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QUANTUM CIRCUITS FOR ASYMMETRIC $1 \to n$ QUANTUM CLONING

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In this paper, we considered asymmetric $1 \to n$ cloning circuits generalized from the asymmetric $1 \to 2$ cloning circuit proposed by Bužek *et al.* [Phys. Rev. A 56, 3446 (1997)]. The generalization is based on an information flux insight of the original cloning circuit. Specifically, the circuit separately and sequentially transfers the Z-type information and X-type information of the input state to the output copies with controlled-not gates. The initial input state of the clones defines the asymmetry among all output clones. Although the generalized circuits do not perform universally, the averaged fidelities over a uniform distribution of all possible input cloning states saturate the optimal fidelity tradeoff relations of universal asymmetric cloning.

Keywords: asymmetric quantum cloning, controlled quantum gate, quantum circuit
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1 Introduction

Unknown quantum states cannot be faithfully copied as a result of the quantum superposition principle, which is the well known quantum no-cloning theorem [1]. However, copies with reduced fidelities can be obtained through the so called quantum cloning machine [2, 3]. Quantum cloning, approximately or probabilistically, is related with various topics of quantum information processing such as protocols of quantum cryptography. The quantum cloning machines can be implemented by various schemes depending on specific physical systems. The quantum circuit scheme takes the advantage that it depends only on the realization of the universal quantum computation, without considering the details of physical implementation. On the other hand, the design of simple quantum circuit relies on our understanding of the investigated quantum information processing task [4]. Here, we will consider the circuit realization of asymmetric quantum cloning.

After the introduction of universal $1 \to 2$ quantum cloning [5], Bužek *et al.* developed a quantum circuit for this cloner [6]. The circuit consists of only controlled-not (CNOT) gates and distributes the quantum information to the clones with respect to their initial input states. Therefore, with specially chosen input states, the circuit can be programmed to realize asymmetric $1 \to 2$ cloning. Another route to asymmetric quantum cloning is introduced by

Cerf [7], where the clones went through different Pauli channels and achieved different fidelities with the cloned state. Niu and Griffiths [8] also derived the optimal asymmetric universal quantum cloner in their comprehensive study of the $1 \to 2$ cloning.

Later, the optimality of the above universal asymmetric cloners and generalized asymmetric $1 \to 3$ cloning fidelity tradeoff relations were constructed by Fiurášek *et al.* [9] with operator methods. Their results were also generalized to asymmetric $1 \to 4$ cloning [10, 11]. The most general asymmetric $1 \to n$ cloning fidelity tradeoff relation was built by Key *et al.* [12] with the concept of singlet monogamy and fully proved by them through Lieb-Mattis Theorem in [13]. Besides these results on single input cloning, there are also many researches on multiple input cloning [2, 3]. An interesting unified universal cloning construction with multiple input to multiple output was introduced in Ref.[14], which has potential for asymmetric generalization. Unlike the single input cloning, we generally do not have expressible tradeoff relations for multiple input to multiple output cloning. The only exception was derived in [13] for $n \to n+1$ universal asymmetric cloning. In this paper, we limit to the single input case.

In [15], Franco *et al.* introduced an interesting insight on the information flux in the cloning circuit proposed in [6]. They investigated the transformations of the operators under the circuit operation, which showed that the circuit performed universally on all possible cloning states. In this circuit, the Z-type and X-type information of the qubit state were separately and sequentially transferred to the clones by two groups of CNOT gates. In light of these insights, we will construct and analyze the $1 \to n$ generalization of the original $1 \to 2$ circuit. Our results show that the generalized cloning circuit does not perform universally, where the output clone fidelity depends on the input cloning states. However, if the cloning state is chosen from a uniform distribution, we find that the averaged fidelities of the output clones saturate the fidelity tradeoff relation in [13]. We also generalize our discussions to higher dimensional discrete systems.

2 Qubit case

Firstly, we give an analysis on the $1 \to 2$ quantum cloning circuit for qubit system as shown in Fig.1a [5].

