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SPIN SQUEEZING OF ONE-AXIS TWISTING MODEL IN THE PRESENCE OF PHASE DEPHASING

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We present a detailed analysis of spin squeezing of the one-axis twisting model with a many-body phase dephasing, which is induced by external field fluctuation in a two-mode Bose-Einstein condensates. Even in the presence of the dephasing, our analytical results show that the optimal initial state corresponds to a coherent spin state $|\theta_0, \phi_0\rangle$ with the polar angle $\theta_0 = \pi/2$. If the dephasing rate $\gamma \ll S^{-1/3}$, where S is total atomic spin, we find that the smallest value of squeezing parameter (i.e., the strongest squeezing) obeys the same scaling with the ideal one-axis twisting case, namely $\xi^2 \propto S^{-2/3}$. While for a moderate dephasing, the achievable squeezing obeys the power rule $S^{-2/5}$, which is slightly worse than the ideal case. When the dephasing rate $\gamma > S^{1/2}$, we show that the squeezing is weak and neglectable.

Keywords: Quantum noise, Bose-Einstein condensates, Phase dephasing, Spin squeezed states
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1 Introduction

Spin squeezing of an ensemble of spin-1/2 particles have attracted considerable attention for decades because they are not only important in view point of fundamental physics but also have a lot of applications. For instance, a quantum interferometer felt with spin squeezed states (SSS) or multi-particle entangled states (MES) in the input ports can improve phase sensitivity beyond standard quantum limit (SQL) [1, 2, 3]. Dynamical generation of the SSS [4, 5] and the MES [6, 7, 8] has been proposed by using the 'one-axis twisting' (OAT) interaction, which leads to quantum correlation among individual spin in a collective spin system. In addition, the OAT scheme of spin squeezing can be transformed into the so-called two-axis twisting with a sequence of $\pi/2$ pulses [9].

Under governed by the OAT Hamiltonian, the spin system can evolve from a coherent spin state (CSS) into a spin squeezed state, which shows the reduced variance V_- below standard quantum limit (SQL)—$S/2$, where S is total atomic spin. The degree of spin squeezing ξ^2 ($= 2V_-/S$) can reach the power rule $S^{-2/3}$, which is the ideal OAT result [4]. Experimental realizations of the OAT model has been proposed [10, 11, 12, 13, 14, 15, 16, 17] and demonstrated [18, 19, 20, 21, 22, 23, 24, 25, 26, 27] in a two-mode Bose-Einstein condensate (BEC).

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Due to experimental imperfections in coupling pulses, atom losses, technique and quantum noises, etc., the achievable squeezing is worse than the theoretical prediction [28]. Recently, we investigated the dependence of spin squeezing on the initial CSS $|\theta_0, \phi_0\rangle$. Our results show that the scaling of ξ^2 depends sensitively upon the polar angle θ_0; it becomes $\xi^2 \propto S^{-1/3}$ [29] even when θ_0 is slightly deviated from its optimal value $\pi/2$.

In this paper, we investigate the effect of phase dephasing on spin squeezing of the OAT model. The dephasing process considered here is induced by external field fluctuation [30, 31, 32, 33, 34, 35, 36, 37, 38], which gives rise to an enhanced decay of the phase coherence between two modes of the BEC [39]. We present the details of Ref. [29] to obtain analytical results of the strongest squeezing $\xi^2_{\rm min}$. For the optimal initial CSS with the polar angle $\theta_0 = \pi/2$, we find that the dephasing effect can be negligible as long as the dephasing rate $\gamma \ll S^{-1/3}$. The ideal one-axis twisting is attainable with the best squeezing $\xi^2_{\rm min} \propto S^{-2/3}$ [4]. For a moderate dephasing rate (i.e., $S^{-1/3} < \gamma < S^{1/2}$), our analytical result indicates that the achievable squeezing scales as $S^{-2/5}$, which is slightly worse than the ideal OAT case. As the dephasing rate increases up to $S^{1/2}$, we find that the squeezing becomes very weak as $\xi^2 \sim 1$.

This paper is arranged as follows. In Sec. II, we consider quantum dynamics of the one-axis twisting model with a many-body phase dephasing, which describes quantitatively a two-mode BEC in the presence of the phase diffusion. The density-matrix elements are obtained by solving quantum master equation. In Sec. III, we investigate the OAT-induced spin squeezing by calculating exact solutions of five relevant quantities, $\langle \hat{S}_z\rangle$, $\langle \hat{S}_+\rangle$, $\langle \hat{S}_z^2\rangle$, $\langle \hat{S}_+^2\rangle$, and $\langle \hat{S}_+(2\hat{S}_z+1)\rangle$. With these expectation values at hands, one can numerically calculate the squeezing parameter ξ^2. To get scaling behavior of ξ^2, in Sec. IV, we consider short-time limit and large enough particle number. Analytical expression of the squeezing parameter is obtained, with which we analyze power rules of ξ^2 according to the role of the phase dephasing. Finally, we conclude in Sec. V with the main results of our work.

