MANY-TO-ONE REMOTE INFORMATION CONCENTRATION
FOR QUDITS AND MULTIPARTITE ENTANGLEMENT

XIN-WEN WANG[a]   SHI-QING TANG

Department of Physics and Electronic Information Science, Hengyang Normal University
Hengyang 421002, People's Republic of China

Key Laboratory of Low-Dimensional Quantum Structures and
Quantum Control of Ministry of Education, and Department of Physics
Hunan Normal University, Changsha 410081, People's Republic of China

LI-JUN XIE   DENG-YU ZHANG

Department of Physics and Electronic Information Science, Hengyang Normal University
Hengyang 421002, People's Republic of China

LE-MAN KUANG[b]

Key Laboratory of Low-Dimensional Quantum Structures and
Quantum Control of Ministry of Education, and Department of Physics
Hunan Normal University, Changsha 410081, People's Republic of China

Telecloning and its reverse process, referred to as remote information concentration (RIC), have attracted considerable interest because of their potential applications in quantum-information processing. We here present a general scheme for RIC in d-level systems (qudits), in which the quantum information initially distributed in many spatially separated qudits can be remotely and deterministically concentrated to a single qudit via an entangled channel without performing any global operations. We show that the entangled channel of RIC can be different types of entangled states, including mixed states as well as pure ones. More interestingly, these mixed states include a bound entangled state which has a similar form to the generalized Smolin state but has different characteristics from it. We also show that there exists a multipartite entangled state which can be used to implement both telecloning and RIC in the two-level system. Our many-to-one RIC protocol could be slightly modified to perform some types of many-to-many RIC tasks.

Keywords: Remote information concentration; multipartite entanglement; qudit

Communicated by: S Braunstein

1   Introduction

Quantum mechanics prohibits an unknown quantum state from being perfectly copied [1, 2]. However, an unknown quantum state can be copied approximately with a certain fidelity [3, 4, 5], referred to as (approximate) quantum cloning. Furthermore, when an unknown

---

[a]xwwang@mail.bnu.edu.cn

[b]lmkuang@hunnu.edu.cn

state comes from a restricted set of quantum states, it can be faithfully cloned with a certain probability [6, 7, 8], referred to as probabilistic quantum cloning. Since the seminal work of Bužek and Hillery [3], quantum cloning has attracted considerable attention (see Refs. [9, 10] for a review), due to the fact that it has wide potential applications in quantum-information science as well as could help us understand quantum mechanics itself more well (see, e.g., [11, 12, 13, 14, 15, 16, 17, 18]). Although the fidelities of clones relative to the original state are less than one, the quantum information of the input system is not degraded but only distributed into a larger quantum system. That is, the quantum cloning process can be regarded as the distribution of quantum information from an initial system to final ones. Thus, quantum cloning combined with remote quantum-information processing (QIP) may have potential applications in multiparty quantum communication and distributed quantum computation. This leads to the advent of the concept of quantum telecloning [19], which is the combination of quantum cloning and quantum teleportation [20], and functions as simultaneously distributing the copies of an unknown quantum state to spatially separated sites, i.e., realizing nonlocal quantum cloning, via a previously shared multipartite entangled state. Telecloning has been widely studied and many idiographic schemes have been presented, including $1 \rightarrow N$ telecloning of an arbitrary state or a phase-covariant state [19, 21, 25, 23, 24, 22, 27, 26, 28]. As the reverse process of telecloning, remote information concentration (RIC) was first introduced by Murao and Vedral [29]. They demonstrated that the quantum information originally distributed into three spatially separated qubits from a single qubit can be remotely concentrated back to a single qubit via a four-qubit unlockable bound entangled state (UBES) [30, 31, 32] (the four-qubit UBES was first found by Smolin and is referred to as *Smolin state* or *Smolin UBES*) without performing any global operations. Telecloning and RIC processes could be regarded as, respectively, remote information depositing and withdrawing processes, or remote information encoding and decoding processes, which is expected to find useful applications in network-based QIP [29]. A scheme for the reverse process of $1 \rightarrow 2$ telecloning via a four-qubit Greenberger-Horne-Zeilinger (GHZ) state [33] has also been proposed [34]. Not long before, schemes for the reverse process of $1 \rightarrow N$ telecloning in two-level systems have been presented [35, 36]. Recently, the reverse process of $1 \rightarrow 2$ telecloning in multilevel systems has been studied by part of our authors [37].

In this paper, we present a general scheme for implementing the reverse process of $1 \rightarrow N$ telecloning of an arbitrary quantum state in d-level systems, which are applicable to arbitrary $N \geq 2$ and $d \geq 2$ in principle. It will be shown that the RIC scheme relies on the establishment of special multiparticle entangled states that function as multiuser quantum-information channels. Particularly, the quantum channel of RIC can be different types of entangled states, including mixed states as well as pure ones; more interestingly, these pure states include different classes of genuine multipartite entangled states which are inequivalent under local operations and classical communication (LOCC), and these mixed states include an UBES which has a similar form to the generalized Smolin UBES [35, 38] but has different features from it. All these entangled states have d^2 common commuting stabilizers. In addition, we show that there exists a multiqubit ($d = 2$) entangled state which can be utilized to implement both telecloning and RIC. We also discuss the possibility of generalizing our many-to-one RIC protocol to perform some types of many-to-many RIC tasks.

2   Many-to-one RIC in multilevel systems

2.1   *A brief review of* $1 \to N$ *universal telecloning*

Before describing our RIC protocol in the next section, we here briefly summarize the forward process, $1 \to N$ universal telecloning [21]. The telecloning scheme aims at simultaneously distributing the optimal clones of an arbitrarily unknown qudit state

$$|\varphi\rangle_t = \sum_{j=0}^{d-1} x_j |j\rangle_t \tag{1}$$

($\sum_{j=0}^{d-1} |x_j|^2 = 1$) from a distributor (Alice) to N spatially separated receivers (Bob_1, Bob_2, $\cdots$, Bob_N) with only LOCC. The quantum channel (resource state) can be the $(2N)$-qudit entangled state

$$|\Phi\rangle_{t'12\cdots NA_1A_2\cdots A_{N-1}} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_{t'} |\phi_j\rangle_{12\cdots NA_1A_2\cdots A_{N-1}}, \tag{2}$$

where

$$|\phi_j\rangle_{12\cdots NA_1A_2\cdots A_{N-1}} = \sum_{n_j=1}^{N} \alpha_{n_j} |\{0,n_0\}, \{1,n_1\}, \cdots, \{j,n_j\}, \cdots, \{d-1,n_{d-1}\}\rangle_{12\cdots N} \otimes$$
$$|\{0,n_0\}, \{1,n_1\}, \cdots, \{j,n_j-1\}, \cdots, \{d-1,n_{d-1}\}\rangle_{A_1A_2\cdots A_{N-1}}, \tag{3}$$

