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CONTROLLING THE COHERENCE IN A PURE DEPHASING MODEL FOR AN ARBITRARILY PRESCRIBED TIME SPAN

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We present an open-loop unitary strategy to control the coherence in a pure dephasing model (related to the phase-flip channel) that is able to recover, for whatever prescribed time span, the initial coherence at the end of the control process. The strategy's key idea is to steer the quantum state to the subset of invariant states and keep it there the necessary time, using a fine tuned control Hamiltonian.

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1 Introduction

The open-loop unitary controlling is an important methodology of quantum control, having the characteristic of avoiding totally any perturbation of systems during the control process, feature that simplifies the technological apparatus required to implement the control in practice. In spite of its limitations, it has a vast range of applications, including quantum chemistry, quantum optics, quantum information and also biophysics.

The unitary control of Markovian quantum systems is strongly motivated because the Markovian approximation can be used to describe a wide class of open quantum systems (besides the closed ones), enabling the theory to be used in many practical problems [1, 2]. Such systems are also particularly amenable because their dynamics can be suitably transformed into real linear dynamical systems, through coherent vector representation [2, pp.50-57].

Finally, the control of coherence in quantum systems is a demanding task for the development of quantum information and computation technologies, fact evidenced by the vast literature on the subject – see [3, 4, 5, 6] and references quoted therein. This subject has been massively studied but there are many open questions even in the most simple situations. For example, the unitary tracking-control strategy to stabilize (keep constant) the coherence of a pure dephasing model presented in [7] suffers from a severe limitation, unavoidable for all unitary control strategies which stabilize the coherence in this model (whether performed in a closed-loop or in an open-loop fashion): the control can be carried out only within a finite time span, at the end of which the control fields diverge. Nevertheless, *it's possible to control the quantum state in order to recover the initial coherence after an arbitrary prescribed*

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time span if one is allowed to use control Hamiltonians that don't keep the coherence constant (necessarily). The contribution of this paper is twofold: the definition of a general strategy to find a fine tuned control Hamiltonian to recover the coherence of a given initial state after any prescribed time span, and the explicit application of such strategy in the model just mentioned, called here *dephasing qubit*.

The structure of the paper is simple. In Section 2 we review basic concepts in order to give a short and precise formulation of our problem in Section 3. In Section 4 we define a general strategy to tackle such kind of problem, we apply it to solve the specific problem stated previously and give a numerical example. In the final Section 5 we discuss our results and comment related issues. Appendix A focuses the concept of *limit time*, related to the definition of the control Hamiltonian.

2 Dephasing qubit

We start recalling basic definitions and results concerning the *dephasing qubit* model, using a notation borrowed from [7].

A general quantum state (density matrix) of a qubit can be written in terms of the *identity operator* I and *Pauli matrices* $(\sigma_x, \sigma_y, \sigma_z)$, whose coefficients define the so called *Bloch vector*:

$$\rho = \frac{1}{2}\left(I + v_x\sigma_x + v_y\sigma_y + v_z\sigma_z\right),\ v = (v_x, v_y, v_z) \in B := \left\{v \in \mathbb{R}^3;\ \|v\| \leq 1\right\}. \tag{1}$$

The purity and coherence are defined, respectively, by

$$P(\rho) := v_x^2 + v_y^2 + v_z^2\ ,\ C(\rho) := v_x^2 + v_y^2. \tag{2}$$

The *free dynamics* is given by the master equation

$$\frac{d}{dt}\rho(t) = \frac{\gamma}{2}\left(\sigma_z\rho(t)\sigma_z - \rho(t)\right), \tag{3}$$

where $\gamma > 0$ is a damping coefficient. A control Hamiltonian

$$H(t) = \frac{1}{2}\left(u_1(t)\sigma_x + u_2(t)\sigma_y + u_3(t)\sigma_z\right);\ u_1(t), u_2(t), u_3(t) \in \mathbb{R}^3. \tag{4}$$

affects the free dynamics according with

$$\frac{d}{dt}\rho(t) = \frac{\gamma}{2}\left(\sigma_z\rho(t)\sigma_z - \rho(t)\right) - i\left[H(t), \rho(t)\right]. \tag{5}$$

