3^{(n)} \psi_3, \\ \phi_3^{(n)} = \delta_{3,f_n(3)} k_3^{(n)} \psi_3. \end{cases} \tag{11}$$

Applying $\sum_n |\cdot|^2$ to above equations, we have

$$\begin{cases} |\psi_1|^2 + \sum_n \delta_{1,f_n(2)}|k_2^{(n)}|^2|\psi_2|^2 \\ \quad + \sum_n \delta_{1,f_n(3)}|k_3^{(n)}|^2|\psi_3|^2 = \sum_n |\phi_1^{(n)}|^2, \\ \sum_n \delta_{2,f_n(2)}|k_2^{(n)}|^2|\psi_2|^2 \\ \quad + \sum_n \delta_{2,f_n(3)}|k_3^{(n)}|^2|\psi_3|^2 = \sum_n |\phi_2^{(n)}|^2, \\ \sum_n \delta_{3,f_n(3)}|k_3^{(n)}|^2|\psi_3|^2 = \sum_n |\phi_3^{(n)}|^2. \end{cases} \tag{12}$$

Together with Eqs.(7) and (9), (12) implies that

$$\begin{array}{rl} & ((|\psi_1|^2, |\psi_2|^2, |\psi_3|^2)^t \\ \prec & (\sum_n |\phi_1^{(n)}|^2, \sum_n |\phi_2^{(n)}|^2, \sum_n |\phi_3^{(n)}|^2)^t. \end{array} \tag{13}$$

Here "$\prec$" is the majorization relation between vectors, the definition and properties of which can be found in [22]. From Eqs.(2) and (3), it follows that

$$\begin{array}{rl} & \sum_n p_n C_f(\rho_n) \\ = & \sum_n p_n f(P_{\pi_n}(\frac{|\phi_1^{(n)}|^2}{p_n}, \frac{|\phi_2^{(n)}|^2}{p_n}, \frac{|\phi_3^{(n)}|^2}{p_n})^t) \\ = & \sum_n p_n f((\frac{|\phi_1^{(n)}|^2}{p_n}, \frac{|\phi_2^{(n)}|^2}{p_n}, \frac{|\phi_3^{(n)}|^2}{p_n})^t) \\ \leq & f((\sum_n |\phi_1^{(n)}|^2, \sum_n |\phi_2^{(n)}|^2, \sum_n |\phi_3^{(n)}|^2)^t) \\ \leq & f((|\psi_1|^2, |\psi_2|^2, |\psi_3|^2)^t) = C_f(\rho). \end{array} \tag{14}$$

The last inequality is from [22, Theorem II.3.3], that is, any symmetric concave function is Schur-concave, i.e., $f(\mathbf{x}) \geq f(\mathbf{y})$ if $\mathbf{x} \prec \mathbf{y}$.

Suppose now that ρ is any mixed state. Let $\rho = \sum_i q_i \sigma_i$ is an optimal pure-state ensemble with $C_f(\rho) = \sum_i q_i C_f(\sigma_i)$. Then

$$\begin{array}{rl} & \sum_n p_n C_f(\rho_n) = \sum_n p_n C_f(\frac{\sum_i q_i K_n \sigma_i K_n^\dagger}{p_n}) \\ \leq & \sum_n p_n \sum_i q_i \frac{tr(K_n\sigma_i K_n^\dagger)}{p_n} C_f(\frac{K_n \sigma_i K_n^\dagger}{tr(K_n\sigma_i K_n^\dagger)}) \\ = & \sum_i q_i (\sum_n tr(K_n \sigma_i K_n^\dagger) C_f(\frac{K_n \sigma_i K_n^\dagger}{tr(K_n\sigma_i K_n^\dagger)})) \\ \leq & \sum_i q_i C_f(\sigma_i) = C_f(\rho). \end{array} \tag{15}$$

Two inequalities follow from Eq.(5) and Eq.(14), respectively.