$\alpha_{n_j} = \sqrt{\frac{n_j d! (N-1)!}{(N+d-1)!}}$ [39, 40], and $|\{0,n_0\}, \{1,n_1\}, \cdots, \{j,n_j\}, \cdots, \{d-1,n_{d-1}\}\rangle$ denotes a completely symmetric (normalized) state with n_j particles in the state $|j\rangle$ and $\sum_{j=0}^{d-1} n_j = N$. Here particle t' is on the sender Alice's hand, particle s is held by the sth recipient Bob_s ($s = 1, 2, \cdots, N$), and the ancillary particles $\{A_1, A_2, \cdots, A_{N-1}\}$ are arbitrarily distributed among these parties (or even be placed elsewhere). For example, when $N = 2$ the state in Eq. (3) can be explicitly expressed as

$$|\phi_j\rangle_{12A} = \frac{1}{\sqrt{2(d+1)}} \sum_{r=1}^{d-1} (|j\rangle_1 |\overline{j+r}\rangle_2 + |\overline{j+r}\rangle_1 |j\rangle_2) |\overline{j+r}\rangle_A + \sqrt{\frac{2}{d+1}} |j\rangle_1 |j\rangle_2 |j\rangle_A, \tag{4}$$

where $\overline{j+r} = j + r$ modulo d. Using the results

$$R^{m,n} |\overline{j+n}\rangle = \omega^{jm} |j\rangle,$$
$$R^{m,n} = \sum_{j=0}^{d-1} \omega^{jm} |j\rangle \langle \overline{j+n}|, \tag{5}$$

with $\omega = e^{2\pi i/d}$, it is easy to prove that

$$R_1^{m,n} \otimes R_2^{m,n} \cdots R_N^{m,n} \otimes R_{A_1}^{-m,n} \otimes R_{A_2}^{-m,n} \cdots R_{A_{N-1}}^{-m,n} |\phi_{\overline{j+n}}\rangle_{12\cdots NA_1A_2\cdots A_{N-1}}$$
$$= \omega^{jm} |\phi_j\rangle_{12\cdots NA_1A_2\cdots A_{N-1}}. \tag{6}$$

The state of the whole system of the $2N + 1$ particles $|\Psi\rangle_{tt'1\cdots NA_1\cdots A_{N-1}} = |\varphi\rangle_t \otimes |\Phi\rangle_{t'12\cdots NA_1A_2\cdots A_{N-1}}$ can be expressed as

$$|\Psi\rangle_{tt'1\cdots NA_1\cdots A_{N-1}} = \frac{1}{d} \sum_{m,n=0}^{d-1} |B^{m,n}\rangle_{tt'} \sum_{j=0}^{d-1} \omega^{-jm} x_j |\phi_{\overline{j+n}}\rangle_{12\cdots NA_1A_2\cdots A_{N-1}}, \tag{7}$$

where $\{|B^{m,n}\rangle : m, n = 0, 1, \cdots, d-1\}$ are the generalized Bell-basis states given by

$$|B^{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle,$$

$$|B^{m,n}\rangle = I \otimes U^{m,n}|B^{0,0}\rangle,$$

$$U^{m,n} = \sum_{k=0}^{d-1} \omega^{km}|\overline{k+n}\rangle\langle k|. \tag{8}$$

The telecloning can now be accomplished by the following simple procedure: (i) Alice performs a generalized Bell-basis measurement (GBM) on particles t and t', obtaining one of the d^2 outcomes $\{(m,n) : m, n = 0, 1, \cdots, d-1\}$, and informs all Bobs of the outcome; (ii) Depending on Alice's outcome (m, n), each Bob performs a local operation $R^{m,n}$ on his particle. According to Eq. (6), if every ancillary particle is also made a corresponding local operation $R^{-m,n}$, the particles $\{1, 2, \cdots, N\}$ and $\{A_1, A_2, \cdots, A_{N-1}\}$ end in the state

$$|\psi\rangle_{12\cdots N A_1 A_2 \cdots A_{N-1}} = \sum_{j=0}^{d-1} x_j |\phi_j\rangle_{12\cdots N A_1 A_2 \cdots A_{N-1}}. \tag{9}$$

According to Ref. [41], it can be easily verified that the collective output state of N clones ρ_N^{out} and the individual output state of one clone ρ_1^{out} are the same as that of Refs. [39, 42]. Thus, each Bob finally obtains a clone with the optimal fidelity $F = (2N + d - 1)/N(d+1)$. It is worth pointing out that the local operations on the ancillary particles are not necessary since the individual output state of a particle is not related to the local operations on the other particles.

2.2   A general scheme for RIC

In this section, we describe the reverse process of the aforementioned telecloning, i.e., RIC. After telecloning operations, the initial single-particle (t) quantum information is remotely distributed into $2N - 1$ spatially separated particles $(1, 2, \cdots, N, A_1, A_2, \cdots, A_{N-1})$, represented by the collective quantum state in Eq. (9). The ownership of particles $1, 2, \cdots, N$ is the same as the preceding section; i.e., they are still held by Bob_1, Bob_2, $\cdots$, Bob_N, respectively. Without loss of generality, we assume particles $A_1, A_2, \cdots, A_{N-1}$ are held by Charlie_1, Charlie_2, $\cdots$, Charlie_{N-1}, respectively. The RIC is aim to concentrate the information initially distributed in $(2N - 1)$-particle cloning state of Eq. (9) back to a remote particle (N', held by Diana) with only LOCC: $|\psi\rangle_{12\cdots N A_1 A_2 \cdots A_{N-1}} \rightarrow |\varphi\rangle_{N'}$.

In order to show clearly the RIC process and how to construct the entangled channel, we rewrite the cloning state in Eq. (9) as (see Appendix A)

$$|\psi\rangle_{12\cdots N A_1 A_2 \cdots A_{N-1}} = \frac{1}{\sqrt{d}} \sum_{m,n=0}^{d-1} \beta_n |\overline{B_{mn}}\rangle_{12\cdots N-1, A_1 A_2 \cdots A_{N-1}} U_N^{-m,n}|\varphi\rangle_N, \tag{10}$$

where $\sum_{n=0}^{d-1} \beta_n^2 = 1$ and

$$|\overline{B_{mn}}\rangle_{1\cdots N-1, A_1 \cdots A_{N-1}} = \sum_{j_1, \cdots, j_{2N-2}=0}^{d-1} \sqrt{P_{j_1 \cdots j_{2N-2}}} |B^{j_1, j_2}\rangle_{1A_1} \cdots |B^{j_{2N-3}, j_{2N-2}}\rangle_{N-1, A_{N-1}}$$

$$\tag{11}$$

with the constraints

$$\sum_{s=1}^{N-1} j_{2s-1} \bmod d = m, \qquad \sum_{s=1}^{N-1} j_{2s} \bmod d = n. \tag{12}$$

Note that particle N in Eq. (10) can be interchanged with any one of particles $1, 2, \cdots, N-1$ because of the permutability of them.