The model has a *decoherence-free subset of states*[a] defined by:

$$V_z := \left\{\frac{1}{2}\left(I + \xi\sigma_z\right);\ -1 \leq \xi \leq 1\right\}.$$

For time-dependent states evolving within V_z, the dynamics is reduced to the Liouville-von Neumann equation (meaning that its time evolution is unitary):

$$\frac{d}{dt}\rho(t) = -i\left[H(t), \rho(t)\right],\ \text{if } \rho(t) \in V_z.$$

[a]We use the term "*decoherence-free subset of states*" to distinguish it from the related concept of *decoherence free subspaces*, for which we refer to [8, 9].

Equation (5) turns out to be equivalent to the following system for the Bloch vector's coordinates:

$$\begin{cases} 2\dot{v}_x = -\gamma v_x + u_y v_z - u_z v_y \\ 2\dot{v}_y = -\gamma v_y - u_x v_z + u_z v_x \\ 2\dot{v}_z = u_x v_y - u_y v_x . \end{cases} \tag{6}$$

Given the values of purity and coherence of an initial state $\rho(0)$,

$$p = v_x^2(0) + v_y^2(0) + v_z^2(0), \quad c = v_x^2(0) + v_y^2(0), \tag{7}$$

the corresponding *breakdown time* is defined by

$$t_b := \frac{p-c}{\gamma c}. \tag{8}$$

Theorem 1 *In the dephasing qubit, the coherence of a time-dependent state cannot be stabilized (kept constant) by unitary controlling for a time span greater then the breakdown time Eq. (8).*

Proof. Let $H(t)$ be the Hamiltonian of a unitary control and let $\rho(t)$ be a solution of the system (5) having constant coherence, $C(\rho(t)) = C(\rho(0)) = c$. Assume that $H(t)$ and $\rho(t)$ are defined for $t \in [0, T]$, for some $T > 0$. The dynamical equations for the Bloch coordinates (6) imply

$$\frac{d}{dt}\left(v_x^2 + v_y^2 + v_z^2\right) = -\gamma\left(v_x^2 + v_y^2\right). \tag{9}$$

So, the coherence (given by Eq. (2)) is kept invariant if, and only if,

$$\frac{d}{dt}\left(v_x^2 + v_y^2\right) = 0, \ \frac{d}{dt}v_z^2 = -\gamma\left(v_x^2 + v_y^2\right). \tag{10}$$

In this case, it follows that

$$v_z^2(t) = v_z^2(0) - c\gamma t, \ \forall t \in [0, T].$$

Since $v_z(t)$ must be real and $v_z^2(0) = p - c$, the condition $v_z^2(t) \geq 0$ implies that $T \leq t_b$; this means that the time span $H(t)$ and $\rho(t)$ are defined cannot be greater than the breakdown time.□

Due to *Theorem 1*, to recover the coherence of an initial state after a time span greater than the *breakdown time* one must accomplish a control strategy that do not keep constant the coherence; so, it is worthwhile to consider the problem formulated in the next section.

3 The Problem

***Problem*:** In the *dephasing qubit*, for a given $T > 0$ and initial state $\rho(0)$, set a control Hamiltonian to steer the state's evolution according with Eq. (5) in such a way that the coherence of the system' state after the time span T turns out to be equal to the coherence of the initial state, *i.e.*,

$$C(\rho(T)) = C(\rho(0)).$$

Remark 1 *This problem cannot be solved using only unitary controlling if coherence and purity start equal: according with Theorem 1 and Eq. (8), nothing can be done in this way if $v_z(0) = 0$. Also, the same theorem and equation imply that there is nothing to do if the initial coherence is zero.*

4 The Solution

In this section, we define and apply a simple and general strategy to solve the specific *Problem* previously stated. This strategy uses the *decoherence-free subset of states* of the *dephasing qubit*.

