Quantum Information and Computation, Vol. 15, No. 11&12 (2015) 951–961

GEOMETRIC PHASE CARRIED BY THE OBSERVABLES AND ITS APPLICATION TO QUANTUM COMPUTATION

ZISHENG WANG
College of Physics and Communication Electronics, Jiangxi Normal university
Nanchang 330022, P. R. China
Institute of Applied Physics and Materials Engineering, Faculty of Science and Technology
University of Macau, Macao SAR, P. R. China

HUI PAN
Institute of Applied Physics and Materials Engineering, Faculty of Science and Technology
University of Macau, Macao SAR, P. R. China

Received April 5, 2013
Revised March 10, 2015

We investigate geometric phases in terms of Heisenberg equation. We find that, equivalently to Schrödinger picture with a memory of its motion in terms of the geometric phase factor contained in the wave function, the observales carry with the geometric message under their evolutions in the Heisenberg picture. Such an intrinsic geometric feature may be particularly useful to implement the multi-time correlation geometric quantum gate in terms of the observables, which leads to a possible reduction in experimental errors as well as gate timing. An application is discussed for nuclear-magnetic-resonance system, where the geometric quantum gate is proposed.

Keywords: Geometric phase, Heisenberg picture, Observales, Nuclear-magnetic-resonance system, Geometric quantum gate.
Communicated by: R Jozsa & G Milburn

1 Introduction

As early as 1956, Pancharatnam anticipated the quantum geometric phase in terms of the experiment involving a sequence of changes in polarization of a beam of classical light by sending it through suitable polarizers [1].

One of the most important progresses in the geometric phase related to the formal structure of quantum mechanics was discovered in the context of adiabatic and cyclic evolution of wave function obeying the Schrödinger equation with time-dependent Hamiltonian by Berry in 1984 [2]. This Berry phase can be interpreted as a holonomy of the Hermitian fibre bundle over the parameter space [3]. Especially, the phase is proportional to the area spanned in parameter space and independent of the path passed by the system during its evolution, which means that the geometric phase has an observable consequence. Thus the wave function of a quantum system retains a memory of its motion in the form of the Berry phase. Therefore, the geometric phases are interesting in both fundamental physical concepts and their applications [4].

The importance of Berry phase on various areas of physics has naturally resulted in the generalization to the nonadiabatic cyclic [5] and non-clyclic [6] evolutions. This nonadiabatic geometric phase, especially, showed a geometric picture of quantum dynamics, which can be defined for any closed or open curve in the Hilbert space. Thus, the geometric phase is independent of the Hamiltonian of the physical system for a given closed curve.

The geometric phase has recently been generalized in the context of non-adiabatic quantum computation both theoretically [7] and experimentally [8, 9]. As an application of the geometric nature, the geometric quantum computation is a potentially intrinsical fault tolerant and therefore resilient to certain types of the experimental and fluctuational errors [10, 11, 12, 13]. This potential value of the quantum holonomy and topology phenomenon has attracted great interests in quantum computation [14, 15], gauge theory [16, 17], fractional statistics [18, 19], quantum Hall effect [20], quantum Bohm theory [21, 22], nonunitary evolution [23, 24] and open quantum system [25, 26, 27, 28, 29, 30, 31]. The geometric phases in the wave functions have been observed in many different experiments [32, 33, 34, 35].

Up to today, the developments and applications of the geometric phases have been studied in the so-called Schrödinger picture. In the Schrödinger picture, any physical observables are visualized as a fixed set of values and the state vector evolves in time according to the Schrödinger equation. On the other hand, the same system can be described equally well by setting the time-dependent observables to evolve in terms of the Heisenberg equation and the state vector remains stationary. This mode of formulating quantum mechanics is physically equivalent to the Schrödinger picture and called as the Heisenberg picture. It is clear that the geometric phases in the stationary wave function disappear in the Heisenberg picture. In physical principle, the two descriptions are physically equivalent. Therefore, the geometric phase should not disappear as an intrinsical property. It is known that, on the other hand, it is particularly useful in the study of the multi-time correlation function [36, 37] in the Heisenberg picture. Thus it is interesting to investigate the geometric phase in the Heisenberg picture, which may be extremely helpful to observe and further apply it.

In this work, we firstly expand the observable and Hamiltonian in the Heisenberg picture in terms of a complete basis with a set of the Hermitian operators and then seek for an analytic solution of the observable by rescaling the Heisenberg equation as a Schrödinger-like matrix form. We find that the observables hide in the message of geometric phases when the system evolves in time and the motion memory of quantum system is preserved in the physical observables. We apply the method to the nuclear magnetic resonance system and find the multi-time correlation geometric quantum gate can be directly implemented in terms of the physical observables.