We first consider employing the following $2N$-particle entangled pure state as the quantum channel (resource state):

$$|\Psi^g\rangle_{A_1'1'A_2'2'\cdots A_N'N'} = \sum_{k_1,\cdots,k_{2N}=0}^{d-1} \sqrt{P_{k_1\cdots k_{2N}}} |B^{k_1,k_2}\rangle_{A_1'1'} \cdots |B^{k_{2N-1},k_{2N}}\rangle_{A_N'N'},$$

$$\sum_{s=1}^{N} k_{2s-1} \bmod d = u, \qquad \sum_{s=1}^{N} k_{2s} \bmod d = v, \tag{13}$$

where u and v are two arbitrarily given nonnegative integers that are less than d. We assume that particles $1', 2', \cdots, (N-1)', A_N'$ are held by Bob_1, Bob_2, $\cdots$, Bob_N, respectively; particles $A_1', A_2', \cdots, A_{N-1}'$ are held by Charlie_1, Charlie_2, $\cdots$, Charlie_{N-1}, respectively; particle N' belongs to Diana. A schematic picture of the RIC protocol is shown in Fig. 1. The procedure is as follows. (S1) All Bobs and Charlies perform GBMs on their own particles, respectively. (S2) Each of them tells Diana the measurement outcome by sending $2\log d$ bits of classical information. (S3) Diana performs a conditional local operation on particle N'.
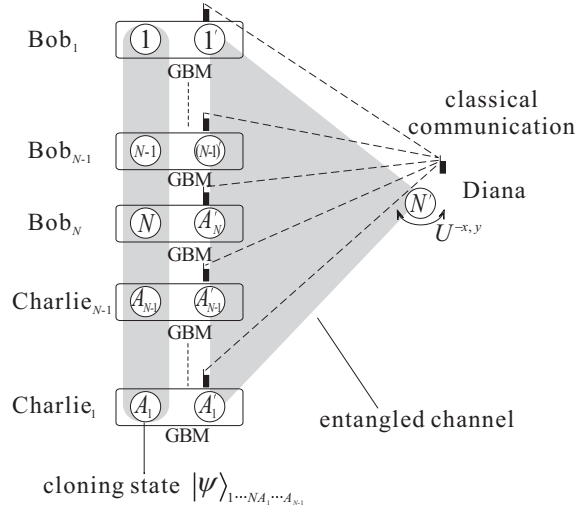

Fig. 1. Schematic picture showing the concentration of information from N Bobs (Bob_1, Bob_2, $\cdots$, Bob_N) and $N-1$ Charlies (Charlie_1, Charlie_2, $\cdots$, Charlie_{N-1}) to the remote receiver, Diana, using an entangled channel. The gray rectangle denotes the cloning state of $2N-1$ qudits, the gray triangle denotes the entangled channel, the blank rectangles denote the generalized Bell-state measurements, and the dashed lines mean the classical communications.

In (S1), the GBMs of all Bobs and Charlies are independent, and thus the sequence can be arbitrary. For clarity, we here assume that Bob_N performs the GBM after the others. Based on the identity

$$|B^{m,n}\rangle_{XY}|B^{m',n'}\rangle_{X'Y'} = \frac{1}{d}\sum_{m'',n''=0}^{d-1} \omega^{m''n''}|B^{\overline{m+m'',n'+n''}}\rangle_{XY'}|B^{\overline{m'-m'',n-n''}}\rangle_{X'Y} \qquad (14)$$

with $\overline{m'-m''} = m' - m'' + d$ modulo d, we can obtain the relationship of the measurement outcomes of these parties. Without loss of generality, we particularly assume the measurement outcomes of Bob_s and $Charlie_s$ ($s = 1, 2, \cdots, N-1$) are $(\overline{j_{2s-1}+l_{2s-1}}, \overline{k_{2s}+l_{2s}})$ and $(\overline{k_{2s-1}-l_{2s-1}}, \overline{j_{2s}-l_{2s}})$, respectively. Then Diana can obtain the results

$$\sum_{s=1}^{N-1}(j_{2s-1}+k_{2s-1}) \bmod d = \left(m + \sum_{s=1}^{N-1}k_{2s-1}\right) \bmod d = u',$$
$$\sum_{s=1}^{N-1}(j_{2s}+k_{2s}) \bmod d = \left(n + \sum_{s=1}^{N-1}k_{2s}\right) \bmod d = v'. \qquad (15)$$

As a consequence, N, A'_N, and N' are projected in the state

$$U_N^{-m,n}|\varphi\rangle_N|B^{k_{2N-1},k_{2N}}\rangle_{A'_N N'}$$
$$= \frac{1}{d}\sum_{x,y=0}^{d-1}\omega^{n(m-k_{2N-1})+(k_{2N}-n)x}|B^{\overline{x+k_{2N-1}-m},\overline{y+k_{2N}-n}}\rangle_{NA'_N}U_{N'}^{-x,y}|\varphi\rangle_{N'}. \qquad (16)$$

Next Bob_N performs a GBM on particles N and A'_N, which can be regarded as being equivalent to Bob_N and Diana together performing the teleportation protocol with a local error-correction operation on particle N'. Assume that the outcome is $(u'' = \overline{x+k_{2N-1}-m}, v'' = \overline{y+k_{2N}-n})$ and particle N' is projected in the state $U_{N'}^{-x,y}|\varphi\rangle_{N'}$. After receiving all the measurement outcomes sending from the other participants, Diana can deduce the result $(x = \overline{u''+u'-u}, y = \overline{v''+v'-v})$. Then, Diana performs the local operation $(U_{N'}^{-x,y})^+ = R_{N'}^{x,y}$ and obtains the state $|\varphi\rangle_{N'}$. As a consequence, the information initially distributed in $2N-1$ spatially separated particles is now remotely concentrated in a single particle.

Equation (13) contains a broad family of entangled pure states. We now consider some special cases. Assuming $k_2 \equiv k_4 \equiv \cdots \equiv k_{2N} \equiv 0$, $P_{k_1\cdots k_{2N}} \equiv 1/d^{N-1}$, and $u = 0$, the state in Eq. (13) reduces to (see Appendix B)

$$|\Psi^{s_1}\rangle_{A'_1 1' A'_2 2'\cdots A'_N N'} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle_{A'_1}|j\rangle_{1'}|j\rangle_{A'_2}|j\rangle_{2'}\cdots|j\rangle_{A'_N}|j\rangle_{N'}, \qquad (17)$$

i.e., a generalized GHZ state [43]. Assuming $\sum_{s=1}^{N-1}k_{2s-1} \bmod d = x$, $k_{2N-1} = d-x$, $\sum_{s=1}^{N-1}k_{2s} \bmod d = y$, $k_{2N} = d-y$, and $\sqrt{P_{k_1\cdots k_{2N}}} = \beta_y\sqrt{P_{k_1\cdots k_{2N-2}}}/\sqrt{d}$ [the definition of β_y is the same as Eq. (10)], the entangled channel in Eq. (13) reduces to

$$|\Psi^{s_2}\rangle_{A'_1 1' A'_2 2'\cdots A'_N N'} = \frac{1}{\sqrt{d}}\sum_{x,y=0}^{d-1}\beta_y|\overline{B_{xy}}\rangle_{A'_1 1' A'_2 2'\cdots A'_{N-1}(N-1)'}|B^{\overline{-x},\overline{-y}}\rangle_{A'_N N'}. \qquad (18)$$