To simplify the calculations, we deal first with a special initial state and then generalize the result. After due developments, the solution will be presented in the form of an algorithm.

4.1 The Strategy

(i) first, steer the qubit's state to the *decoherence-free subset of states*;

(ii) second, keep the state within V_z for the period needed;

(iii) finally, bring the system to some final state which has coherence equals to the initial value at the end of the process.

4.2 Solving the Problem for special initial state

Consider an initial state $\rho(0)$ with purity p greater than a positive coherence c which has the following special form[b]

$$\rho(0) = \frac{1}{2}I + \frac{1}{2}v_x(0)\,\sigma_x + \frac{1}{2}v_z(0)\,\sigma_z,\ \ v_z(0) \neq 0 < v_x(0)\,. \tag{11}$$

In this case, we can use control fields having y-component being the only nonzero – a choice that confines the time-dependent Bloch vector to the xz-plane during its entire evolution:

$$u_x = 0 = u_z;\ \ u_y =: \epsilon u,\ \ \epsilon = \pm 1,\ \ u > 0. \tag{12}$$

For convenience we have introduced the signal ϵ which determines de direction the state's Bloch vector rotates in the xz-plane due to the action of the control Hamiltonian: $\epsilon = +1$ corresponds to clockwise direction and $\epsilon = -1$ corresponds to counterclockwise direction.

The dynamics of Bloch vector Eq. (6) under action of the control fields Eq. (12) with initial conditions at t_0 added turns to

$$\begin{cases} 2\dot{v}_x = -\gamma v_x + \epsilon u v_z \\ 2\dot{v}_y = -\gamma v_y \\ 2\dot{v}_z = -\epsilon u v_x \\ v_y(t_0) = 0,\ \ v_x^2(t_0) + v_z^2(t_0) \leq 1. \end{cases} \tag{13}$$

Assuming the control field u to be constant and

$$u > \frac{\gamma}{2}, \tag{14}$$

[b] As we already have said: if $v_x(0) = 0$, there is nothing to be done; if $v_z(0) = 0$, there is nothing which can be done.

the solution of Eq. (13) is given by

$$\begin{cases} v_x(t) = e^{-\gamma(t-t_0)/4}\left(v_x(t_0)\cos\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}(t-t_0)\right)+\right. \\ \qquad\qquad \left.+\frac{2\epsilon u v_z(t_0)-\gamma v_x(t_0)}{\sqrt{4u^2-\gamma^2}}\sin\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}(t-t_0)\right)\right) \\ v_y(t) = 0 \\ v_z(t) = e^{-\gamma(t-t_0)/4}\left(v_z(t_0)\cos\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}(t-t_0)\right)+\right. \\ \qquad\qquad \left.-\frac{2\epsilon u v_x(t_0)-\gamma v_z(t_0)}{\sqrt{4u^2-\gamma^2}}\sin\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}(t-t_0)\right)\right). \end{cases} \tag{15}$$

Now, we describe separately the evolution of the controlled state $\rho(t)$ during the first and the third stages of our control process, starting from the initial state Eq. (11).

The shortest time span $\Delta t_1 > 0$ we need to steer $\rho(0)$ to V_z is given by the first positive zero of $v_x(t)$ in Eq. (15) with $t_0 = 0$ and $t = \Delta t_1$; after some algebraic manipulation, we get Δt_1 explicitly:

$$\Delta t_1 = \frac{4}{\sqrt{4u^2-\gamma^2}}\arctan\left(\frac{\sqrt{4u^2-\gamma^2}v_x(0)}{\gamma v_x(0)-2\epsilon_1 u v_z(0)}\right). \tag{16}$$