2 The one-axis twisting Model

We focus on a two-component Bose-Einstein condensates with the internal states $|\uparrow\rangle$ and $|\downarrow\rangle$ that is confined in a deep three-dimensional harmonic potential. Quantum dynamics of the total system can be described by the Lindblad equation ($\hbar = 1$):

$$\frac{d\hat{\rho}}{dt} = i[\hat{\rho}, \hat{H}] + \Gamma_p(2\hat{S}_z\hat{\rho}\hat{S}_z - \hat{S}_z^2\hat{\rho} - \hat{\rho}\hat{S}_z^2), \tag{1}$$

where $\hat{H} = \chi \hat{S}_z^2$, known as the one-axis twisting Hamiltonian [4], can be realized in a two-mode BECs [10, 11, 12] with the interaction strength χ tunable via the Feshbach resonance technique [25] or the BEC spatial splitting [26]. Atomic spin operators $\hat{S}_z = (\hat{b}_\uparrow^\dagger \hat{b}_\uparrow - \hat{b}_\downarrow^\dagger \hat{b}_\downarrow)/2$ and $\hat{S}_+ = (\hat{S}_-)^\dagger = \hat{b}_\uparrow^\dagger \hat{b}_\downarrow$ represent atomic population imbalance and phase coherence between the two bosonic modes. The second term in Eq. (1) simulates a phase dephasing of the BEC due to magnetic-field fluctuations [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. Such a kind of many-body decoherence has also been studied in cavity-QED system [40, 41].

Due to particle-number conservation, the OAT Hamiltonian and the super-operator in the Eq. (1) commutes with total angular momentum operator $\hat{\mathbf{S}}^2$. Consequently, the state at any time t can be expanded in terms of common eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$, i.e., $\{|S, m\rangle\}$ with total angular momentum $S = N/2$ and $m = -S, -S+1, \cdots, S$. Using the basis, the density-matrix operator $\hat{\rho}$ reads

$$\hat{\rho} = \sum_{m,n=-S}^{S} \rho_{m,n} \left|S, m\right\rangle \left\langle S, n\right| , \tag{2}$$

with the elements $\rho_{m,n} = \langle S, m|\hat{\rho}|S, n\rangle$ satisfying

$$\frac{d\rho_{m,n}}{dt} = \left[i\chi(n^2 - m^2) - \Gamma_p(n-m)^2\right] \rho_{m,n}. \tag{3}$$

Assume that the BEC system evolves from a coherent spin state (CSS) [42]: $|\theta_0, \phi_0\rangle \equiv \exp\{i\theta_0[\hat{S}_x \sin(\phi_0) - \hat{S}_y \cos(\phi_0)]\}|S, S\rangle = \sum_m c_m|S, m\rangle$, with the probability amplitudes

$$c_m = \begin{pmatrix} 2S \\ S+m \end{pmatrix}^{1/2} \left(\cos\frac{\theta_0}{2}\right)^{S+m} \left(\sin\frac{\theta_0}{2}\right)^{S-m} e^{i(S-m)\phi_0}. \tag{4}$$

Exact solutions of Eq. (3) can be obtained with the density-matrix elements

$$\rho_{m,n}(\tau) = \rho_{m,n}(0)e^{-i(m^2-n^2)\tau}e^{-(m-n)^2\gamma\tau}, \tag{5}$$

where $\tau = \chi t$ and $\gamma = \Gamma_p/\chi$. The first term on the right-hand side of Eq. (5) $\rho_{m,n}(0) = c_m c_n^*$, with c_m given by Eq. (4). The second term arises from time evolution of the density matrix under the OAT Hamiltonian $\chi\hat{S}_z^2$. The last one is the dephasing term due to magnetic-field fluctuation and has been obtained previously [43, 44, 45, 46]. Particularly, Takeuchi *et al.* [43] considered a light-induced spin squeezing in an atomic gas and obtained almost the same result with ours. However, their result is based upon an approximated commutation relation of the Stokes operators of light. Here we present exact solution of the density-matrix elements, which describes quantitatively the OAT-induced spin squeezing in a two-mode BEC in the presence of the phase dephasing.