In this circuit, the first qubit takes in the state to be cloned and the other two qubits act as the clones. Throughout this paper, the first qubit denoted with number 0 is reserved as the original qubit. The function of this circuit is determined by the input states of the three qubits. For quantum cloning, there are two important states for the two clones, $\{|+\rangle_z|+\rangle_x, |+\rangle_x|+\rangle_z\}$, with the subscripts denoting the Pauli basis. With a randomly chosen qubit input state, $|\psi\rangle_0$, the cloning circuit transforms the inputs into,

$$|\psi\rangle_0|+\rangle_{z,1}|+\rangle_{x,2} \to |\psi\rangle_1|\Phi^+\rangle_{02}, \quad |\psi\rangle_0|+\rangle_{x,1}|+\rangle_{z,2} \to |\psi\rangle_2|\Phi^+\rangle_{01}, \tag{1}$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. We notice that the cloning state is transferred completely to the two clones with the above preparations. Therefore, with an asymmetric superposition state, $|\phi\rangle_{12} = a|+\rangle_{z,1}|+\rangle_{x,2} + b|+\rangle_{x,1}|+\rangle_{z,2}$, with a and b real numbers, the linear cloning circuit performs the following transformation,

$$|\psi\rangle_0|\phi\rangle_{12} \to a|\psi\rangle_1|\Phi^+\rangle_{02} + b|\psi\rangle_2|\Phi^+\rangle_{01}. \tag{2}$$

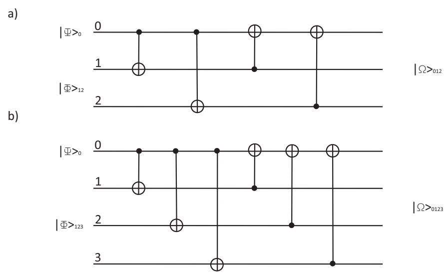

Fig. 1. Quantum circuits for $1 \to 2$ qubit cloning (a) and $1 \to 3$ qubit cloning (b).

where a, b satisfy the normalization condition $a^2 + b^2 + ab = 1$ which confines the cloning fidelities. Here, the initial state of clones acts as a software which determines how much input information are separately transferred to the clones. The output states of the two qubit clones are, $\rho_1 = [I+(a^2+ab)\hat{n}\cdot\vec{\sigma}]/2, \rho_2 = [I+(b^2+ab)\hat{n}\cdot\vec{\sigma}]/2$, where $|\psi\rangle\langle\psi| = [I+\hat{n}\cdot\vec{\sigma}]/2$, $|\hat{n}| = 1$. It is important to notice that the two cloning outputs have the same direction with the cloning input state while their lengths are reduced (< 1). The contraction rate is independent of the direction of the input cloning state and only depends on the software part of the circuit, i.e., the input states of the clones, so the circuit performs universally.

Table 1. (taken from [15]). The evolved operators after time t_j [*i.e.*, after the application of the j-th CNOT in Fig.1a] expressed in terms of the operators before the copying stage.

	t_1	t_2	t_3	t_4
$\hat{\hat{X}}_0(t)$	$\hat{X}_0\hat{X}_1$	$\hat{X}_0\hat{X}_1\hat{X}_2$	$\hat{X}_0\hat{X}_1\hat{X}_2$	$\hat{X}_0\hat{X}_1\hat{X}_2$
$\hat{\hat{Z}}_0(t)$	$\hat{Z}_0$	$\hat{Z}_0$	$\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1\hat{Z}_2$
$\hat{\hat{X}}_1(t)$	$\hat{X}_1$	$\hat{X}_1$	$\hat{X}_0\hat{X}_2$	$\hat{X}_0\hat{X}_2$
$\hat{\hat{Z}}_1(t)$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$
$\hat{\hat{X}}_2(t)$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_0\hat{X}_1$
$\hat{\hat{Z}}_2(t)$	$\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$