For Δt_1 to be positive, the argument of arctan in (16) has to be positive, so we must set

$$\epsilon_1 := -signal\,(v_z(0)). \tag{17}$$

Analogously, the shortest time span $\Delta t_3 > 0$ we need to steer $\rho(\Delta t_1)$ from V_z to some state having coherence equals to that of $\rho(0)$, with the innocuous option to get the final state having its σ_x-component equals to that of $\rho(0)$, is given by the first positive solution of the following transcendent equation for Δt_3, obtained from Eq. (15) by setting $t_0 = \Delta t_1$, $t = \Delta t_3 + \Delta t_1$ and $\epsilon_3 = -\epsilon_1$:

$$v_x(0) = e^{-\gamma\Delta t_3/4}\left(\frac{2\epsilon_3 u v_z(\Delta t_1)}{\sqrt{4u^2-\gamma^2}}\sin\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}\Delta t_3\right)\right), \tag{18}$$

where

$$v_z(\Delta t_1) = v_z(0)\,e^{-\gamma\Delta t_1/4}\cos\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}\Delta t_1\right)\left(1+\frac{\gamma v_z(0)-2\epsilon u v_x(0)}{\sqrt{4u^2-\gamma^2}v_z(0)}\tan\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}\Delta t_1\right)\right).$$

Remark 2 *Note the consistence of taking $\epsilon_3 = signal\,(v_x(0)\,v_z(0))$ in order for Δt_3 to be positive in Eq. (18), since $v_z(\Delta t_1)$ has the same signal that $v_z(0)$; this choice can be verified by taking into account the definition of ϵ_1 in Eq. (17).*

Now, to write down our control Hamiltonian which solves the *Problem,* we have to find a control field's intensity u that guarantees the implicit equation Eq. (18) has a positive solution and such that

$$T \geq \Delta t_1 + \Delta t_3. \tag{19}$$

This amounts to solve for Δt_1, Δt_3 and u the system constituted by Equations (16) and (18) and Inequality (19). Finally, by setting

$$\epsilon = -signal\left(v_z\left(0\right)\right), \ \Delta t_2 := T - \Delta t_1 - \Delta t_3,$$

we can define the control Hamiltonian:

$$H\left(t\right) = \epsilon u\left[\mathfrak{h}(\Delta t_1 - t) - \mathfrak{h}(t - \Delta t_2)\right]\sigma_y, \ 0 \le t \le T, \tag{20}$$

where $\mathfrak{h}$ denotes the Heaviside Step Function,

$$\mathfrak{h}\left(t\right) = \begin{cases} 0, & t < 0 \\ 1, & t > 1. \end{cases}$$

4.3 Solving the Problem for a general initial state

Here, we present the control Hamiltonian which solves the *Problem* for an arbitrary initial state with purity greater than a non-zero coherence, *viz.,*

$$\rho\left(0\right) = \frac{1}{2}I + \frac{1}{2}\left(v_x\left(0\right)\sigma_x + v_y\left(0\right)\sigma_y + v_z\left(0\right)\sigma_z\right), \tag{21}$$

where

$$0 < c = v_x\left(0\right)^2 + v_y\left(0\right)^2 < \ p = v_x\left(0\right)^2 + v_y\left(0\right)^2 + v_z\left(0\right)^2.$$

Now, we define the unitary operator

$$U_\theta := \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} = \left(\cos\frac{\theta}{2}\right)I - i\left(\sin\frac{\theta}{2}\right)\sigma_z,$$

where $\theta \in [0, 2\pi)$ is such that

$$v_x\left(0\right) = \sqrt{c}\cos\theta \ , \ v_y\left(0\right) = \sqrt{c}\sin\theta.$$

Using U_θ, we define the following state which has the previous special form as well as the same purity and coherence of $\rho\left(0\right)$:

$$\tilde{\rho}\left(0\right) := U_\theta^*\rho\left(0\right)U_\theta = \frac{1}{2}I + \frac{1}{2}\sqrt{c}\sigma_x + \frac{1}{2}v_z\left(0\right)\sigma_z.$$

Now, let $\tilde{H}\left(t\right)$ be the control Hamiltonian that solves the *Problem* for $\tilde{\rho}\left(0\right)$ and time span $T > 0$. Since U_θ is constant and commutes with σ_z, the control Hamiltonian which solves the *Problem* for $\rho\left(0\right)$ and time span $T > 0$ is given by:[c]

$$H\left(t\right) := U_\theta\tilde{H}\left(t\right)U_\theta^*.$$

Remark 3 *Naturally, a control Hamiltonian which solves the* Problem *for a general initial state must be unitarily equivalent to the control Hamiltonian which solves the* Problem *for some special initial state, because general states are related to the special ones by a change of variables (specifically, a suitable rotation in the xy-plane).*

[c]See the explicit expression in Eq. (4.4) below.