2 Rescaling to Heisenberg equation

Consider the Heisenberg equation of motion for an arbitrary observable $\mathcal{Q}$ that is independent of time in the Schrödinger picture,

$$\frac{d}{dt}\mathcal{Q}(t) = -\frac{i}{\hbar}[\mathcal{Q}(t), \mathcal{H}(R(t))], \tag{1}$$

where the Hamiltonian of physical system $\mathcal{H}(R(t))$ is usually time-dependent because of interacting with external potentials and $R(t) = (R_1(t), R_2(t), ..., R_l(t))$ are a set of slowly varying classical parameters (adiabatic evolution).

It is known that in a complex Hilbert space, one can find a set of traceless and Hermitian matrices with the normalization condition $\{\mathcal{O}_l = \mathcal{O}_l^+, l = 1, 2, \ldots, N^2-1 : Tr\mathcal{O}_l = 0, Tr(\mathcal{O}_k\mathcal{O}_l) = 2\delta_{kl}\}$ [38], where the closed algebra is satisfied by the commutative $[\mathcal{O}_l, \mathcal{O}_k] = 2if_{lkj}\mathcal{O}_j$ (f_{lkj} are the totally antisymmetric structure constants) and anticommutative $\{\mathcal{O}_l, \mathcal{O}_k\} = \frac{4}{N}\delta_{lk} + 2d_{lkj}\mathcal{O}_j$ (d_{lkj} are the totally symmetric N-tensor components) relations. The complex $N \times N$ matrices $\{\mathcal{O}_l, l = 1, 2, \ldots, N^2-1\}$, together with the identity matrix $1_{N\times N} = \mathcal{O}_{N^2}$ form a complete basis for a space of $N \times N$ matrices. So, any observable $\mathcal{Q}$ and Hamiltonian $\mathcal{H}$ can be expanded as

$$\mathcal{Q}(t) = \sum_{k=1}^{N^2} \mathcal{Q}_k(t)\mathcal{O}_k, \mathcal{H}(R(t)) = \sum_{k=1}^{N^2} \mathcal{H}_k(R(t))\mathcal{O}_k, \tag{2}$$

where $\mathcal{Q}_k(t)$ and $\mathcal{H}_k(t)$ are corresponding expansive coefficients, respectively. Inserting Eq. (2) into Eq. (1), one has

$$\dot{\mathcal{Q}}_i(t) = \frac{1}{2}\sum_{k,j=1}^{N^2} \mathrm{Tr}\left(-\frac{i}{\hbar}\mathcal{H}_k(R(t))\mathcal{O}_i[\mathcal{O}_j, \mathcal{O}_k]\right)\mathcal{Q}_j(t), \tag{3}$$

which can be rescaled in terms of a matrix form, i.e.,

$$\frac{d}{dt}\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_{N^2} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N^2} \\ M_{21} & M_{22} & \cdots & M_{2N^2} \\ \cdots & \cdots & \cdots & \cdots \\ M_{N^21} & M_{N^22} & \cdots & M_{N^2N^2} \end{pmatrix}\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_{N^2} \end{pmatrix}, \tag{4}$$

where the matrix elements are defined by

$$M_{ij}(t) = \frac{1}{2}\sum_{k=1}^{N^2} \mathrm{Tr}\,(-\frac{i}{\hbar}\mathcal{H}_k(R(t))\mathcal{O}_i[\mathcal{O}_j, \mathcal{O}_k]). \tag{5}$$

Eq. (4) is a Schrödinger-like matrix equation by taking $(\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_{N^2})^t$ as a vector in the N^2-dimensional complex Hilbert space. The matrix M defined by Eq. (5), however, may not be a Hermitian matrix in the general case. Fortunately, the matrix MM^+ is a Hermitian matrix and can be diagonalized by a unitary matrix U_L with the relation $U_LU_L^+ = U_L^+U_L = 1$, i.e.,

$$U_L^+MM^+U_L = D^2, \tag{6}$$

with

$$D_{ij}^2 = d_i^2\delta_{ij}, \tag{7}$$

where

$$d_i^2 = \sum_j (U_L^+M)_{ij}(M^+U_L)_{ji} = \sum_j |(U_L^+M)_{ij}|^2, \tag{8}$$

are positive real number. Thus the matrix M may be denoted as

$$M = U_L D U_R^+, \tag{9}$$

in terms of another unitary matrix $U_R = M^{-1} U_L D$ with the relation $U_R U_R^+ = U_R^+ U_R = 1$. Thus the matrix M can be diagonalized by using the two unitary matrices U_L and U_R, or a set of biorthonormal eigenvectors [23]. By applying the matrices U_L and U_R into the unitary transformations, in fact, we can get a set of biorthonormal eigenvectors.