Now we prove C3 holds true. Let $\rho = \sum_i p_i \rho_i$ be any ensemble of ρ. And let $\rho_i = \sum_j q_{ij} \rho_{ij}$ be an optimal pure-state ensemble of ρ_i, i.e., $C_f(\rho_i) = \sum_j q_{ij} C_f(\rho_{ij})$. Then

$$\begin{array}{rl} & C_f(\rho) = C_f(\sum_i p_i \sum_j q_{ij} \rho_{ij}) \\ = & C_f(\sum_{ij} p_i q_{ij} \rho_{ij}) \leq \sum_{ij} p_i q_{ij} C_f(\rho_{ij}) \\ = & \sum_i p_i C_f(\rho_i). \end{array} \tag{16}$$

The inequality follows from Eq.(5). □

As examples of coherence measures built using Theorem 1 consider, for any nonnegative function $\widehat{f}(x)$ concave in the interval $x \in [0, 1]$ with $\widehat{f}(0) = \widehat{f}(1) = 0$, the function $f : \mathcal{R}^d \to \mathcal{R}^+$ defined by $f((x_1, x_2, \cdots, x_d)^t) = \sum_i \widehat{f}(x_i)$. Then f satisfies Eqs.(1)-(3). Taking

$$\widehat{f}(x) = -x \log_2 x, \tag{17}$$

we can induce the coherence measure $\mathcal{C}_f$ which is identical with the relative entropy coherence measure [13] on pure states. Generally speaking, they are different on mixed states. Choosing

$$f((x_1, x_2, \cdots, x_d)^t) = (\sum_{i=1}^{d} \sqrt{x_i})^2 - 1, \tag{18}$$

we can easily check that the coherence measure $\mathcal{C}_f$ is identical with l_1-norm coherence measure [13] on pure states. They are indeed different on mixed states. It is well known that α-entropy are used to measure the uncertainty. We can also define α-entropy coherence measure. Let

$$f((x_1, x_2, \cdots, x_d)^t) = \frac{1}{1-\alpha} \log_2 \sum_{i=1}^{d} x_i^{\alpha}, \quad 0 < \alpha < 1. \tag{19}$$

It follows from the fact that the logarithm is a concave, non-decreasing function and therefore preserves concavity, that the α-entropy is a concave function. The Eqs.(1)(2) are easy to check. Consequently, Theorem 1 can be applied directly to prove that the α-entropy can derive a coherence measure.

In the following, we will show that one can construct any coherence measure for pure states by our strategy.

Theorem 2. The restriction of any coherence measure (satisfying C1,C2b and C3) to pure states can be derived by a function $f : \Omega \to \mathcal{R}^+$ satisfying Eqs.(1)-(3).

Proof: Let μ be an arbitrary coherence measure and let U be a diagonal unitary matrix. From the monotonicity of coherence measures under incoherent operations, it is evident that $\mu(U\rho U^\dagger) \leq \mu(\rho)$. Symmetrically, one can see that $\mu(\rho) = \mu(U^\dagger(U\rho U^\dagger)U) \leq \mu(U\rho U^\dagger)$. So $\mu(U\rho U^\dagger) = \mu(\rho)$. Define $f : \Omega \to \mathcal{R}^+$ by $f((x_1, x_2, \cdots, x_d)^t) = \mu(|\psi\rangle\langle\psi|)$, where $|\psi\rangle = \sum_{i=1}^{d} \sqrt{x_i}|i\rangle$. For any pure state $|\psi\rangle = \sum_i \psi_i|i\rangle$, there exists diagonal unitary matrix U such that $U|\psi\rangle = \sum_i |\psi_i||i\rangle$. It follows that

$$\mu(|\psi\rangle\langle\psi|) = \mu(U|\psi\rangle\langle\psi|U^\dagger) = f((|\psi_1|^2, |\psi_2|^2, \cdots, |\psi_d|^2)^t). \tag{20}$$

In the following, we will check that f satisfies Eqs.(1)-(3). The Eq.(1) follows from C1. Let π be a permutation of $\{1, 2, \cdots, d\}$, by the definition of f, we have

$$f((x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(d)})^t) = \mu(P_\pi|\psi\rangle\langle\psi|P_\pi). \tag{21}$$

By the same argument as the diagonal unitary matrix case, one can obtain

$$\mu(P_\pi|\psi\rangle\langle\psi|P_\pi) = \mu(|\psi\rangle\langle\psi|). \tag{22}$$