The above results can also be conceived from another viewpoint. In the above discussions, we use Schrödinger picture where the input state is changed by the circuit. However, we can also use Heisenberg picture where the operators are changed while the state remains unchanged. Table 1 lists all the operator changes after each CNOT gate. After cloning, we can carry out the calculations with the transformed operators and the initial state. Here, we are interested in the fidelities between the output clone states, $\tilde{\rho}_{1,2} = [I+\sum_{\Sigma=X,Y,Z} tr(\rho_{1,2}\hat{\hat{\Sigma}}_{1,2})\hat{\Sigma}]/2$, with the input cloning state $|\psi\rangle\langle\psi|$, where $\hat{\hat{\Sigma}}_{1,2}$ are the operators at the output clones. In order to obtain these operators, we notice the following facts. For a general CNOT gate acting on its control and target qubits, the following operator changes can be directly checked with the corresponding operator basis,

$$\hat{\hat{X}}_c = \hat{X}_c\hat{X}_t, \quad \hat{\hat{Z}}_c = \hat{Z}_c, \quad \hat{\hat{X}}_t = \hat{X}_t, \quad \hat{\hat{Z}}_t = \hat{Z}_c\hat{Z}_t, \tag{3}$$

with $\hat{X}_c$, $\hat{Z}_c$ ($\hat{X}_t$, $\hat{Z}_t$) the operators of the control (target) qubit before the action of the gate and $\hat{\hat{X}}_c$, $\hat{\hat{Z}}_c$ ($\hat{\hat{X}}_t$, $\hat{\hat{Z}}_t$) the analogous operators after the action of the gate. Here we only need to consider $\hat{X}_{1,2}(t)$ and $\hat{Z}_{1,2}(t)$, since $\hat{Y}_{1,2}(t)$ is found as $i\hat{X}_{1,2}(t)\hat{Z}_{1,2}(t)$. As noted in [15], the Z-type information flows through the first two CNOT gates and the X-type information flows through the last two CNOT gates. Now we make a close check on the output operators $\hat{\hat{X}}_{1,2}(t_4)$, $\hat{\hat{Z}}_{1,2}(t_4)$ and the initial states $|+\rangle_{z,1}|+\rangle_{x,2}$, $|+\rangle_{x,1}|+\rangle_{z,2}$. Firstly, Operators $\hat{\hat{X}}_{1,2}(t_4)$, $\hat{\hat{Z}}_{1,2}(t_4)$ always depend upon the input qubit operators $\hat{X}_0$, $\hat{Z}_0$ respectively, which shows that the quantum information is transferred to these clones. Secondly, $|+\rangle_{z,1}|+\rangle_{x,2}$ is a simultaneous eigenvector of $\hat{\hat{X}}_1(t_4)$ and $\hat{\hat{Z}}_1(t_4)$ (also $\hat{\hat{Y}}_1(t_4)$) with eigenvalue $+1$ and it changes into an orthogonal state under the operation of $\hat{\hat{X}}_2(t_4)$ and $\hat{\hat{Z}}_2(t_4)$ (also $\hat{\hat{Y}}_2(t_4)$). Similarly, $|+\rangle_{x,1}|+\rangle_{z,2}$ is an eigenvector of both $\hat{\hat{X}}_2(t_4)$ and $\hat{\hat{Z}}_2(t_4)$ (also $\hat{\hat{Y}}_2(t_4)$) with eigenvalue $+1$ and it changes into an orthogonal state under the operation of $\hat{\hat{X}}_1(t_4)$ and $\hat{\hat{Z}}_1(t_4)$ (also $\hat{\hat{Y}}_1(t_4)$). This confirms the following fact: when $|+\rangle_{z,1}|+\rangle_{x,2}$ ($|+\rangle_{x,1}|+\rangle_{z,2}$) is used as the input, the cloning state comes up at clone 1(2). Taking into account of the unbiasedness between $\hat{X}$ and $\hat{Z}$, ${}_x\langle+|+\rangle_z = 1/\sqrt{2}$, we can carry out the calculations of $tr(\rho_{1,2}\hat{\hat{\Sigma}}_{1,2})$ to determine $\tilde{\rho}_{1,2}$. The results show that the two cloning outputs have the same direction with the cloning input state and different reduced lengths. This confirms the universality of the circuit just as we have already found out via Schrödinger picture.