3 Geometric phase in Heisenberg picture

In the Schrödinger picture, the Berry phases depend on the area spanned in evolving path of a dynamic system. A general approach to determine the possible geometric phases for a physical system is to consider the eigenvalues associated with the effective dynamics of the system.

In order to investigate the geometric phase in the Heisenberg picture, we firstly consider the eigenvectors of $N^2 \times N^2$ time-dependent matrix $M(R(t)$. Suppose that a complete set of biorthonormal eigenvectors [23], $|\tilde{\overrightarrow{\lambda}}(R(t))\rangle = \{|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle, i = 1, 2, \ldots, N^2\}$ and $|\tilde{\overrightarrow{\eta}}(R(t))\rangle = \{|\tilde{\overrightarrow{\eta}}_i(R(t))\rangle, i = 1, 2, \ldots, N^2\}$, obeys the instantaneous eigenequations,

$$M(R(t))|\tilde{\overrightarrow{\lambda}}(R(t))\rangle = \tilde{\lambda}(R(t))|\tilde{\overrightarrow{\lambda}}(R(t))\rangle, \tag{10}$$

and

$$M^+(R(t))|\tilde{\overrightarrow{\eta}}(R(t))\rangle = \tilde{\eta}(R(t))|\tilde{\overrightarrow{\eta}}(R(t))\rangle, \tag{11}$$

where $\tilde{\lambda}(R(t)) = \tilde{\eta}^*(R(t))$ with the complete and normalized conditions [23],

$$\sum_i |\tilde{\overrightarrow{\eta}}_i(R(t))\rangle\langle\tilde{\overrightarrow{\lambda}}_i(R(t))| = 1, \tag{12}$$

and

$$\langle\tilde{\overrightarrow{\eta}}_i(R(t))|\tilde{\overrightarrow{\lambda}}_j(R(t))\rangle = \delta_{ij}. \tag{13}$$

In order to find an exact solution of Eq. (4) by using Eqs. (10)-(13), we set

$$|\overrightarrow{\lambda}_i(R(t))\rangle = e^{i\alpha_i(t)}|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle, \tag{14}$$

and then substitute it into Eq. (4). After a straightforward algebra calculation, we find

$$\alpha_i(t) = -i\int_0^t dt \left(\langle\tilde{\overrightarrow{\eta}}_i|M|\tilde{\overrightarrow{\lambda}}_i\rangle - \langle\tilde{\overrightarrow{\eta}}_i|\frac{\partial}{\partial t}|\tilde{\overrightarrow{\lambda}}_i\rangle\right), \tag{15}$$

where the phase factor $\alpha_i(t)$ can be separated into the two parts. The first term,

$$\gamma_i^d(t) = -i\int_0^t dt\langle\tilde{\overrightarrow{\eta}}_i(t)|M|\tilde{\overrightarrow{\lambda}}_i(t)\rangle, \tag{16}$$

is relative to the Hamiltonian of system as shown in Eq. (5) and therefore called as the dynamic phase. The second term,

$$\begin{aligned}\gamma_i^g(t) &= i\int_0^t dt\langle\tilde{\overrightarrow{\eta}}_i(R(t))|\frac{\partial}{\partial t}|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle \\ &= i\sum_\beta\int_{R(0)}^{R_\beta(t)} dR^\beta\langle\tilde{\overrightarrow{\eta}}_i(R(t))|\frac{\partial}{\partial R^\beta}|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle \\ &= i\int_0^{R(t)}\langle\tilde{\overrightarrow{\eta}}_i(R(t))|d|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle,\end{aligned} \tag{17}$$

is an integral of the local differential one-form,

$$A_i(R(t)) = \langle\tilde{\overrightarrow{\eta}}_i(R(t))|d|\tilde{\overrightarrow{\lambda}}_i(R(t))\rangle, \tag{18}$$

which is a N^2-dimensional vector potential. From Eqs. (17) and (18), we know that $\gamma_i^g(t)$ is independent of the dynamics of system.

It is interesting to note that Eq. (14) defines a projective map Π, i.e.,

$$|\tilde{\overrightarrow{\lambda}}_i\rangle \to \Pi(|\tilde{\overrightarrow{\lambda}}_i\rangle) = \{|\overrightarrow{\lambda}_i\rangle : |\overrightarrow{\lambda}_i\rangle = e^{i\alpha_i(t)}|\tilde{\overrightarrow{\lambda}}_i\rangle\}, \tag{19}$$

and

$$|\tilde{\overrightarrow{\eta}}_i\rangle \to \Pi(|\tilde{\overrightarrow{\eta}}_i\rangle) = \{|\overrightarrow{\eta}_i\rangle : |\overrightarrow{\eta}_i\rangle = e^{i\alpha_i(t)}|\tilde{\overrightarrow{\eta}}_i\rangle\}, \tag{20}$$