3 Spin squeezing of the OAT model

Starting from a CSS $|\theta_0, \phi_0\rangle$, unitary evolution of the spin system under the OAT Hamiltonian leads to spin squeezing and multipartite entanglement. Both of them can be quantified by the variances of a spin component $\hat{S}_\psi = \hat{\mathbf{S}} \cdot \mathbf{n}_\psi$ that is normal to the mean-spin vector $\langle\hat{\mathbf{S}}\rangle \equiv (\langle\hat{S}_x\rangle, \langle\hat{S}_y\rangle, \langle\hat{S}_z\rangle)$, i.e., $\mathbf{n}_\psi \cdot \langle\hat{\mathbf{S}}\rangle = 0$. As usual, the expectation value of an operator $\hat{\mathcal{O}}$ is defined by $\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}})$. For a given state $\hat{\rho}$, the mean spin and its direction $\mathbf{n}_3 = \langle\hat{\mathbf{S}}\rangle/|\langle\hat{\mathbf{S}}\rangle|$ can be determined uniquely, which in turn gives $\mathbf{n}_\psi = \mathbf{n}_1 \cos\psi + \mathbf{n}_2 \sin\psi$, with three orthogonal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ (for details see Ref. [29]). The increased and the reduced variances of the spin component $\hat{S}_\psi$ are defined, respectively, as $V_+ = \max_\psi(\Delta\hat{S}_\psi)^2$ and $V_- = \min_\psi(\Delta\hat{S}_\psi)^2$, with

$$V_\pm = \frac{1}{2}\left(\mathcal{C} \pm \sqrt{\mathcal{A}^2 + \mathcal{B}^2}\right), \tag{6}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ depend only on five expectation values (see Appendix A): $\langle\hat{S}_z\rangle$, $\langle\hat{S}_+\rangle$, $\langle\hat{S}_z^2\rangle$, $\langle\hat{S}_+^2\rangle$, and $\langle\hat{S}_+(2\hat{S}_z + 1)\rangle$. According to Kitagawa and Ueda [4], a spin state is squeezed only if the variance V_- is smaller than the SQL, $S/2$, namely the squeezing parameter

$$\xi^2 = \frac{2V_-}{S} < 1. \tag{7}$$

The spin squeezed state is useful to improve frequency resolution in spectroscopy provided that $\zeta_{\rm S}^2 = 2SV_-/|\langle\hat{\mathbf{S}}\rangle|^2 < 1$ [3], which provides a sufficient criterion for the degree of multipartite entanglement [12]. In addition, the squeezing parameter can be defined as $\zeta^2 = 2V_-/|\langle\hat{\mathbf{S}}\rangle| = (S/|\langle\hat{\mathbf{S}}\rangle|)\xi^2$ [47]. The three definitions are slightly different in magnitude, $\xi^2 \le \zeta^2 \le \zeta_{\rm S}^2$ due to $|\langle\hat{\mathbf{S}}\rangle| \le S$. For large enough particle number N $(> 10^2)$, the minimum values of them obey almost the same power rule [43], so we only focus on the squeezing parameter ξ^2.

Based upon Eq.(5), we now calculate exact solutions of the mean spin and the variances $V_\pm$. After some straightforward calculations, we can obtain (see Append. A, or Ref.[29])

$$\langle\hat{S}_z\rangle = S\cos(\theta_0), \quad \langle\hat{S}_+\rangle = S\sin(\theta_0)e^{i\phi_0}e^{-\gamma\tau}[R(\tau)]^{2S-1}, \tag{8}$$

where the population imbalance $\langle\hat{S}_z\rangle$ is time-independent, and

$$R(\tau) = \cos(\tau) + i\cos(\theta_0)\sin(\tau) = \sqrt{1-\sin^2(\theta_0)\sin^2(\tau)} \cdot e^{i\tan^{-1}[\cos(\theta_0)\tan(\tau)]}. \tag{9}$$

From Eq. (8), we note that the phase dephasing considered in Eq. (1) imposes an exponential decay term $e^{-\gamma\tau}$ to the phase coherence $|\langle S_+\rangle|$, but maintains the population imbalance $\langle\hat{S}_z\rangle$ intact. Moreover, it is easy to obtain $\langle\hat{S}_x\rangle = {\rm Re}\langle\hat{S}_+\rangle = |\langle\hat{S}_+\rangle|\cos(\phi)$ and $\langle\hat{S}_y\rangle = {\rm Im}\langle\hat{S}_+\rangle = |\langle\hat{S}_+\rangle|\sin(\phi)$, with the argument of $\langle\hat{S}_+\rangle$:

$$\phi \equiv \arg\langle\hat{S}_+\rangle = \phi_0 + (2S-1)\tan^{-1}[\cos(\theta_0)\tan(\tau)]. \tag{10}$$