For the case $d = 2$ (qubit), we proved that the state of Eq. (18) is the same as that of Eq. (2) (see Appendix C). This indicates that the multiqubit entangled state in Eq. (2) can be competent for implementing both telecloning and RIC, two inverse processes. In other words, the aforementioned telecloning and RIC for $d = 2$ can be achieved by using the same entangled channel. However, such a result is not applicable to $d > 2$ (qudit). This is an interesting difference between qudit-RIC and qubit-RIC. According to Ref. [19], the states of Eqs. (17) and (18) with $d = 2$ are not equivalent to each other, i.e., cannot be transformed into each other by LOCC. It can be verified that the states of Eqs. (17) and (18) with $d > 2$ are also LOCC inequivalent. This implies that Eq. (13) contains at least two inequivalent classes of genuine $2N$-partite entangled pure states. In other words, different classes of genuine $2N$-qudit entangled pure states can implement a same multiparty QIP task, $(2N - 1) \rightarrow 1$ RIC. Such a phenomenon is counterintuitive, since a given QIP task can be achieved by only typical structure of entangled states and different types of entangled states are usually competent for implementing different QIP tasks. It has been shown [44, 45] that quantum teleportation can be deterministically implemented by using both multiqubit W and GHZ states, two inequivalent genuine multiqubit entangled states [46]. However, teleportation is a two-party communication, and the W and GHZ states in fact play the same role as the bipartite entangled state, i.e., only the bipartite entanglement of them is exploited. In contrast, RIC is a multiparty communication (each party holds one particle of the entangled channel), and the states of Eqs. (17) and (18) play a role of multipartite entanglement.

We now show that the quantum channel of our RIC can also be a broad family of entangled mixed states. Let $P_{k_1 \cdots k_{2N}} = \delta_{k_1,c_1} \cdots \delta_{k_{2N},c_{2N}}$, where $c_1, \cdots, c_{2N}$ are arbitrarily chosen nonnegative integers that are less than d. Then the entangled channel in Eq. (13) reduces to a product state of N generalized Bell states,

$$|\Psi^{s_3}\rangle_{A'_1 1' A'_2 2' \cdots A'_N N'} = |B^{c_1,c_2}\rangle_{A'_1 1'} |B^{c_3,c_4}\rangle_{A'_2 2'} \cdots |B^{c_{2N-1},c_{2N}}\rangle_{A'_N N'}. \tag{19}$$

Because the constants $c_1, \cdots, c_{2N}$ are arbitrary, we deduce that the quantum channel of our RIC can also be the following form of entangled mixed states:

$$\rho_{A'_1 1' \cdots A'_N N'} = \sum_{k_1, \cdots, k_{2N}=0}^{d-1} C_{k_1 \cdots k_{2N}} |B^{k_1 k_2}\rangle_{A'_1 1'} \langle B^{k_1 k_2}| \otimes \cdots \otimes |B^{k_{2N-1} k_{2N}}\rangle_{A'_N N'} \langle B^{k_{2N-1} k_{2N}}|,$$
$$\tag{20}$$

where $\sum C_{k_1 \cdots k_{2N}} = 1$. This can be easily proved by resorting to a purified state of $\rho_{A'_1 1' \cdots A'_N N'}$,

$$|\Psi^\rho\rangle_{A'_1 1' \cdots A'_N N'} = \sum_{k_1, \cdots, k_{2N}=0}^{d-1} \sqrt{C_{k_1 \cdots k_{2N}}} |B^{k_1,k_2}\rangle_{A'_1 1'} \cdots |B^{k_{2N-1},k_{2N}}\rangle_{A'_N N'} |\mathcal{A}_{k_1 \cdots k_{2N}}\rangle_A, \tag{21}$$

where $\{|\mathcal{A}_{k_1 \cdots k_{2N}}\rangle_A\}$ are orthogonal normalized states of an ancillary system A. Particularly, by carrying out the same procedure as before [see (S1)-(S3)], the information of $|\psi\rangle_{1 \cdots N A_1 \cdots A_{N-1}}$ can also be concentrated in particle N' via the entangled channel $|\Psi^\rho\rangle_{A'_1 1' \cdots A'_N N'}$. In the whole process, the ancillary system A is not touched, and thus can be traced out at any time. This finishes the proof that the mixed state $\rho_{A'_1 1' \cdots A'_N N'}$ can be used to implement our RIC.

If we set $u = v = 0$ and $C_{k_1 \cdots k_{2N}} = 1/d^{2(N-1)}$, Eq. (20) reduces to

$$\rho'_{A'_1 1' \cdots A'_N N'} = \frac{1}{d^{2(N-1)}} \sum_{k_1, \cdots, k_{2N}=0}^{d-1} |B^{k_1 k_2}\rangle_{A'_1 1'} \langle B^{k_1 k_2}| \otimes \cdots \otimes |B^{k_{2N-1} k_{2N}}\rangle_{A'_N N'} \langle B^{k_{2N-1} k_{2N}}|.$$
(22)

For $d = 2$, $\rho'_{A'_1 1' \cdots A'_N N'}$ is exactly the generalized Smolin state [35, 38], a $2N$-qubit UBES. The generalized Smolin UBES is fully symmetric; i.e., it is unchanged under permutation of any two qubits. This leads to the generalized Smolin UBES being separable with respect to any $2 : 2$ partition of $\{A'_1, A'_2, \cdots, A'_N, 1', 2', \cdots, N'\}$. For $d > 2$, $\rho'_{A'_1 1' \cdots A'_N N'}$ also describes an UBES (see Appendix D); and when $N = 2$ it recovers the results of Ref. [37]. However, $\rho'_{A'_1 1' \cdots A'_N N'}$ with $d > 2$ is an asymmetric but not symmetric UBES, because $\{A'_1, A'_2, \cdots, A'_N, 1', 2', \cdots, N'\}$ are not completely permutable, i.e., particularly, any one of $G_1 = \{A'_1, A'_2, \cdots, A'_N\}$ and any one of $G_2 = \{1', 2', \cdots, N'\}$ are not permutable, as shown in Appendix D. The asymmetry results in the fact that $\rho'_{A'_1 1' \cdots A'_N N'}$ cannot be superactivated for $d > 2$, which presents a striking contrast to the generalized Smolin UBES being superactivable [47]. These results indicate that there exists an analog to the generalized Smolin UBES in multilevel systems; however, it has some different characteristics. Note that the asymmetric $2N$-qudit UBES didn't appear in previous literature, and thus is a "new" asymmetric UBES.

As shown above, different types of entangled states, including both pure and mixed states, can be exploited as the quantum channel of many-to-one RIC. The pure states can be multiple-Bell states and LOCC inequivalent genuine $2N$-partite entangled states. The mixed states could even be bound entangled states. However, it can be verified that all these states have several common properties as follows. (a) All of them are stabilized by the Abelian group $S = \{S^{mn} = \otimes_{s=1}^{N} U_{A'_s}^{-m,n} \otimes U_{s'}^{m,n} : m, n = 0, 1, \cdots, d-1\}$; that is, for any m and n, $\text{tr}(S^{mn}|\Psi^g\rangle_{A'_1 1' \cdots A'_N N'}\langle \Psi^g|) = \text{tr}(S^{mn}\rho_{A'_1 1' \cdots A'_N N'}) = 1$. (b) They can be expanded by the generalized Bell states with the same constraints [see the second row of Eq. (13)]. (c) The amount of entanglement across the $\{A'_1, A'_2, \cdots, A'_N, 1', 2', \cdots, (N-1)'\} : \{N'\}$ cut is $\log_2 d$ ebit, which ensures that the success probability of remotely concentrating one-qudit information is one.