This implies that f is invariant under any permutation transformation. To prove Eq.(3), for

$$\mathbf{x} = (x_1, x_2, \cdots, x_d)^t,$$

$$\mathbf{y} = (y_1, y_2, \cdots, y_d)^t \in \Omega$$

and $\lambda \in [0, 1]$, we define

$$K_1 = \mathrm{diag}(\tfrac{\sqrt{\lambda x_1}}{\sqrt{\lambda x_1 + (1-\lambda) y_1}}, \tfrac{\sqrt{\lambda x_2}}{\sqrt{\lambda x_2 + (1-\lambda) y_2}}, \cdots, \tfrac{\sqrt{\lambda x_d}}{\sqrt{\lambda x_d + (1-\lambda) y_d}}),$$

$$K_2 = \text{diag}(\frac{\sqrt{(1-\lambda)y_1}}{\sqrt{\lambda x_1+(1-\lambda)y_1}}, \frac{\sqrt{(1-\lambda)y_2}}{\sqrt{\lambda x_2+(1-\lambda)y_2}}, \cdots, \frac{\sqrt{(1-\lambda)y_d}}{\sqrt{\lambda x_d+(1-\lambda)y_d}}).$$

It is easy to check that $\Phi(\cdot) = K_1 \cdot K_1^\dagger + K_2 \cdot K_2^\dagger$ is an incoherent operation. And

$$K_1 \sum_i \sqrt{\lambda x_i + (1-\lambda)y_i}|i\rangle = \sqrt{\lambda}\sum_i \sqrt{x_i}|i\rangle, \tag{23}$$

$$K_2 \sum_i \sqrt{\lambda x_i + (1-\lambda)y_i}|i\rangle = \sqrt{1-\lambda}\sum_i \sqrt{y_i}|i\rangle. \tag{24}$$

From (C2b), we get that

$$\begin{array}{ll} & \lambda\mu(\sum_i \sqrt{x_i}|i\rangle) + (1-\lambda)\mu(\sum_i \sqrt{y_i}|i\rangle) \\ \leq & \mu(\sum_i \sqrt{\lambda x_i + (1-\lambda)y_i}|i\rangle). \end{array} \tag{25}$$

That is $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y})$, i.e., f is concave. □

4 Optimal conversion for coherent states

The section is devoted to the optimal conversion probability in a single-copy scenario. In [9], an optimal local conversion strategy between any two pure entangled states of a bipartite system is presented . In [23], Brandão and Gour have proposed a general framework to analyse the conversion in the asymptotic limit and shown that a quantum resource theory is asymptotically reversible if its set of allowed operations is maximal.

For pure states $|\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle, |\phi\rangle = \sum_{i=1}^d \phi_i|i\rangle$, we can assume that $|\psi_1| \geq |\psi_2| \geq \cdots \geq |\psi_d|$ and $|\phi_1| \geq |\phi_2| \geq \cdots \geq |\phi_d|$. Indeed, in general case, there exist two permutations π, σ of $\{1, 2, \cdots, d\}$ such that $|\psi_{\pi(1)}| \geq |\psi_{\pi(2)}| \geq \cdots \geq |\psi_{\pi(d)}|$ and $|\phi_{\sigma(1)}| \geq |\phi_{\sigma(2)}| \geq \cdots \geq |\phi_{\sigma(d)}|$. Let $U = P_\pi$ and $V = P_\sigma$, here P_π and P_σ are permutation matrices corresponding to π and σ, respectively. Note that $U|\psi\rangle \stackrel{ICO}{\longrightarrow} V|\phi\rangle \Leftrightarrow |\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle$, here $|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle$ indicates that $|\psi\rangle\langle\psi|$ is transformed to $|\phi\rangle\langle\phi|$ by an incoherent operation. Therefore we can replace $|\psi\rangle$ and $|\phi\rangle$ by $U|\psi\rangle$ and $V|\phi\rangle$. Furthermore, $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle) = P(U|\psi\rangle \stackrel{ICO}{\longrightarrow} |V\phi\rangle)$. Here $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle)$ denotes the greatest probability of success under incoherent operations transferring $|\psi\rangle$ to $|\phi\rangle$.

Theorem 3. $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle) = \min_{l\in[1,d]} \frac{\sum_{i=l}^d |\psi_i|^2}{\sum_{i=l}^d |\phi_i|^2}$.