where $|\tilde{\overrightarrow{\lambda}}_i\rangle$ and $|\tilde{\overrightarrow{\eta}}_i\rangle$ trace a curve $\mathcal{C}$ in Hilbert space $\mathcal{H}$: $[0,t] \to \mathcal{H}$ with $\hat{\mathcal{C}} = \Pi^{-1}(\mathcal{C})$ being a closed curve in projective Hilbert space in processing of the cyclic evolution of a physical system. Therefore the same $|\overrightarrow{\lambda}_i(t)\rangle$ and $|\overrightarrow{\eta}_i(t)\rangle$ can be chosen for every curve $\mathcal{C}$ for which $\Pi^{-1}(\mathcal{C}) = \hat{\mathcal{C}}$ by an appropriate choice of $\alpha_i(t)$ [39].

Thus, the phase factor γ_i^g from Eq. (17) under the cyclic evolution is rewritten as

$$\gamma_i^g = \oint_{\hat{\mathcal{C}}=\partial S} A_i = \int_S dA_i, \tag{21}$$

where the curve $\hat{\mathcal{C}}$ is traced by the parameters $R(t)$ and the surface S can be arbitrarily chosen as long as it is bounded by the closed curve $\hat{\mathcal{C}}$.

For another choice $\beta_i(t)$ different from $\alpha_i(t)$, such as

$$|\overrightarrow{\lambda}'_i(t)\rangle = e^{i\beta_i(t)}|\tilde{\overrightarrow{\lambda}}_i(t)\rangle, \tag{22}$$

and

$$|\overrightarrow{\eta}'_i(t)\rangle = e^{i\beta_i(t)}|\tilde{\overrightarrow{\eta}}_i(t)\rangle, \tag{23}$$

ones have

$$A_i \to A'_i = \langle\overrightarrow{\eta}'_i(t)|d|\overrightarrow{\lambda}'_i(t)\rangle = A_i - d\beta_i(t). \tag{24}$$

It is interesting to note that an exterior derivative of the connection one-form A_i is invariant, i.e.,

$$dA_i' = dA_i. \tag{25}$$

According to Eq. (21), we see that γ_i^g in Eq. (21) is gauge invariant under the gauge transformations (19) and (20) as well as (22) and (23). Therefore, γ_i^g can not be modified and eliminated by a multiplication of the basis vectors $|\overrightarrow{\tilde{\lambda}}_i(R(t))\rangle$ and $\langle\overrightarrow{\tilde{\eta}}_i(R(t))|$ or by a local (time-dependent) complex factor. In other words, $\gamma_i^g(t)$ is a geometric phase factor in which is independent of $\alpha_i(t)$ and $\mathcal{H}(t)$ for a given closed curve $\hat{\mathcal{C}}$.

From the above analysis, it is known that in the Hilbert space the observables are able to be rescaled in the Heisenberg picture as N^2-state vectors. For a given eigenvalue of the system, however, there exist two cases of degeneracy and nondegeneracy eigenvectors. It is obvious that in the nondegeneracy case, γ_i^g is a Abelian geometric phase.

For the degeneracy eigenvectors $|\overrightarrow{\tilde{\lambda}}_i(R(t)), b\rangle$ and $\langle\overrightarrow{\tilde{\eta}}_i(R(t)), a|$ with the degeneracy labels b and a, the corresponding vector potential A_i, geometric phase γ_i^g and dynamic phase γ_i^d should be modified as

$$A_{iab} = \langle\overrightarrow{\tilde{\eta}}_i(t), a|d|\overrightarrow{\tilde{\lambda}}_i(t), b\rangle, \tag{26}$$

$$\gamma_{iab}^g = i\int_0^t A_{iab}, \tag{27}$$

and

$$\gamma_{iab}^d(t) = -i\int_0^t dt\langle\overrightarrow{\tilde{\eta}}_i, a|M|\overrightarrow{\tilde{\lambda}}_i, b\rangle, \tag{28}$$

respectively. Thus, a useful generic solution of the observable $\mathcal{Q}$ in the Heisenberg picture is given by

$$\mathcal{Q}(t) = \sum_{i,a,b} c_{iab}\mathcal{P}\left(e^{i\alpha_{iab}(t)}\right)\mathcal{O}\cdot|\overrightarrow{\tilde{\lambda}}_i(t), a\rangle, \tag{29}$$

where $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_{N^2}\}$ is written as a vector form, c_{iab} are time-independent constants determined by the initial conditions and $\mathcal{P}$ denotes path-ordering operator. It is noted that the geometric part of the phase factor, $\mathcal{P}\left(e^{i\gamma_{iab}^g(t)}\right)$, is similar to Wilezek and Zee's non-Abelian geometric phase [16] in the Heisenberg picture. Moreover, this phase factor reduces to the Abelian geometric phase factor $\gamma_i^g(t)$ under the case of nondegeneracy with $a = b = 1$.