The key points for the physical or experimental realization of the RIC task above are as follows: (i) preparation of the entangled channel, i.e., the generalized Bell states or GHZ states, or the UBES of Eq. (22); (ii) realization of telecloning (or cloning) of an arbitrary quantum state; (iii) implementation of the GBM. All these building blocks are achievable in quantum optics as discussed in Ref. [37].

3 Discussion and conclusion

A more general RIC protocol should be a many-to-many protocol. However, it will be much more complicated and cannot be obtained by directly generalizing the many-to-one protocol shown above. As a matter of fact, there are two types of many-to-many RIC protocols. One involves more than one receiver. The other aims at concentrating multi-qudit information to a remote site. For the former case, we here consider the reverse process of the "many-to-many" quantum information distribution presented in Ref. [22]. In the "many-to-many" information

distribution protocol of Ref. [22], the information of an entangled state

$$|\varphi'\rangle = \sum_{j=0}^{d-1} x_j |j\rangle_{t_1} |j\rangle_{t_2} \cdots |j\rangle_{t_L} \tag{23}$$

shared by L spatially separated distributors is transmitted by telecloning procedure to M receivers ($M > L$) situated at different locations. Naturally, the reverse process of it is to remotely concentrate the information distributed in M particles back to L spatially separated particles. Let $M = 2N - 1$, this task can be implemented by slightly modifying the aforementioned many-to-one RIC protocol, with the high-dimensional Bell state $|B^{k_{2N-1},k_{2N}}\rangle_{A'_N N'}$ in Eqs. (13), (16), and (20) being replaced by the high-dimensional GHZ state

$$|G^{k_{2N-1},k_{2N}}\rangle = I \otimes U^{k_{2N-1},k_{2N}} \otimes \underbrace{U^{0,k_{2N}} \otimes \cdots \otimes U^{0,k_{2N}}}_{L-1} |G^{0,0}\rangle_{A'_N N'_1 N'_2 \cdots N'_L},$$

$$|G^{0,0}\rangle_{A'_N N'_1 N'_2 \cdots N'_L} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_{A'_N} |j\rangle_{N'_1} |j\rangle_{N'_2} \cdots |j\rangle_{N'_L},$$

$$U^{k_{2N-1},k_{2N}} = \sum_{j=0}^{d-1} \omega^{jk_{2N-1}} |\overline{j+k_{2N}}\rangle\langle j|, \tag{24}$$

and $|\varphi\rangle_{N'}$ and $U_{N'}^{-x,y}$ in Eq. (16) replaced by $|\varphi'\rangle_{N'_1 N'_2 \cdots N'_L}$ and $U_{N'_1}^{-x,y} \otimes U_{N'_2}^{0,y} \otimes \cdots \otimes U_{N'_L}^{0,y}$, respectively.

As a direct extension of the aforementioned many-to-one RIC, the second type of many-to-many RIC, which aims at concentrating multi-qudit information to a remote site, should be the reverse process of $L \to N$ ($N > L$) optimal universal telecloning [48, 49]. However, it is not clear whether the output state of $L \to N$ optimal universal cloning has the form similar to that in Eq. (10). Thus we cannot construct the entangled channel by the idea similar to that used in our many-to-one RIC protocol and choose suitable operations. We here discuss alternatively a simple scenario, i.e., the reverse process of the following many-to-many quantum information distribution. Suppose that Alice had distributed the information of L identical but unknown d-level quantum states $|\varphi\rangle^{\otimes L}$ into a $(2N - L)$-qudit state

$$|\psi\rangle_{\{2N-L\}} = \frac{1}{\sqrt{d}} \sum_{m,n=0}^{d-1} \beta_n |\overline{B_{mn}}\rangle_{\{2N-2L\}} \left(U^{-m,n}\right)^{\otimes L} |\varphi\rangle^{\otimes L} \tag{25}$$

shared by $(2N - L)$ spatially separated clients. Note that this state is not necessarily to be the output state of the so-called $L \to N$ ($N > L$) optimal universal telecloning [48, 49]. The reverse process is to remotely concentrate the distributed information in $2N - L$ spatially separated particles back to L particles held by a receiver. It is easy to verify that such a RIC task can be accomplished by the same procedure as the aforementioned many-to-one RIC via the quantum channel $|B^{0,0}\rangle^{\otimes N}$ shared among the $(2N - L)$ senders (each one holds one particle of a Bell state) and a receiver (holds L particles of L Bell states).

In conclusion, we have studied the many-to-one RIC, i.e., the reverse process of $1 \to N$ universal telecloning, in d-level systems, which are applied to arbitrary $N \geq 2$ and $d \geq 2$ in principle. We have shown that the quantum channel of RIC can be different types of

entangled states, including mixed states as well as pure ones, in contrast to telecloning which requires a certain type of entangled channel. Such a difference may be due to the fact that RIC can be considered to be a disentangling operation, whereas telecloning can be considered to be an entangling operation. Although these entangled states are LOCC inequivalent, they have a common feature, i.e., have d^2 common commuting stabilizers. We have also revealed concomitantly some interesting entanglement phenomena as follows. (a) Similar to qubit-RIC, qudit-RIC can also be implemented by an UBES. Though such a multilevel UBES has a similar form to the generalized Smolin UBES, it has some different features; particularly, the former one has asymmetry and the latter one has symmetry. (b) Telecloning and RIC for qubits can be achieved by using the same entangled channel, but there is no such feature for qudits. Our many-to-one RIC protocol can be slightly modified to implement some many-to-many RIC tasks. These protocols are experimentally achievable in the field of quantum optics.

Subsequent to submitting this manuscript, Zhang *et al.* independently proposed a many-to-one RIC protocol with the generalized Bell states acting as the entangled channel [48]. This paper has shown that many-to-one RIC can be realized by different channels including both pure and mixed entangled states (even bound entangled states). As a matter of fact, the entangled channel used in Ref. [48] is the same as that in Eq. (19) with $c_1 = c_2 = \cdots = c_{2N} = 0$, i.e., a special case of the general channel in Eq. (13), of the present paper.