4.4 Algorithm

To solve the *Problem* for the initial state

$$\rho(0) = \frac{1}{2}I + \frac{1}{2}\left(v_x(0)\sigma_x + v_y(0)\sigma_y + v_z(0)\sigma_z\right)$$

with

$$0 < c = v_x^2(0) + v_y^2(0) < p = c + v_z^2(0),$$

do:

i) Set $\epsilon := -signal\,(v_z(0))$ and $\theta \in [0, 2\pi)$ such that

$$v_x(0) = \sqrt{c}\cos\theta\ ,\ v_y(0) = \sqrt{c}\sin\theta;$$

ii) Solve the following system for u, Δt_1 and Δt_3:

$$\begin{cases} \tan\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}\Delta t_1\right) = \frac{\sqrt{4u^2-\gamma^2}\sqrt{c}}{\gamma\sqrt{c}+2u\sqrt{p-c}} \\ \sin\left(\frac{1}{4}\sqrt{4u^2-\gamma^2}\Delta t_3\right) = \frac{e^{\gamma(\Delta t_1+\Delta t_3)/4}\sqrt{4u^2-\gamma^2}\sqrt{c}}{2\sqrt{u^2p+\gamma u\sqrt{c}\sqrt{p-c}}} \\ u > \gamma/2,\ \Delta t_1 > 0,\ \Delta t_3 > 0,\ \Delta t_1 + \Delta t_3 \leq T; \end{cases} \tag{22}$$

(Alternatively, one can prescribe a positive value for u, determine Δt_1 and Δt_3 from the first and second equations of System (22) and then verify if the inequalities are also satisfied.)

iii) Define

$$\Delta t_2 := T - \Delta t_1 - \Delta t_3 \geq 0;$$

iv) Define the control Hamiltonian by:

$$H(t) = -\epsilon u\left[\mathfrak{h}(\Delta t_1 - t) - \mathfrak{h}(t - \Delta t_2)\right]\left[(\sin\theta)\sigma_x - (\cos\theta)\sigma_y\right],\ 0 \leq t \leq T.$$

Remark 4 *The System of equations (22) has solutions for $u > \gamma/2$ sufficiently large (implying that Δt_1 and Δt_3 are correspondingly small). To verify, we note the following approximations valid under such conditions:*

$$\Delta t_1 \approx \frac{2}{u}\arctan\left(\left|\frac{v_x(0)}{v_z(0)}\right|\right),$$

$$|v_z(\Delta t_1)| \approx e^{-\gamma\Delta t_1/4}\sqrt{v_z^2(0) + v_x^2(0)},$$

$$|v_x(0)| \approx e^{-\gamma\Delta t_3/4}\,|v_z(\Delta t_1)|\sin\left(\frac{u}{2}\Delta t_3\right).$$

The first equation gives an approximation for Δt_1, the second equation implies $|v_z(\Delta t_1)| > |v_x(0)|$ and the third equation has a sine function which oscillates very quickly; therefore, for relatively small values of Δt_3 it follows that $e^{-\gamma\Delta t_3/4} \approx 1$ and

$$\Delta t_3 \approx \frac{2}{u}\arcsin\left(\frac{|v_x(0)|}{\sqrt{v_z^2(0) + v_x^2(0)}}\right).$$

4.5 A numerical example

Let us illustrate the application of the *Solution* using numerical values presented in [7].