Proof: We will show the equation by verifying that $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle) \leq \frac{\sum_{i=l}^d |\psi_i|^2}{\sum_{i=l}^d |\phi_i|^2}$ for each l and giving an optimal incoherent operation.

In the case of $l = 1$, it is trivial, since $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle) \leq 1 = \frac{\sum_{i=1}^d |\psi_i|^2}{\sum_{i=1}^d |\phi_i|^2}$. For the case of $l \neq 1$, define $f_l : \Omega \to \mathcal{R}^+$ by $f_l((x_1, x_2, \cdots, x_d)^t) = \sum_{i=l}^d x_i^\downarrow$, here $(x_1^\downarrow, x_2^\downarrow, \cdots, x_d^\downarrow)^t$ is the vector obtained by rearranging the coordinates of $(x_1, x_2, \cdots, x_d)^t$ in the decreasing order. We firstly check that f_l satisfies Eqs.(1-3). Since $l \geq 2$, $f_l((1, 0, \cdots, 0)^t) = \sum_{i=l}^d 0 = 0$. By the definition of f_l, it is clear that f_l is invariant under any permutation transformation. f_l is a concave function follows from the Ky Fan's maximum principle [22, Page 24].

From Theorem 1, it follows that it can derive a coherence measure C_{f_l}. From the C2b and neglecting positive contributions coming from unsuccessful conversions, it follows that

$$P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle)C_{f_l}(|\phi\rangle\langle\phi|) \leq C_{f_l}(|\psi\rangle\langle\psi|). \tag{26}$$

Now we give the optimal incoherent operation. The strategy is borrowed from [9] which consider similar problem in the frame of entanglement. The key difference lies in replacing the Nielsen Theorem by the corresponding part about coherent transformation which is recently proposed in [20]. For the convenience of readers, we also provide the details.

We divide into two steps. In the first, by using the result in [20], we will show that an incoherent operation transfer the initial state $|\psi\rangle$ into a temporary pure state $|\gamma\rangle$ with certainty. Secondly, $|\gamma\rangle$ is transfered into $|\phi\rangle$ by mean of incoherent operation with the probability $\min_{l\in[1,d]} \frac{\sum_{i=l}^{d}|\psi_i|^2}{\sum_{i=l}^{d}|\phi_i|^2}$.

Let l_1 be the smallest integer in $[1,d]$ such that

$$\frac{\sum_{i=l_1}^{d}|\psi_i|^2}{\sum_{i=l_1}^{d}|\phi_i|^2} = \min_{l\in[1,d]} \frac{\sum_{i=l}^{d}|\psi_i|^2}{\sum_{i=l}^{d}|\phi_i|^2} \equiv r_1. \tag{27}$$

It may happen that $l_1 = r_1 = 1$. If not, it follows from the equivalence

$$\frac{a}{b} < \frac{a+c}{b+d} \Leftrightarrow \frac{a}{b} < \frac{c}{d} (a,b,c,d>0) \tag{28}$$

that for any integer $k \in [1, l_1-1]$ such that $\frac{\sum_{i=k}^{l_1-1}|\psi_i|^2}{\sum_{i=k}^{l_1-1}|\phi_i|^2} > r_1$. Let us then define l_2 as the smallest integer $\in [1, l_1-1]$ such that

$$r_2 = \frac{\sum_{i=l_2}^{l_1-1}|\psi_i|^2}{\sum_{i=l_2}^{l_1-1}|\phi_i|^2} = \min_{l\in[1,l_1-1]} \frac{\sum_{i=l}^{l_1-1}|\psi_i|^2}{\sum_{i=l}^{l_1-1}|\phi_i|^2} \quad (>r_1). \tag{29}$$

Repeating this process until $l_k = 1$ for some k, we obtain s series of $k+1$ integers $l_0 > l_1 > l_2 > \cdots l_k$ $(l_0 = d+1)$, and k positive numbers $0 < r_1 < r_2 < \cdots < r_k$, by the means of which we define our temporary (normalized) state

$$\begin{array}{l} |\gamma\rangle = \sum_{i=1}^{d}\gamma_i|i\rangle, \text{ where} \\ \gamma_i = \sqrt{r_j}\phi_i \quad \text{ if } i \in [l_j, l_{j-1}-1], 1 \le j \le k \end{array} \tag{30}$$

i.e.,

$$\vec{\gamma} = \begin{pmatrix} \sqrt{r_k}\begin{pmatrix} \phi_{l_k} \\ \vdots \\ \phi_{l_{k-1}-1} \end{pmatrix} \\ \vdots \\ \sqrt{r_2}\begin{pmatrix} \phi_{l_2} \\ \vdots \\ \phi_{l_1-1} \end{pmatrix} \\ \sqrt{r_1}\begin{pmatrix} \phi_{l_1} \\ \vdots \\ \phi_{l_0-1} \end{pmatrix} \end{pmatrix} \tag{31}$$