References

1. W. K. Wootters and W. H. Zurek (1982), *A single quantum cannot be cloned*, Nature, Vol. 299, pp. 802-803.
2. D. Dieks (1982), *Communication by EPR devices*, Phys. Lett. A, Vol. 92, pp. 271-272.
3. V. Bužek and M. Hillery (1996), *Quantum copying, Beyond the no-cloning theorem*, Phys. Rev. A, Vol. 54, pp. 1844-1852.
4. J. Fiurášek, R. Filip, and N. J. Cerf (2005), *Highly asymmetric quantum cloning in arbitrary dimension*, Quantum Inf. Comp., Vol. 5, pp. 583-592.
5. H. Fan, B. Liu, and K. Shi (2007), *Quantum cloning of identical mixed qubits*, Quantum Inf. Comp., Vol. 7, pp. 551-558.
6. L. M. Duan and G. C. Guo (1998), *Probabilistic Cloning and Identification of Linearly Independent Quantum States*, Phys. Rev. Lett., Vol. 80, pp. 4999-5002.
7. A. K. Pati (1999), *Quantum Superposition of Multiple Clones and the Novel Cloning Machine*, Phys. Rev. Lett. Vol. 83, pp. 2849-2852.
8. O. Jiménez, J. Bergou, and A. Delgado (2010), *Probabilistic cloning of three symmetric states*, Phys. Rev. A, Vol. 82, pp. 062307.
9. V. Scarani, S. Iblisdir, N. Gisin, and A. Acín (2005), *Quantum cloning*, Rev. Mod. Phys., Vol. 77, pp. 1225-1256.

10. H. Fan, Y. N. Wang, L. Jing, et al. (2013), Quantum Cloning Machines and the Applications, arXiv: quant-ph/1301.2956v3.
11. E. F. Galvao and L. Hardy (2000), *Cloning and quantum computation*, Phys. Rev. A, Vol. 62, pp. 022301.
12. L. P. Lamoureux, H. Bechmann-Pasquinucci, N. J. Cerf, N. Gisin, and C. Macchiavello (2006), *Reduced randomness in quantum cryptography with sequences of qubits encoded in the same basis*, Phys. Rev. A, Vol. 73, pp. 032304.
13. A. Ferenczi and N. Lütkenhaus (2012), *Symmetries in Quantum Key Distribution and the Connection between Optimal Attacks and Optimal Cloning*, Phys. Rev. A, Vol. 85, pp. 052310.
14. M. Ricci, F. Sciarrino, N. J. Cerf, R. Filip, J. Fiurášek, and F. De Martini (2005), *Separating the Classical and Quantum Information via Quantum Cloning*, Phys. Rev. Lett., Vol. 95, pp. 090504.
15. D. Bruß, J. Calsamiglia, and N. Lütkenhaus (2001), *Quantum cloning and distributed measurements*, Phys. Rev. A, Vol. 63, pp. 042308.
16. P. Sekatski, B. Sanguinetti, E. Pomarico, N. Gisin, and C. Simon (2010), *Cloning entangled photons to scales one can see*, Phys. Rev. A, Vol. 82. pp. 053814.
17. N. Spagnolo, F. Sciarrino, and F. De Martini (2010), *Resilience to decoherence of the macroscopic quantum superpositions generated by universally covariant optimal quantum cloning*, Phys. Rev. A, Vol. 82, pp. 032325.
18. E. Nagali, T. De Angelis, F. Sciarrino, and F. De Martini (2007), *Experimental realization of macroscopic coherence by phase-covariant cloning of a single photon*, Phys. Rev. A, Vol. 76, pp. 042126.
19. M. Murao, D. Jonathan, M. B. Plenio, and V. Vedral (1999), *Quantum telecloning and multiparticle entanglement*, Phys. Rev. A, Vol. 59, pp. 156-161.
20. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters (1993), *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett., Vol. 70, pp. 1895-1899.
21. M. Murao, M. B. Plenio, and V. Vedral (2000), *Quantum-information distribution via entanglement*, Phys. Rev. A, Vol. 61, pp. 032311.
22. I. Ghiu (2003), *Asymmetric quantum telecloning of d-level systems and broadcasting of entanglement to different locations using the "many-to-many" communication protocol*, Phys. Rev. A, Vol. 67, pp. 012323.
23. X. W. Wang and G. J. Yang (2009), *Hybrid economical telecloning of equatorial qubits and generation of multipartite entanglement*, Phys. Rev. A, Vol. 79, pp. 062315.
24. I. Ghiu and A. Karlsson (2005),*Broadcasting of entanglement at a distance using linear optics and telecloning of entanglement* , Phys. Rev. A, Vol. 72, pp. 032331.
25. L. Chen and Y. X. Chen (2007), *Asymmetric quantum telecloning of multiqubit states*, Quantum Inf. Comp., Vol. 7, pp. 716-729.
26. X. W. Wang and G. J. Yang (2009), *Probabilistic ancilla-free phase-covariant telecloning of qudits with the optimal fidelity*, Phys. Rev. A, Vol. 79, pp. 064306.
27. W. H. Zhang, J. Wang, L. Ye, and J. Dai (2007), *Suboptimal asymmetric economical phase-covariant quantum cloning and telecloning in d-dimension*, Phys. Lett. A, Vol. 369, pp. 112-119.
28. X. W. Wang, Y. H. Su, and G. J. Yang (2010), *One-to-Many Economical Phase-Covariant Cloning and Telecloning of Qudits*, Chin. Phys. Lett., Vol. 27, pp. 100303.
29. M. Murao and V. Vedral (2001), *Remote Information Concentration Using a Bound Entangled State*, Phys. Rev. Lett., Vol. 86, pp. 352-355.
30. J. A. Smolin (2001), *Four-party unlockable bound entangled state*, Phys. Rev. A, Vol. 63, pp. 032306.
31. M. Horodecki, P. Horodecki, and R. Horodecki (1998), *Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?*, Phys. Rev. Lett., Vol. 80, pp. 5239-5242.
32. P. Horodecki and R. Horodecki (2001), *Distillation and bound entanglement*, Quantum Inf. Comp., Vol. 1, pp. 45-75.
33. D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger (1990), *Bells theorem without*

inequalities, Am. J. Phys., Vol. 58, pp. 1131-1143.

34. Y. F. Yu, J. Feng, and M. S. Zhan (2003), *Remote information concentration by a Greenberger-Horne-Zeilinger state and by a bound entangled state*, Phys. Rev. A, Vol. 68, pp. 024303.

35. R. Augusiak and P. Horodecki (2006), *Generalised Smolin states and their properties*, Phys. Rev. A, Vol. 73, pp. 012318.

36. L. Y. Hsu (2007), *Remote one-qubit information concentration and decoding of operator quantum error-correction codes*, Phys. Rev. A, Vol. 76, pp. 032311.

37. X. W. Wang, D. Y. Zhang, G. J. Yang, S. Q. Tang, and L. J. Xie (2011), *Remote information concentration and multipartite entanglement in multilevel systems*, Phys. Rev. A, Vol. 84, pp. 042310.

38. S. Bandyopadhyay, I. Chattopadhyay, V. Roychowdhury, and D. Sarkar (2005), *Bell-correlated activable bound entanglement in multiqubit systems*, Phys. Rev. A, Vol. 71, pp. 062317.

39. S. Albeverio, and S. M. Fei (2000), *A remark on the optimal cloning of an N-level quantum system*, Eur. Phys. J. B, Vol. 14, pp. 669-672.

40. H. Fan, K. Matsumoto, and M. Wadati (2001), *Quantum cloning machines of a d-level system*, Phys. Rev. A, Vol. 64, pp. 064301.