Table 2. The evolved operators after time t_j [*i.e.*, after the application of the j-th CNOT in Fig.1b] expressed in terms of the operators before the copying stage.

	t_1	t_2	t_3	t_4	t_5	t_6
$\hat{\hat{X}}_0(t)$	$\hat{X}_0\hat{X}_1$	$\hat{X}_0\hat{X}_1\hat{X}_2$	$\hat{X}_0\hat{X}_1\hat{X}_2\hat{X}_3$	$\hat{X}_0\hat{X}_1\hat{X}_2\hat{X}_3$	$\hat{X}_0\hat{X}_1\hat{X}_2\hat{X}_3$	$\hat{X}_0\hat{X}_1\hat{X}_2\hat{X}_3$
$\hat{\hat{Z}}_0(t)$	$\hat{Z}_0$	$\hat{Z}_0$	$\hat{Z}_0$	$\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1\hat{Z}_2$	$\hat{Z}_1\hat{Z}_2\hat{Z}_3$
$\hat{\hat{X}}_1(t)$	$\hat{X}_1$	$\hat{X}_1$	$\hat{X}_1$	$\hat{X}_0\hat{X}_2\hat{X}_3$	$\hat{X}_0\hat{X}_2\hat{X}_3$	$\hat{X}_0\hat{X}_2\hat{X}_3$
$\hat{\hat{Z}}_1(t)$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$
$\hat{\hat{X}}_2(t)$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_0\hat{X}_1\hat{X}_3$	$\hat{X}_0\hat{X}_1\hat{X}_3$
$\hat{\hat{Z}}_2(t)$	$\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$
$\hat{\hat{X}}_3(t)$	$\hat{X}_3$	$\hat{X}_3$	$\hat{X}_3$	$\hat{X}_3$	$\hat{X}_3$	$\hat{X}_0\hat{X}_1\hat{X}_2$
$\hat{\hat{Z}}_3(t)$	$\hat{Z}_3$	$\hat{Z}_3$	$\hat{Z}_0\hat{Z}_3$	$\hat{Z}_0\hat{Z}_3$	$\hat{Z}_0\hat{Z}_3$	$\hat{Z}_0\hat{Z}_3$

Next we consider the generalized $1 \rightarrow 3$ cloning circuit shown in Fig.1b. According to the transformed operators $\hat{\hat{\Sigma}}_i(t)$, $\Sigma = X, Y, Z$ and $i = 1, 2, 3$, in table 2, we choose the following states, $\{|+\rangle_{z,1}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{23}$, $|+\rangle_{z,2}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{13}$, $|+\rangle_{z,3}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{12}\}$, with a,b real and $a^2 + b^2 = 1$, to construct an initial asymmetric state for the clones, $|\phi\rangle_{123} = a_1|+\rangle_{z,1}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{23} + a_2|+\rangle_{z,2}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{13} + a_3|+\rangle_{z,3}(a|+\rangle_x|+\rangle_x + b|-\rangle_x|-\rangle_x)_{12}$, with $a_1, a_2, a_3 \geq 0$ and $a_1^2 + a_2^2 + a_3^2 + a_1a_2 + a_2a_3 + a_1a_3 = 1$.

It can be directly checked that with an initial state, $|+\rangle_{z,i}(a|+\rangle_x|+\rangle_x+b|-\rangle_x|-\rangle_x)_{jk}$, the input cloning state comes out at clone i.

Will this circuit perform universally? It turns out that the answer is NO! Since the three copies behave quite similarly, we take copy 1 as an example. Direct calculations show that,

$$\langle\hat{\hat{X}}_1(t_6)\rangle \equiv tr(|\psi\rangle_0\langle\psi|\hat{X}_0)tr(|\phi\rangle_{123}\langle\phi|\hat{X}_2\hat{X}_3) = (a_1^2+a_1a_2+a_1a_3+a_2a_3)\langle\hat{X}_0\rangle, \quad (4)$$

$$\langle\hat{\hat{Z}}_1(t_6)\rangle \equiv tr(|\psi\rangle_0\langle\psi|\hat{Z}_0)tr(|\phi\rangle_{123}\langle\phi|\hat{Z}_1) = (a_1^2+a_1a_2+a_1a_3+2ab\times a_2a_3)\langle\hat{Z}_0\rangle \quad (5)$$

$$\langle\hat{\hat{Y}}_1(t_6)\rangle \equiv tr(|\psi\rangle_0\langle\psi|\hat{Y}_0)tr(|\phi\rangle_{123}\langle\phi|\hat{Z}_1\hat{X}_2\hat{X}_3) = (a_1^2+a_1a_2+a_1a_3-2ab\times a_2a_3)\langle\hat{Y}_0\rangle \quad (6)$$