4 Nuclear-magnetic-resonance system

It is known that the nuclear-magnetic-resonance (NMR) spectroscopy is a powerful technique that can provide detailed information on the topology, dynamics and three-dimensional structure of molecules in solution and the solid state. For an application, let us consider a nuclear-magnetic-resonance system with the Hamiltonian,

$$\begin{aligned}\mathcal{H}(t) &= -\frac{\hbar}{2}\Omega_0(\sigma_x\sin\theta\cos\omega t + \sigma_y\sin\theta\sin\omega t) \\ &\quad -\frac{\hbar}{2}\Omega_1\sigma_z\cos\theta,\end{aligned} \tag{30}$$

where $\Omega_i = g(\mu)B_i/\hbar (i = 0, 1)$ with $g(\mu)$ are the gyromagnetic, B_i and θ act as the external controllable parameters and can be experimentally changed, and $\sigma_i (i = x, y, z)$ are the Pauli operators satisfying a closed algebra with the commutative $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k (i, j, k = x, y, z)$ and the anticommutative $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ relations. Therefore, σ_x, σ_y and σ_z together with the unit operator $I_{2\times 2}$ construct a complete basis vector for any 2×2 matrices. Thus the arbitrary observable $\mathcal{Q}(t)$ in the system can be expanded as $\mathcal{Q}(t) = \mathcal{Q}_x(t)\sigma_x + \mathcal{Q}_y(t)\sigma_y + \mathcal{Q}_z(t)\sigma_z + \mathcal{Q}_I(t)I$. $\mathcal{Q}_I$ is independent of time from the structure of Eq. (3) whereas $\mathcal{Q}_i(t)$ (i=x,y,z) satisfy

$$\frac{d}{dt}\begin{pmatrix} \mathcal{Q}_x \\ \mathcal{Q}_y \\ \mathcal{Q}_z \end{pmatrix} = \begin{pmatrix} 0 & \Omega_1 \cos\theta & -\Omega_0 \sin\theta \sin(\omega t) \\ -\Omega_1 \cos\theta & 0 & \Omega_0 \sin\theta \cos(\omega t) \\ \Omega_0 \sin\theta \sin(\omega t) & -\Omega_0 \sin\theta \cos(\omega t) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}_x \\ \mathcal{Q}_y \\ \mathcal{Q}_z \end{pmatrix}, \quad (31)$$

which defines the matrix M in Eq. (4). It is direct to obtain its biorthonormal eigenvalues and corresponding eigenvectors. The eigenvalues are $\tilde{\lambda}_0 = \tilde{\eta}_0 = 0$ and $\tilde{\lambda}_\pm = \tilde{\eta}_\pm^* = \pm i\lambda$ with $\lambda = \sqrt{\Omega_0^2 \sin^2\theta + \Omega_1^2 \cos^2\theta}$ and corresponding normalized and biorthonormal eigenvectors,

$$|\vec{\tilde{\lambda}}_0(t)\rangle = |\vec{\tilde{\eta}}_0(t)\rangle = \frac{1}{\lambda}\begin{pmatrix} \Omega_0 \sin\theta \cos(\omega t) \\ \Omega_0 \sin\theta \sin(\omega t) \\ \Omega_1 \cos\theta \end{pmatrix}, \quad (32)$$

and

$$|\vec{\tilde{\lambda}}_\pm(t)\rangle = |\vec{\tilde{\eta}}_\mp(t)\rangle = \frac{1}{\sqrt{2}\lambda}\begin{pmatrix} -\Omega_1 \cos\theta \cos(\omega t) \pm i\lambda \sin(\omega t) \\ -\Omega_1 \cos\theta \sin(\omega t) \mp i\lambda \cos(\omega t) \\ \Omega_0 \sin\theta \end{pmatrix}, \quad (33)$$

respectively.

Inserting Eqs. (32) and (33) into Eq. (21), the cyclic geometric phases can be obtained for the different biorthonormal eigenvectors as

$$\begin{aligned} \gamma_0^g(\mathcal{C}) &= i\oint_0^{2\pi/\omega} dt \langle \vec{\tilde{\eta}}_0(t)|\frac{\partial}{\partial t}|\vec{\tilde{\lambda}}_0(t)\rangle \\ &= 0, \end{aligned} \quad (34)$$

and

$$\begin{aligned} \gamma_\pm^g(\mathcal{C}) &= i\oint_0^{2\pi/\omega} dt \langle \vec{\tilde{\eta}}_\mp(t)|\frac{\partial}{\partial t}|\vec{\tilde{\lambda}}_\pm(t)\rangle \\ &= \pm\frac{\pi}{\lambda}\Omega_1 \cos\theta, \end{aligned} \quad (35)$$