41. Y. N. Wang, H. D. Shi, Z. X. Xiong, L. Jing, X. J. Ren, L. Z. Mu, and H. Fan (2011), *Unified universal quantum cloning machine and fidelities*, Phys. Rev. A, Vol. 84, pp. 034302.

42. R. F. Werner (1998), *Optimal cloning of pure states*, Phy. Rev. A, Vol. 58, pp. 1827-1832.

43. A. Cabello (2001), *Multiparty multilevel Greenberger-Horne-Zeilinger states*, Phys. Rev. A, Vol. 63, pp. 022104.

44. P. Agrawal and A. Pati (2006), *Perfect teleportation and superdense coding with W states*, Phys. Rev. A, Vol. 74, pp. 062320.

45. X. W. Wang, G. J. Yang, Y. H. Su, and M. Xie (2009), *Simple schemes for quantum information processing with W-type entanglement*, Quantum Inf. Process., Vol. 8, pp. 431-442.

46. W. Dür, G. Vidal, and J. I. Cirac (2000), *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A, Vol. 62, pp. 062314.

47. L. Y. Hsu (2007), *Mixed entangled states with two or more common stabilizers*, Phys. Rev. A, Vol. 76, pp. 022322.

48. Y. L. Zhang, Y. N. Wang, X. R. Xiao, L. Jing, L. Z. Mu, V. E. Korepin, and H. Fan (2013), Quantum network teleportation for quantum information distribution and concentration, Phys. Rev. A, Vol. 87, pp. 022302.

49. W. Dür and J. I. Cirac (2000), *Multiparty teleportation*, J. Mod. Opt., Vol. 47, pp. 247-255.

50. G. Wang and M. Ying (2007), *Multipartite unlockable bound entanglement in the stabilizer formalism*, Phys. Rev. A, Vol. 75, pp. 052332.

Appendix A

In this appendix, we demonstrate that the cloning state of Eq. (9) can be rewritten as form of Eq. (10). To satisfy Eq. (6), $|\phi_j\rangle$ can be rewritten as

$$|\phi_j\rangle_{12\cdots N A_1 A_2 \cdots A_{N-1}} = \sum_{n=0}^{d-1} \beta_n |\lambda_{j_n}\rangle_{12\cdots N-1, A_1 A_2 \cdots A_{N-1}} |\overline{j+n}\rangle_N, \tag{A.1}$$

where

$$R_1^{k,l} \otimes R_2^{k,l} \cdots R_{N-1}^{k,l} \otimes R_{A_1}^{-k,l} \otimes R_{A_2}^{-k,l} \cdots R_{A_{N-1}}^{-k,l} |\lambda_{j_n}\rangle_{12\cdots N-1, A_1 A_2 \cdots A_{N-1}}$$
$$= \omega^{-nk} |\lambda_{\overline{(j-l)_n}}\rangle_{12\cdots N-1, A_1 A_2 \cdots A_{N-1}}. \tag{A.2}$$

Now let

$$|\overline{B_{mn}}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{jm} |\lambda_{j_n}\rangle. \tag{A.3}$$

It can be verified that

$$R_1^{k,l} \otimes R_2^{k,l} \cdots R_{N-1}^{k,l} \otimes R_{A_1}^{-k,l} \otimes R_{A_2}^{-k,l} \cdots R_{A_{N-1}}^{-k,l} |\overline{B_{mn}}\rangle_{12\cdots N-1,A_1 A_2 \cdots A_{N-1}}$$

$$= \omega^{lm-nk} |\overline{B_{mn}}\rangle_{12\cdots N-1,A_1 A_2 \cdots A_{N-1}}. \tag{A.4}$$

We notice that

$$R^{k,l} \otimes R^{-k,l} |B^{x,y}\rangle = \omega^{lx-yk} |B^{x,y}\rangle. \tag{A.5}$$

Therefore, $|\overline{B_{mn}}\rangle_{12\cdots N-1,A_1 A_2 \cdots A_{N-1}}$ can also be expressed as the form of Eq. (11). From Eq. (A.3), we obtain

$$|\lambda_{j_n}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega^{-jm} |\overline{B_{mn}}\rangle. \tag{A.6}$$

Then Eq. (10) can be obtained by substituting Eqs. (A.1) and (A.6) into Eq. (9).

Appendix B

If $k_2 \equiv k_4 \equiv \cdots \equiv k_{2N} \equiv 0$, $P_{k_1 \cdots k_{2N}} \equiv 1/d^{N-1}$, and $u = 0$, the state in Eq. (13) can be expressed as

$$
\begin{aligned}
&|\Psi^{s_1}\rangle_{A_1' 1' A_2' 2' \cdots A_N' N'} \\
=\ & \frac{1}{\sqrt{d^{N-1}}} \sum_{k_3, k_5, \cdots, k_{2N-1}=0}^{d-1} |B^{-k_3 - k_5 \cdots - k_{2N-1}, 0}\rangle_{A_1', 1'} \\
& \otimes |B^{k_3, 0}\rangle_{A_2', 2'} \otimes \cdots \otimes |B^{k_{2N-1}, 0}\rangle_{A_N', N'} \\
=\ & \frac{1}{\sqrt{d^{2N-1}}} \sum_{k_3, k_5, \cdots, k_{2N-1}=0}^{d-1} \sum_{j_1, j_3, \cdots, j_{2N-1}=0}^{d-1} \omega^{j_1(-k_3 - k_5 \cdots - k_{2N-1})} |j_1\rangle_{A_1'} |j_1\rangle_{1'} \\
& \otimes \omega^{j_3 k_3} |j_3\rangle_{A_2'} |j_3\rangle_{2'} \otimes \cdots \otimes \omega^{j_{2N-1} k_{2N-1}} |j_{2N-1}\rangle_{A_N'} |j_{2N-1}\rangle_{N'} \\
=\ & \frac{1}{\sqrt{d^{2N-1}}} \sum_{j_1, j_3, \cdots, j_{2N-1}=0}^{d-1} |j_1\rangle_{A_1'} |j_1\rangle_{1'} |j_3\rangle_{A_2'} |j_3\rangle_{2'} \cdots |j_{2N-1}\rangle_{A_N'} |j_{2N-1}\rangle_{N'} \\
& \times \sum_{k_3=0}^{d-1} \omega^{(j_1 - j_3)k_3} \sum_{k_5=0}^{d-1} \omega^{(j_1 - j_5)k_5} \cdots \sum_{k_{2N-1}=0}^{d-1} \omega^{(j_1 - j_{2N-1})k_{2N-1}} \\
=\ & \frac{1}{\sqrt{d}} \sum_{j_1=0}^{d-1} |j_1\rangle_{A_1'} |j_1\rangle_{1'} |j_1\rangle_{A_2'} |j_1\rangle_{2'} \cdots |j_1\rangle_{A_N'} |j_1\rangle_{N'}. \tag{B.1}
\end{aligned}
$$

Here we have used the identity $\sum_{k=0}^{d-1} \omega^{jk} = d\delta_{j,0}$, where $\delta_{j=0,0} = 1$ and $\delta_{j\neq 0,0} = 0$. Obviously, the state of Eq. (B.1) is the same as that of Eq. (17), i.e., a normal generalized GHZ state.