Obviously, the state of the output clone contracts differently along different axis. Therefore, unlike the $1 \rightarrow 2$ case, the output is now dependent on the input state and the circuit does not operate universally. However, if the input state $|\psi\rangle$ is uniformly distributed on the Bloch Sphere, i.e., $|\psi\rangle = \cos(\theta/2)|+\rangle_z + e^{i\varphi}\sin(\theta/2)|-\rangle_z$, with $\theta \in [0,\pi], \varphi \in [0,2\pi]$, then we can show that the output fidelities saturate the fidelity tradeoff relation for optimal asymmetric universal quantum cloning. The output states are, $\rho_i = (I + \langle\hat{\hat{X}}_i(t_6)\rangle\hat{X} + \langle\hat{\hat{Y}}_i(t_6)\rangle\hat{Y} + \langle\hat{\hat{Z}}_i(t_6)\rangle\hat{Z})/2$, $i = 1,2,3$. Their fidelities with the cloning state are, $F_i = \langle\psi|\rho_i|\psi\rangle$. For example,

$$F_1 = \langle\psi|\rho_1|\psi\rangle = \frac{1}{2}(1+a_1^2+a_1a_2+a_1a_3+a_2a_3\langle\hat{X}_0\rangle^2+2ab\times a_2a_3\langle\hat{Z}_0\rangle^2-2ab\times a_2a_3\langle\hat{Y}_0\rangle^2). \quad (7)$$

Taking the average over the Bloch sphere, we get

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\varphi\langle\hat{\Sigma}_0\rangle^2 = \frac{1}{3}, \quad (8)$$

with $\Sigma = X, Y, Z$. Hence, $\langle F_1\rangle = \frac{1}{2}(1+a_1^2+a_1a_2+a_1a_3+\frac{1}{3}a_2a_3)$, similarly, $\langle F_2\rangle = \frac{1}{2}(1+a_2^2+a_1a_2+a_2a_3+\frac{1}{3}a_1a_3)$, $\langle F_3\rangle = \frac{1}{2}(1+a_3^2+a_1a_3+a_2a_3+\frac{1}{3}a_1a_2)$, which saturate the optimal fidelity tradeoff relation for asymmetric quantum cloning in [9].

An interesting special case is, $a = b = 1/\sqrt{2}$, then the contraction rates of $\langle\hat{Z}_0\rangle$ and $\langle\hat{X}_0\rangle$ are equal. If the input state lies on the XZ plane, $\langle\hat{Y}_0\rangle = 0$, it will be contracted identically at the output ports. In other words, the clone states will have the same direction with the input state, only with contracted length.

Further generalizations to more copies are direct, where we only need to add more copy qubits and CNOT operations. It should be noted that the CNOT gates are separated into two groups, where the first group all having the input qubit as control transfer the Z-type information and the second group all having the input qubit as target transfer the X-type information. With this generalized circuit, we choose the following state as the initial state for asymmetric $1 \rightarrow n$ cloning,

$$|\phi\rangle_{1,\ldots,n} = a_1|+\rangle_{z,1}|+\rangle_{x,2}\ldots|+\rangle_{x,n} + a_2|+\rangle_{x,1}|+\rangle_{z,2}\ldots|+\rangle_{x,n} + \ldots + a_n|+\rangle_{x,1}|+\rangle_{x,2}\ldots|+\rangle_{z,n} \quad (9)$$

with normalization $\sum_{i=1}^n a_i^2 + \sum_{i<j} a_ia_j = 1$. After the circuit operation, all the clone operators change as,

$$\hat{\hat{X}}_i = \hat{X}_0\prod_{k\neq i,k=1}^{n}\hat{X}_k, \quad \hat{\hat{Z}}_i = \hat{Z}_0\hat{Z}_i. \quad (10)$$