respectively. In the geometric quantum computation, however, the nonadiabatic evolutions result in the errors that typically destroy cyclicity [40]. Therefore, it is interesting to discuss the noncyclic case, the nontrivial noncyclic dynamic phases are given by

$$\gamma_\pm^d(t) = -i\int_0^t dt \langle \vec{\tilde{\eta}}_\mp(t)|M|\vec{\tilde{\lambda}}_\pm(t)\rangle = \pm\gamma_d(t), \quad (36)$$

where $\gamma_d(t) = \lambda t/2$. According to the gauge invariant, the noncyclic geometric phase is expressed as [41]

$$\begin{aligned}\gamma^g_\pm(t) &= Arg\langle\overrightarrow{\tilde{\eta}}_\mp(0)|\overrightarrow{\tilde{\lambda}}_\pm(t)\rangle - Im\int_0^t dt\langle\overrightarrow{\tilde{\eta}}_\mp(t)|\frac{\partial}{\partial t}|\overrightarrow{\tilde{\lambda}}_\pm(t)\rangle \\ &= \mp\tan^{-1}\frac{2\lambda\Omega_1\cos\theta\sin\omega t}{(\Omega_1^2\cos^2\theta+\lambda^2)\cos\omega t+\Omega_0^2\sin^2\theta} \pm \frac{1}{2\lambda}\Omega_1\cos\theta\omega t, \end{aligned} \tag{37}$$

which is called as a Pancharatnam phase [1, 28]. In Eq. (37), the first term is total phase. Under the situation of the cyclic evolution with the cyclicity $T = 2\pi/\omega$, it is equal to a constant $\mp 2\pi$, which is not important and can be dropped off in quantum computation. The second term is exactly the same as the geometric phase in Eq. (35) and therefore noted as $\gamma_g(t) = (2\lambda)^{-1}\Omega_1\cos\theta\omega t$.

Thus the solution of Eq. (31) is given by

$$\begin{aligned}\begin{pmatrix}\mathcal{Q}_x\\ \mathcal{Q}_y\\ \mathcal{Q}_z\end{pmatrix} &= c_0|\overrightarrow{\tilde{\lambda}}_0(t)\rangle + c_+e^{i(\gamma_d(t)+\gamma_g(t))}|\overrightarrow{\tilde{\lambda}}_+(t)\rangle \\ &\quad +c_-e^{-i(\gamma_d(t)+\gamma_g(t))}|\overrightarrow{\tilde{\lambda}}_+(t)\rangle, \end{aligned} \tag{38}$$

where c_0 and $c_\pm$ are constants determined by the initial conditions. Eq. (38) implies that there indeed exist nontrivial geometric phases depending on the particular set of eigenvalues at the observable $\mathcal{Q}$ in the Heisenberg picture.

In terms of Eq. (38), it is possible to implement the conditional geometric quantum computation by using the observables. An effective way is to let the dynamic phase be proportional to the geometric phase by tuning experimentally the external controllable parameters [42], such as $\Omega_0, \omega_1, \theta$ and ω. We find that when $\lambda^2 = r\Omega_1\cos\theta\omega$, the dynamic phase is proportional to the geometric phase, i.e., $\gamma_d = r\gamma_g$ with a given controllable proportional constant r. Thus the time-dependent quantum gate may be written as

$$U(t) = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{i(1+r)\gamma_g(t)} & 0\\ 0 & 0 & e^{-i(1+r)\gamma_g(t)}\end{pmatrix}, \tag{39}$$

under the biorthonormal basis $\{|\overrightarrow{\tilde{\lambda}}_0(t)\rangle, |\overrightarrow{\tilde{\lambda}}_\pm(t)\rangle\}$ and $\{|\overrightarrow{\tilde{\eta}}_0(t)\rangle, |\overrightarrow{\tilde{\eta}}_\pm(t)\rangle\}$. Under the cyclic evolution with the cyclicity $T = 2\pi/\omega$, $U(t = T)$ is a geometric quantum gate. In this case, the proposed approach can avoid the problems about some types of errors that do not directly observe in the experiments and therefore reduces experimental operations, which is useful to analyze the fault tolerance associated with such errors. Especially, the observable $\mathcal{Q}(t)$ can be obtained in the geometric quantum computation. For example, under the initial conditions $\mathcal{Q}_z(0) = 1, \mathcal{Q}_x(0) = \mathcal{Q}_y(0) = \mathcal{Q}_I(0) = 0$, we find that $c_+ = c_- = \frac{1}{\sqrt{2}}\lambda^{-1}\Omega_0\sin\theta, c_0 = \lambda^{-1}\Omega_1\cos\theta$, and $c_I = 0$. In this case, $\mathcal{Q}(t) = \sigma_z(t)$, i.e.,

$$\begin{aligned}\sigma_z(t) &= \frac{\Omega_0\sin\theta}{\lambda^2}\{\Omega_1\cos\theta(1-\cos(1+r)\gamma_g)\sigma^+_{xy}(t) \\ &\quad -\lambda\sin(1+r)\gamma_g\sigma^-_{xy}(t)\} + \sigma_z, \end{aligned} \tag{40}$$