Consider a system with damping coefficient $\gamma = 0.1$ and assume the initial state has purity $p = 0.8$ and coherence $c = 0.3$,

$$\rho(0) = \frac{I}{2} + \frac{\sqrt{0.3}}{2}\sigma_x + \frac{\sqrt{0.5}}{2}\sigma_z.$$

In this case, the breakdown time is

$$t_b = \frac{p - c}{\gamma c} \approx 16.67.$$

If we set $u = 0.2$, then the system of equations (22) implies

$$\Delta t_1 \approx 5.79, \quad \Delta t_3 \approx 9.11.$$

Then, $\Delta t_1 + \Delta t_3 = 14.90$. For $T = 20 > t_b$, the application of our control strategy gives the following results: *the purity evolves from the initial value* 0.8 *to the final value* ≈ 0.63*; the coherence evolves from the initial value* 0.3 *to the final (and same) value* 0.3*, decreasing to zero during the first stage (between* $t = 0$ *and* $t \approx 5.8$*), staying equals to zero during the second stage (between* $t \approx 5.8$ *and* $t \approx 10.9$*) and increasing to* 0.3 *during the third stage (between* $t \approx 10.9$ *and* $t = 20$*).*

Figure 1 gives the graph of the y-component of the control Hamiltonian, the path of the Bloch vector in the xz-plane during the control process and the graph of purity and coherence as functions of time:

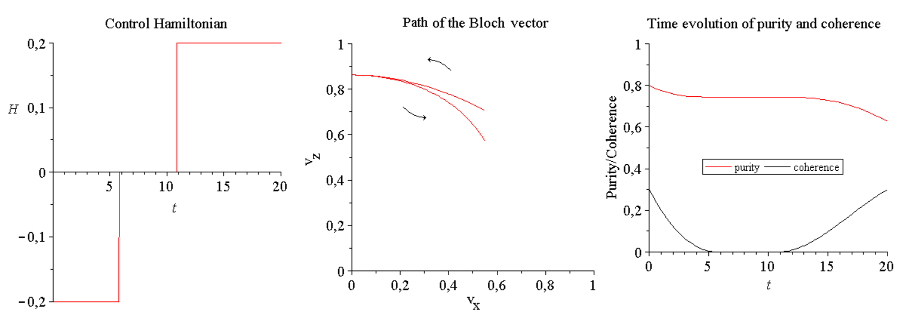

Fig. 1. Example of controlling the coherence in the dephasing qubit.

5 Conclusions

Characteristic of many problems in control theory is the need to develop idiosyncratic strategies – even for situations in which there are general procedures to solve them, because the advantages of a specific procedure may be worthwhile in a particular application. We think this fact is well illustrated here by our open-loop strategy to control the coherence in the dephasing qubit. For a comparison with the tracking-control strategy of [7], we remark that: the tracking-control can be applied to stabilize the coherence only for a time span smaller than the breakdown time (with energetic expenditure reducing as the control period decreases), while our strategy can be applied for any prescribed time span (with the control fields becoming larger as the control period decreases). The trick of our control strategy lies in the first and third stages, which must be performed as quickly as necessary since purity decreases during them; for this strategy to be successful, the control field's intensity must reach sufficiently large values, as one can deduce from Eq. (16).

We believe the reasoning presented here can be naturally adapted to control the coherence of other Markovian quantum systems having a *decoherence-free subset of states*, with the help of a *coherent vector representation*. The *Strategy* (Sec. 4.1) is general, in the sense that it may be applied to recover the coherence in models other than the *dephasing qubit*; nevertheless, the first and third stages must be carried out taking into account specific details of each model. A natural development of this work is the application of the *Strategy* in more complex and realistic situations, what can be more interesting and more useful, but more laborious too.