From the construction, it follows that

$$\sum_{i=k}^{d}|\psi_i|^2 \ge \sum_{i=k}^{d}|\gamma_i|^2 \quad \forall k \in [1,d], \tag{32}$$

which is equivalent that $\sum_{i=1}^{k}|\psi_i|^2 \leq \sum_{i=1}^{k}|\gamma_i|^2 \quad \forall k \in [1,d]$. By [20, Theorem 1], there exists an incoherent operation transferring $|\psi\rangle$ into $|\gamma\rangle$ with certainty.

Define the positive operator $M : \mathcal{C}^d \to \mathcal{C}^d$ by

$$M = \begin{pmatrix} M_k & & & \\ & \ddots & & \\ & & M_2 & \\ & & & M_1 \end{pmatrix}, \tag{33}$$

where

$$M_j = \sqrt{\frac{r_1}{r_j}} I_{[l_{j-1}-l_j]}, \quad j = 1, 2, \cdots, k, \tag{34}$$

is proportional to the identity in $(l_{j-1} - l_j)$-dimensional subspace of $\mathcal{C}^d$. So that $M, \sqrt{I - M^2}$ define an incoherent operation satisfying $M|\gamma\rangle = \sqrt{r_1}|\phi\rangle$. $\square$

At the end of the section, we consider two alternative scenarios where Theorem 3 can be applied. At first, we consider the greatest probability of copies of state $|\phi\rangle$ transferred from $|\psi\rangle$, denote it by $m^{max}_{|\psi\rangle\to|\phi\rangle}$, i.e., $m^{max}_{|\psi\rangle\to|\phi\rangle} = \max_n P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle^{\otimes n})$. In general, this cannot be obtained by Theorem 3 directly. However, there are circumstances in which $m^{max}_{|\psi\rangle\to|\phi\rangle} = P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle)$. Indeed, let $n_{|\psi\rangle}$ denote the number of nonvanishing coefficients of the entangled state $|\psi\rangle$, and recall that $n_{|\psi\rangle^{\otimes N}} = n^N_{|\psi\rangle}$. Then,

$$n_{|\psi\rangle} < n^2_{|\phi\rangle} \Rightarrow P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle^{\otimes N}) = 0 \quad N \geq 2 \tag{35}$$

implies that $m^{max}_{|\psi\rangle\to|\phi\rangle} = P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle)$ when $n_{|\psi\rangle} < n^2_{|\phi\rangle}$. Secondly, from Theorem 3, we also get that one can often extract more coherence from two copies of a given state $|\psi\rangle$, i.e., $|\psi\rangle^{\otimes 2}$, than twice what they can obtain from one single copy $|\psi\rangle$. For example, $|\psi\rangle = (\frac{1}{\sqrt{2}})(|1\rangle + |2\rangle)$ and $|\phi\rangle = (\frac{1}{\sqrt{3}})(|1\rangle + |2\rangle + |3\rangle)$. Then $1 = P(|\psi\rangle^{\otimes 2} \stackrel{ICO}{\longrightarrow} |\phi\rangle) > P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle) = 0$.

5 Conclusion

This paper is focused on quantification of coherence. We have provided a tool to build an important class of coherence measures which cover the relative entropy measure for pure states, the l_1-norm measure for pure states, and the α-entropy measure. Furthermore, any coherence measure on pure coherent states can be constructed in this way. Using a set of coherence measure and constructing the optimal conversion, we give the explicit expression of the greatest probability $P(|\psi\rangle \stackrel{ICO}{\longrightarrow} |\phi\rangle)$ of success in the conversion of given states $|\psi\rangle$ and $|\phi\rangle$ under incoherent operations.

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