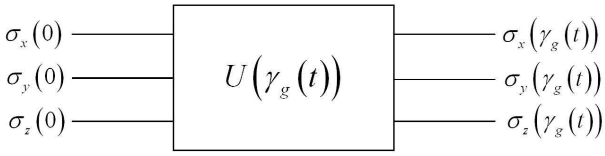

Fig. 1. (Color online) Time-dependent quantum gate from the observables σ_x, σ_y and σ_x for the nuclear-magnetic-resonance system. For the cyclic evolution, $U(\gamma(t = 2\pi/\omega))$ is a geometric quantum gate.

where $\sigma_{xy}^{+}(t) = \cos(\omega t)\sigma_x + \sin(\omega t)\sigma_y$ and $\sigma_{xy}^{-}(t) = \sin(\omega t)\sigma_x - \cos(\omega t)\sigma_y$.

Under the conditions of $Q_x(0) = 1, Q_y(0) = Q_z(0) = Q_I(0) = 0$, one has $c_+ = c_- = -\frac{1}{\sqrt{2}}\lambda^{-1}\Omega_1 \cos\theta, c_0 = \lambda^{-1}\Omega_0 \sin\theta$, and $c_I = 0$. In this case, $\mathcal{Q}(t) = \sigma_x(t)$, we have

$$\begin{aligned}\sigma_x(t) &= \frac{1}{\lambda^2}\{(\Omega_0^2 \sin^2\theta + \Omega_1^2 \cos^2\theta \cos(1+r)\gamma_g)\sigma_{xy}^{+}(t) \\ &\quad + \lambda\Omega_1 \cos\theta \sin(1+r)\gamma_g \sigma_{xy}^{-}(t)\}. \end{aligned} \tag{41}$$

The similar case for $Q_y(0) = 1, Q_x(0) = Q_z(0) = Q_I(0) = 0$, $c_+ = \frac{i}{\sqrt{2}}$. Thus $c_- = \frac{-i}{\sqrt{2}}$ and $c_0 = c_I = 0$. In this case, $\mathcal{Q}(t) = \sigma_y(t)$, we find

$$\begin{aligned}\sigma_y(t) &= \frac{1}{\lambda}\{\Omega_1 \cos\theta \sin(1+r)\gamma_g \sigma_{xy}^{+}(t) \\ &\quad - \lambda \cos(1+r)\gamma_g \sigma_{xy}^{-}(t)\}. \end{aligned} \tag{42}$$

From Eqs. (40)-(42), we see that the observables carry with the geometric phase in the cyclic evolution. In the Heisenberg picture, therefore, the quantum system retains its memory of evolution in terms of this phase. Therefore a quantum geometric gate can be implemented in terms of the observables σ_x, σ_y and σ_x as shown in Fig. 1.

Comparing the geometric phases from the observables satisfied the Heisenberg equation with the ones from the wave functions satisfied the Schrödinger equation [42], we find that they are not fully the same. The results can be understood because the observables in the Heisenberg picture are different from the wave functions in the Schr̈dinger picture, which provides the other choices in the geometric quantum computation.

5 Conclusions

In summary, an exact solution of observables under time evolution is given in the Heisenberg picture, where the Heisenberg equation is rescaled as a Schrödinger-like matrix form in terms of a complete basis with a set of the Hermitian operators. We find that the observales carry with a geometric message in their evolutions in the Heisenberg picture. Especially, this geometric phase shows a geometric structure picture of quantum dynamics so as to be defined for any closed curve in the projective Hilbert space. Similarly to the Schrödinger picture, thus, the geometric phase is independent of the Hamiltonian of the physical system for a given closed curve and proportional to the area spanned in the parameter space. Therefore the evolving memory of a quantum system is kept in terms of the geometric phase carried by the observables in the Heisenberg picture.

Such a geometric nature provides another clue to implement the geometric quantum calculation in terms of the observables as shown for the nuclear-magnetic-resonance system. In particular it may be helpful to reduce the experimental and fluctuational errors as well as gate timing for the multi-time correlation geometric quantum computation.

Acknowledgments

This work is supported by the Natural Science Foundation of China under Grant No.11365012, the Natural Science Foundation of Jiangxi Province, China under Grant No. 20132BAB202008, the Foundation of Science and Technology of Education Office of Jiangxi Province under No. GJJ13235.