Now we can calculate all the expectations needed for fidelities,

$$\langle \hat{\hat{X}}_i \rangle = (a_i^2 + \sum_{k\neq i,k=1}^{n} a_i a_k + \frac{1}{2}\sum_{k,l\neq i,k\neq l} a_k a_l)\langle \hat{X}_0\rangle, \tag{11}$$

$$\langle \hat{\hat{Z}}_i \rangle = (a_i^2 + \sum_{k\neq i,k=1}^{n} a_i a_k)\langle \hat{Z}_0\rangle, \tag{12}$$

$$\langle \hat{\hat{Y}}_i \rangle = (a_i^2 + \sum_{k\neq i,k=1}^{n} a_i a_k)\langle \hat{Y}_0\rangle. \tag{13}$$

Finally, we get all the clone fidelities for this asymmetric cloning circuit,

$$F_i = \frac{1}{2}(1 + a_i^2 + \sum_{k\neq i,k=1}^{n} a_i a_k + \sum_{k,l\neq i,k<l} a_k a_l \langle \hat{X}_0\rangle^2). \tag{14}$$

The last term in the bracket proportional to $\langle \hat{X}_0\rangle^2$ shows that the generalized cloning circuit is not universal. However, if the cloning state is chosen from uniform distribution over all pure qubit states, after averaging, we finally get

$$F_i = \frac{1}{2}(1 + a_i^2 + \sum_{k\neq i,k=1}^{n} a_i a_k + \frac{1}{3}\sum_{k,l\neq i,k<l} a_k a_l). \tag{15}$$

These fidelities saturate the optimal tradeoff relation for $1 \to n$ asymmetric universal quantum cloner and can be checked with [13]

$$\sum_{i=1}^{n}(3F_i - 1) = 1 + \frac{1}{n+1}(\sum_{i=1}^{n}\sqrt{3F_i - 1})^2. \tag{16}$$

3 Qudit case

3.1 Preliminary

Before the generalization of the above discussions to d-dimensional systems (qudit), we give some preliminary concepts. Firstly, we introduce the generalized Pauli operators for state description, $\hat{Z} = \sum_{k=0}^{d-1} e^{i2\pi k/d}|k\rangle_z\langle k|$, $\hat{X} = \sum_{l=0}^{d-1} e^{i2\pi l/d}|l\rangle_x\langle l|$, with $|k\rangle_z = \sum_{l=0}^{d-1} e^{-i2\pi kl/d}|l\rangle_x/\sqrt{d}$, $|l\rangle_x = \sum_{k=0}^{d-1} e^{i2\pi kl/d}|k\rangle_x/\sqrt{d}$. An arbitrary state of a d-dimensional system can be represented with $\{\hat{X}, \hat{X}^2, ..., \hat{X}^{d-1}\}$, $\{\hat{Z}, \hat{Z}^2, ..., \hat{Z}^{d-1}\}$ and all their products. We denote this set as $\mathcal{C}$.

$$\rho = \frac{1}{d}(I + \sum_{\hat{\Sigma}\in\mathcal{C}} tr(\rho\hat{\Sigma})\hat{\Sigma}). \tag{17}$$

A pure state has, $tr(\rho^2) = 1$, and $\sum_{\hat{\Sigma}\in\mathcal{C}} tr(\rho\hat{\Sigma})^2 = d-1$. If ρ is randomly taken from a uniform distribution over all pure qudit states, then after taking average, we get $\langle tr(\rho\hat{\Sigma})^2\rangle = 1/(d+1)$ for all $\Sigma \in \mathcal{C}$ since there are $d^2 - 1$ elements in this set.

In the circuit (Fig.2), we use two generalized CNOT gates acting on one control qudit and one target qudit [16]. The two gates are mutually converse and changes the operators of the control and target in the following way,

$$\overrightarrow{\hat{X}}_c = \hat{X}_c\hat{X}_t^{-1}, \quad \overrightarrow{\hat{Z}}_c = \hat{Z}_c, \quad \overrightarrow{\hat{X}}_t = \hat{X}_t, \quad \overrightarrow{\hat{Z}}_t = \hat{Z}_c\hat{Z}_t, \tag{18}$$