Turning to the important question about the energy expenditure of the control process, we close the paper stating a new problem:

> **Optimal Control Problem**: In the *dephasing qubit*, for a given $T > 0$ and an initial state $\rho(0)$, set a control Hamiltonian to steer the state's evolution according with the dynamics given by Eq. (5) so that (i) the coherence of the system' state after the time span T be equal to the coherence of the initial state and (ii) the expenditure of energy in the process is minimum, with this expenditure being defined by a quadratic form on the control fields [10, 11], *e.g.*,
>
> $$K_u = \int_0^T \left(u_1^2(t) + u_2^2(t) + u_3^2(t)\right) dt.$$

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Appendix A *The limit time*

For the *dephasing qubit*, we define the "*limit time*" by the maximum time span $\tilde{T}$ that one can spend on steering an initial state to the decoherence-free subset of states V_z and, after, to some final state having coherence equals to the initial value, using solely a control Hamiltonian as given by Eq. (4). Specializing this definition for Hamiltonians having the shape (4.4, iv), we define the "*limit control field*" as the minimal value that a (constant) control field u can assume in the solutions of the system (22) when $T = \tilde{T}$.

The relevance of these concepts is the following: for a control period $T \geq \tilde{T}$, the control of coherence can be done using a control field $u = \tilde{u}$, while for a control period $T < \tilde{T}$, the control field must satisfy $u > \tilde{u}$.

Quantities $\tilde{T}$ and $\tilde{u}$ are mutually dependent and are characterized by the property that *the purity of the initial state is fully reduced to the initial value of the coherence at the end of the corresponding control process*, namely:

$$v_z\left(\tilde{T}\right) = 0. \tag{A.1}$$

To calculate $\tilde{T}$ and $\tilde{u}$, we combine the two equations of the system (22) with condition (A.1); after some algebraic manipulation, we get the following system for $\tilde{u}$ and $\Delta\tilde{t}_1$ and $\Delta\tilde{t}_3$, where $\tilde{T} = \Delta\tilde{t}_1 + \Delta\tilde{t}_3$:[d]

$$\begin{cases} \Delta\tilde{t}_1 = \frac{4}{\sqrt{4\tilde{u}^2-\gamma^2}} \arctan\left(\frac{\sqrt{4\tilde{u}^2-\gamma^2}\,v_x(0)}{\gamma v_x(0)-2\epsilon\tilde{u}v_z(0)}\right) \\ \Delta\tilde{t}_3 = \frac{4}{\sqrt{4\tilde{u}^2-\gamma^2}} \arcsin\left(\frac{e^{\gamma(\Delta\tilde{t}_1+\Delta\tilde{t}_3)/4}\sqrt{4\tilde{u}^2-\gamma^2}\,|v_x(0)|}{2\sqrt{p_0\tilde{u}^2-\epsilon\gamma v_z(0)v_x(0)\tilde{u}}}\right) \\ \tan\left(\frac{\sqrt{4\tilde{u}^2-\gamma^2}}{4}\Delta\tilde{t}_3\right) = -\frac{\sqrt{4\tilde{u}^2-\gamma^2}}{\gamma}. \end{cases} \tag{A.2}$$

Since this system is very complicated, it is useful to know that $\tilde{T}$ is greater then the breakdown time, given by Eq. (8). This fact is easy to prove and it implies a super estimation of $\tilde{u}$, to which we now turn (with some omissions in the argument). Using that Δt_1 and Δt_3 are decreasing functions of u, a sub estimation of $\Delta\tilde{t}_3$ implies a super estimation of $\tilde{u}$; since, in general, $\Delta t_3 \geq \Delta t_1$ and $\Delta t_1 + \Delta t_3 > t_b$, it follows

$$\Delta\tilde{t}_3 > \frac{t_b}{2} = \frac{p-c}{2\gamma c}$$

Inserting this sub estimation for $\Delta\tilde{t}_3$ in the third equation of the System (A.2), we conclude that *the minimal control field $\tilde{u}$ is not greater than the solution ξ of the following equation*:

$$\tan\left(\sqrt{4\xi^2-\gamma^2}\,\frac{p-c}{8\gamma c}\right) = -\frac{\sqrt{4\xi^2-\gamma^2}}{\gamma}. \tag{A.3}$$

[d]We remark that Eq. (A.1) is equivalent to the third equation of System (A.2).