Hui Pan thanks the supports of the Science and Technology Development Fund from Macao SAR (FDCT-068/2014/A2 and FDCT-132/2014/A3), and Multi-Year Research Grants (MYRG2014-00159-FST and MYRG2015-0015-FST) and Start-up Research Grant (SRG-2013-00033-FST) from Research and Development Office at University of Macau.

1. S. Pancharatnam, Proc. Ind. Acad. Sci. A 44, 247 (1956).
2. M.V. Berry, Proc. R. Soc. A 392, 45 (1984).
3. B. Simon, Holonomy, Phys. Rev. Lett. 51, 2167 (1983).
4. Geometric Phases in Physics, edited by A. Shapere and F. Wilczek, (World Scientific, Singapore, 1989).
5. Y. Aharonov, J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
6. J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
7. Erik Sjöqvist, D M Tong, L Mauritz Andersson, Björn Hessmo, Markus Johansson, and Kuldip Singh, New J. Phys. 14, 103035 (2012).
8. A. A. Abdumalikov Jr, J. M. Fink, K. Juliusson, M. Pechal, S. Berger, A.Wallraff, and S. Filipp, Nature 496, 482 (2013).
9. Guanru Feng, Guofu Xu, and Guilu Long, Phys. Rev. Lett. 110, 190501 (2013).
10. P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999).
11. R. Das, S.K.K. Kumar, A. Kumar, J. Magn. Reson. 177, 318 (2005).
12. T. Gopinath, A. Kumar, J. Magn. Reson. 193, 168-176 (2008).
13. Z. S. Wang, Phys. Rev. A **79**, 024304 (2009).
14. Austin G. Fowler and Kovid Goyal, Quantum Information and Computation 9, 0721 (2009).
15. Mark R. Dowling and Michael A. Nielsen, Quantum Information and Computation 8, 0861 (2008)
16. F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
17. H. Z. Li, Phys. Rev. Lett. 58, 539 (1987).
18. D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).
19. F. D. Haldane and Y. S. Wu, Phys. Rev. Lett. 55, 2887 (1985).
20. G. W. Semenoff and P. Sodano, Phys. Rev. Lett. 57, 1195 (1986).
21. Z.S. Wang, C.F. Wu, X.-L. Feng, L.C. Kwek, C.H. Lai, C.H. Oh, V. Vedral, Phys. Lett. A 372, 775 (2008).
22. Z.S. Wang, L.C. Kwek, C.H. Lai, C.H. Oh, Phys. Lett. A 359, 608 (2006).
23. J. C. Garrison and E. M. Wright, Phys. Lett. A 128, 177 (1988).
24. D.M. Tong, E. Sjöqvist, L. C. Kwek, and C.H. Oh, Phys. Rev. Lett. 93, 080405 (2004).
25. A. Uhlmann, Rep. Math. Phys. 24, 229 (1986).
26. E. Sjövist et al., Phys. Rev. Lett. 85, 2845 (2000).
27. A. Carollo, I. Fuentes-Guridi, M. Franca Santos, V. Vedral, Phys. Rev. Lett. 90, 160402 (2003).
28. Z. S. Wang et al., Europhys. Lett. 74, 958 (2006).
29. M. S. Sarandy and D. A. Lidar, Phys. Rev. A 73, 062101 (2006).
30. Hualan Xu, Dan Fu, Z. S. Wang, Hui Pan, J. Magn. Reson. 223, 25 (2012).
31. F. Gaitan, J. Magn. Reson. 139, 152 (1999).
32. T. Bitter and D. Dubbers, Phys. Rev. Lett. 59, 251 (1987).
33. A. Tomita and R. Y. Chiao, Phys. Rev. Lett. 57, 937 (1986).
34. D. Suter, K. T. Mueller, and A. Pines, Phys. Rev. Lett. 60, 1218 (1988).
35. M. Möttönen, J. J. Vartiainen, and J. P. Pekola, Phys. Rev. Lett. 100, 177201 (2008).

36. Hamed Ahmadi and Chen-Fu Chiang, Quantum Information and Computation 12, 0864 (2012)
37. P. Echternach, C.P. Williams, S.C. Dultz, S. Braunstein, and J.P. Dowling, Quantum Information and Computation 1, 143 (2001).
38. M. S. Byrd, and N. Khaneja, Phys. Rev. A 68, 062322 (2003).
39. Z. S. wang and Qian Liu, Phys. Lett. A 377, 3272 (2013).
40. A. Blais and A.-M. S. Tremblay, Phys. Rev. A 67, 012308 (2003).
41. N. Mukunda and R. Simon, Ann. Phys. (N.Y.) 228, 205 (1993).
42. Z. S. Wang, G. Q. Liu, Y. H. Ji, Phys. Rev. A 79, 054301 (2009).