Appendix C

This appendix shows the equivalence of the state in Eq. (2) to the state in Eq. (18) for $d = 2$. By substituting Eqs. (A.1) and (A.6) into Eq. (2), the telecloning state $|\Phi\rangle_{t'12\cdots N A_1 A_2 \cdots A_{N-1}}$

reads

$$
\begin{aligned}
|\Phi\rangle_{t'12\cdots NA_1A_2\cdots A_{N-1}} &= \frac{1}{d}\sum_{j=0}^{d-1}|j\rangle_{t'}\sum_{y=0}^{d-1}\beta_y\sum_{x=0}^{d-1}\omega^{-jx}|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}|j+y\rangle_N \\
&= \frac{1}{d\sqrt{d}}\sum_{x,y,j=0}^{d-1}\beta_y\omega^{-jx}|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}\sum_{l=0}^{d-1}\omega^{-lj}|B^{l,y}\rangle_{t'N} \\
&= \frac{1}{d\sqrt{d}}\sum_{x,y,j,l=0}^{d-1}\beta_y\omega^{-j(x+l)}|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}\sum_{l=0}^{d-1}|B^{l,y}\rangle_{t'N} \\
&= \frac{1}{\sqrt{d}}\sum_{x,y=0}^{d-1}\beta_y|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}|B^{\overline{-x},y}\rangle_{t'N}. \quad (C.1)
\end{aligned}
$$

Here we have used the identity

$$
|j\rangle_{t'}|k\rangle_N = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\omega^{-jl}|B^{l,\overline{k-j}}\rangle_{t'N} \quad (0\le j,k\le d-1), \quad (C.2)
$$

which can be obtained from Eq. (8). For $d=2$, Eq. (C.1) reduces to

$$
\begin{aligned}
|\Phi\rangle_{t'12\cdots NA_1A_2\cdots A_{N-1}} &= \frac{1}{\sqrt{d}}\sum_{x,y=0}^{1}\beta_y|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}|B^{\overline{-x},y}\rangle_{t'N} \\
&= \frac{1}{\sqrt{d}}\sum_{x,y=0}^{1}\beta_y|\overline{B_{xy}}\rangle_{12\cdots N-1,A_1A_2\cdots A_{N-1}}|B^{\overline{-x},\overline{-y}}\rangle_{t'N}, \quad (C.3)
\end{aligned}
$$

which is obviously the same as the state of Eq. (18) with $d=2$.

Appendix D

We here prove that the state $\rho'_{A_1'1'A_2'2'\cdots A_N'N'}$ in Eq. (22) is an asymmetric UBES for any $d>2$, by using some results of Ref. [50]. We define an Abelian subgroup of the generalized Pauli group [50],

$$
S = \{S^{mn} = \bigotimes_{s=1}^{N}U_{A_s'}^{-m,n}\otimes U_{s'}^{m,n} : m,n = 0,1,\cdots,d-1\}, \quad (D.1)
$$

which is composed of d^2 commuting operators. A state $|\psi\rangle$ is said to be stabilized by S, if $S^{mn}|\psi\rangle = 1$, $\forall\, m,n = 0,1,\cdots,d-1$. All the states stabilized by S constitute a subspace, denoted by H_S, of the Hilbert space of n qudits. Define $T_s = \{A_s',s'\}$ $(s=1,2,\cdots,N)$ and $S_{T_s}^{mn} = U_{A_s'}^{-m,n}\otimes U_{s'}^{m,n}$. It can be verified that any two operators $S_{T_s}^{mn}$ and $S_{T_s}^{m'n'}$ are commutable, $\forall\, s=1,2,\cdots,N$. Then the two operators $S^{mn},S^{m'n'}\in S$ are said to commute locally with respect to the partition $\{T_1,T_2,\cdots,T_N\}$ of $\{A_1',A_2',\cdots,A_N',1',2',\cdots,N'\}$, and S is said to be separable with respect to this partition [50].

It can be verified that

$$
S_{T_s}^{mn}|B^{x_s,y_s}\rangle_{A_s's'} = \omega^{y_sm-x_sn}|B^{x_s,y_s}\rangle_{A_s's'}, \quad (D.2)
$$

$\forall\ s\ =\ 1, 2, \cdots, N$; i.e., $\{|B^{x_s,y_s}\rangle_{A'_s s'} : x_s, y_s = 0, 1, \cdots, d-1\}$ are the simultaneous eigenstates of $S^{mn}_{T_s}$ corresponding to the eigenvalues $\{\omega^{y_s m - x_s n} : x_s, y_s = 0, 1, \cdots, d-1\}$ for each $m, n = 0, 1, \cdots, d-1$. Then it is obvious that the $2N$-qudit states $\{\otimes^N_{s=1}|B^{x_s,y_s}\rangle_{A'_s,s'} : x_s, y_s = 0, 1, \cdots, d-1\}$ are the simultaneous eigenstates of S^{mn} with the eigenvalues $\{\omega^{\sum^N_{s=1} y_s m - \sum^N_{s=1} x_s n} : x_s, y_s = 0, 1, \cdots, d-1\}$ for each $m, n = 0, 1, \cdots, d-1$. In particular, each term of the state $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ in Eq. (22) is the simultaneous eigenstate of S^{mn} with eigenvalue 1 for each $m, n = 0, 1, \cdots, d-1$. These eigenstates also form an orthonormal basis of the stabilized space H_S. According to *Lemma 1* of Ref. [50], the state $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ in Eq. (22) is the maximally mixed state over H_S.

As have been shown that S is separable with respect to the partition $\{T_1, T_2, \cdots, T_N\}$. It can also be verified that for any $X \neq Y \in \{A'_1, A'_2, \cdots, A'_N, 1', 2', \cdots, N'\}$, there exists at least one partition $\{g_1, g_2, \cdots, g_f\}$ with $X \in g_1, Y \in g_2$ such that S is separable with respect to this partition. These results satisfy the condition 1 in *Theorem 1* of Ref. [50], which indicates that $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ is a bound entangled state. The unlockability or activability of $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ is obvious. For example, it can be unlocked as follows: let qudits A'_s and s' ($s = 2, \cdots, N$) join together and perform a GBM on them; then depending on the measurement outcome qudits A'_1 and $1'$ is projected in a generalized Bell state, i.e., pure entanglement is distilled out between qudits A'_1 and $1'$. In fact $S_{T_s} = \{S^{mn}_{T_s} : m, n = 0, 1, \cdots, d-1\}$ is obviously inseparable, $\forall\ s = 1, 2, \cdots, N$, which satisfies the condition 2 in *Theorem 1* of Ref. [50]. Thus $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ is an UBES.

We now classify the $2N$ qudits of the state $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ into two groups $G_1 = \{A'_1, A'_2, \cdots, A'_N\}$ and $G_2 = \{1', 2', \cdots, N'\}$. It is obvious that S acts symmetrically on the N qudits of each group, which indicates that the state remains invariant when exchanging any two qudits inside the same group. However, when we exchange two qudits that belong to two different groups, the state will change. Therefore, the UBES $\rho'_{A'_1 1' A'_2 2' \cdots A'_N N'}$ is asymmetric when $d > 2$.