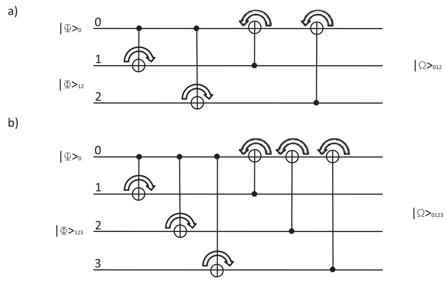

Fig. 2. Quantum circuits for $1 \to 2$ qudit cloning (a) and $1 \to 3$ qudit cloning (b). The controlled gate with right or left arrow operates according to Eq.(18) or Eq.(19) respectively.

$$\overleftarrow{\hat{X}}_c = \hat{X}_c \hat{X}_t, \quad \overleftarrow{\hat{Z}}_c = \hat{Z}_c, \quad \overleftarrow{\hat{X}}_t = \hat{X}_t, \quad \overleftarrow{\hat{Z}}_t = \hat{Z}_c^{-1} \hat{Z}_t, \tag{19}$$

where the arrow towards the right, or left, defines the transformation $|k\rangle|m\rangle \to |k\rangle|m+k\rangle$, or $|k\rangle|m\rangle \to |k\rangle|m-k\rangle$, respectively, sums and differences in the kets are modulo d.

3.2 $1 \to 2$ *cloning*

We consider the following state as the initial input state for the circuit (Fig.2a), $|\psi\rangle_0|\phi\rangle_{12} = |\psi\rangle_0[a_1|0\rangle_{z,1}|0\rangle_{x,2} + a_2|0\rangle_{x,1}|0\rangle_{z,2}]$, with normalization condition $a_1^2 + a_2^2 + 2a_1a_2/d = 1$. The evolution of the operators are listed in Table 3.

Table 3. The evolved operators after time t_j [*i.e.*, after the application of the j-th CNOT in Fig.2a] expressed in terms of the operators before the copying stage.

	t_1	t_2	t_3	t_4
$\hat{\hat{X}}_0(t)$	$\hat{X}_0\hat{X}_1^{-1}$	$\hat{X}_0\hat{X}_1^{-1}\hat{X}_2^{-1}$	$\hat{X}_0\hat{X}_1^{-1}\hat{X}_2^{-1}$	$\hat{X}_0\hat{X}_1^{-1}\hat{X}_2^{-1}$
$\hat{\hat{Z}}_0(t)$	$\hat{Z}_0$	$\hat{Z}_0$	$\hat{Z}_1^{-1}$	$\hat{Z}_0^{-1}\hat{Z}_1^{-1}\hat{Z}_2^{-1}$
$\hat{\hat{X}}_1(t)$	$\hat{X}_1$	$\hat{X}_1$	$\hat{X}_0\hat{X}_2^{-1}$	$\hat{X}\hat{X}_2^{-1}$
$\hat{\hat{Z}}_1(t)$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$	$\hat{Z}_0\hat{Z}_1$
$\hat{\hat{X}}_2(t)$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_2$	$\hat{X}_0\hat{X}_1^{-1}$
$\hat{\hat{Z}}_2(t)$	$\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$	$\hat{Z}_0\hat{Z}_2$

After the circuit operation, the expectations of the clone operators are calculated as,

$$\langle\hat{\hat{\Sigma}}_1\rangle = (a_1^2 + \frac{2}{d}a_1a_2)\langle\psi|\hat{\Sigma}|\psi\rangle, \quad \langle\hat{\hat{\Sigma}}_2\rangle = (a_2^2 + \frac{2}{d}a_1a_2)\langle\psi|\hat{\Sigma}|\psi\rangle, \tag{20}$$

for all $\Sigma \in \mathcal{C}$. Therefore, this circuit realizes optimal universal asymmetric quantum cloning, with tradeoff fidelities,

$$F_1 = \frac{1}{d}(1 + (d-1)(a_1^2 + \frac{2a_1a_2}{d})), \quad F_2 = \frac{1}{d}(1 + (d-1)(a_2^2 + \frac{2a_1a_2}{d})), \tag{21}$$

where we used the identity $\sum_{i=1}^{d^2-1} tr(\rho\hat{\Sigma}_i)^2 = d-1$ since the input cloning state $|\psi\rangle$ is pure.

3.3 General $1 \to n$ *cloning*

Now, we directly come to the $1 \to n$ cloning. In Fig.2b, we draw the $1 \to 3$ cloning circuit from which circuits with more output clones can be similarly constructed. After the operation of the generalized cloning circuit, the operators of the output qudits change to,

$$\hat{\hat{X}}_i = \hat{X}_0 \prod_{k\neq i,k=1}^{n} \hat{X}_k^{-1}, \quad \hat{\hat{Z}}_i = \hat{Z}_0\hat{Z}_i. \tag{22}$$

The states of the output qudits can be obtained with the initial state of the copy qudits,

$$|\phi\rangle_{1,\ldots,n} = a_1|0\rangle_{z,1}|0\rangle_{x,2}\ldots|+\rangle_{x,n} + a_2|0\rangle_{x,1}|0\rangle_{z,2}\ldots|0\rangle_{x,n} + \ldots + a_n|0\rangle_{x,1}|0\rangle_{x,2}\ldots|0\rangle_{z,n}, \tag{23}$$

with normalization $\sum_{j=1}^{n} a_j^2 + \frac{2}{n}\sum_{j<l} a_j a_l = 1$. For output copy i, we have

$$\langle \hat{\hat{X}}_i^k \rangle = (a_i^2 + \frac{2}{d}a_i \sum_{j\neq i} a_j + \frac{2}{d}a_i \sum_{j,l\neq i,j<l} a_j a_l)\langle \hat{X}_0^k \rangle \tag{24}$$

with $k = 1, \ldots, d-1$. All the other operators, i.e., $\{\hat{Z}, \hat{Z}^2, \ldots, \hat{Z}^{d-1}\}$ and the product operators between $\hat{X}$ and $\hat{Z}$ have the same expectation relation,

$$\langle \hat{\hat{\Sigma}}_i \rangle = (a_i^2 + \frac{2}{d}a_i \sum_{j\neq i} a_j)\langle \hat{\Sigma}_0 \rangle. \tag{25}$$

Now, we get the fidelity,

$$F_i = \frac{1}{d}[1 + (d-1)(a_i^2 + \frac{2}{d}\sum_{j\neq i,j=1}^{n} a_i a_j) + \frac{2}{d}\sum_{k=1}^{d-1}\sum_{j,l\neq i,j<l} a_j a_l \langle \hat{X}_0^k \rangle^2] \tag{26}$$

Because of the term with $\langle \hat{X}_0^k \rangle^2$, this cloning circuit is not universal, however after averaging over the uniform distribution, we get

$$F_i = \frac{1}{d}[1 + (d-1)(a_i^2 + \frac{2}{d}\sum_{k\neq i,k=1}^{n} a_i a_k + \frac{2}{d(d+1)}\sum_{k,l\neq i,k<l} a_k a_l)] \tag{27}$$

Now, these fidelities satisfy the optimal tradeoff relation for universal asymmetric $1 \to n$ quantum cloner, which generalizes the previous results on $1 \to 3$ [9] and $1 \to 4$ [10] cloners. It can be directly checked with the tradeoff relation founded in [13]

$$\sum_{i=1}^{n}[(d+1)F_i - 1] = d - 1 + \frac{1}{n-d+1}(\sum_{i=1}^{n}\sqrt{(d+1)F_i - 1})^2. \tag{28}$$

4 Conclusion

In this paper, we considered a generalization of the $1 \to 2$ cloning circuit proposed by Bužek *et al.*[5, 16] for asymmetric $1 \to n$ quantum cloning. This generalization is based on an insight of information flux in the original circuit. Here, the Z-type and X-type information of the input state flow through the circuit separately, therefore we can generalize the original circuit with 2 clones to circuit with n clones. The generalized circuit does not work universally on the input state since it contracts differently along different axis. However, after taking average over the uniform distribution of the input state, the fidelities of the output copies saturate the optimal universal fidelity tradeoff relation, which implies the conservation of quantum information. Besides, under some circumstances, the generalized circuit operates universally. The success of the scheme in this paper involves the consideration of transferring quantum operators in the circuit. Similar methods and considerations may also apply in the study of fundamental quantum theory and other tasks of quantum information processing.

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