Here, ϕ_0 is the azimuth angle of the initial CSS. The variances $V_\pm$ depend upon the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ (see Append. A). In real calculations of them, only $\cos(\phi)$ and $\sin(\phi)$ are required, which depends on $\langle\hat{S}_+\rangle$. In addition, we need to solve the following expectation values:

$$\langle\hat{S}_z^2\rangle = \frac{S}{2} + S\left(S-\frac{1}{2}\right)\cos^2(\theta_0), \tag{11}$$

$$\langle\hat{S}_+^2\rangle = S\left(S-\frac{1}{2}\right)\sin^2(\theta_0)e^{2i\phi_0}e^{-4\gamma\tau}[R(2\tau)]^{2S-2}, \tag{12}$$

and

$$\langle\hat{S}_+(2\hat{S}_z+1)\rangle = 2S\left(S-\frac{1}{2}\right)\sin(\theta_0)e^{i\phi_0}e^{-\gamma\tau}[R(\tau)]^{2S-2}[i\sin(\tau)+\cos(\theta_0)\cos(\tau)]. \tag{13}$$

Substituting the above results into the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, one can obtain the variances $V_\pm$ and also the squeezing parameter ξ^2. In Fig. 1, we plot exact numerical results of ξ^2 (solid lines) for different values of the dephasing rate γ. We find that the strongest squeezing occurs at a certain time $\tau_{\rm min}$ $(= \chi t_{\rm min})$, with its position indicated by the arrows of Fig. 1.

4 Scaling behaviors of the squeezing parameter

In order to analyze scaling behaviors of ξ^2, we now consider the short-time limit (i.e., $\tau = \chi t \ll 1$) and large enough particle-number $(S = N/2 \gg 1)$, so Eq. (9) can be approximated as $R(\tau) \approx S\exp(-\frac{1}{2}\tau^2\sin^2\theta_0)e^{i\tau\cos\theta_0}$, which in turn yields

$$\langle\hat{S}_+\rangle \approx S\sin(\theta_0)e^{i\phi}e^{-\beta}, \tag{14}$$

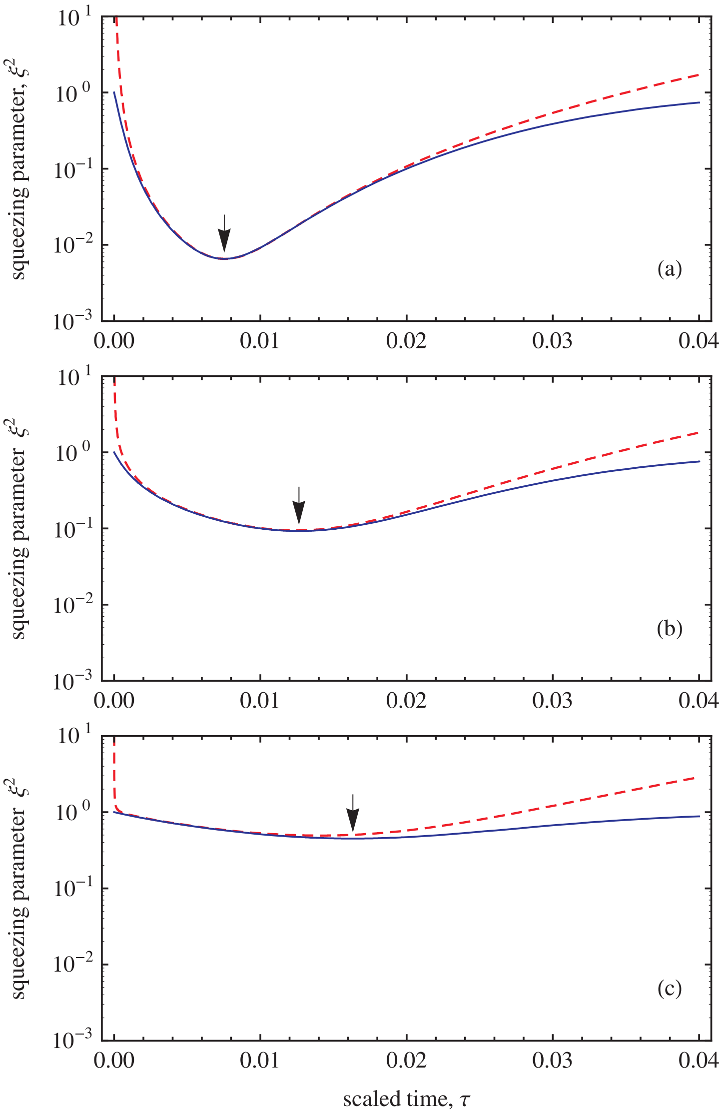

Fig. 1. (color online) The squeezing parameter ξ^2 as a function of scaled time τ $(= \chi t)$ for various dephasing rates $\gamma = \Gamma_p/\chi = 0$ (a), 1 (b), and 10 (c). Solid blue lines are given by exact numerical simulations and red dashed lines are predicated by Eq. (18). Other parameters are $S = N/2 = 10^3$, $\theta_0 = \pi/2$, and $\phi_0 = 0$. The arrows indicate the location of maximal-squeezing times $\tau_{\rm min} = 7.536 \times 10^{-3}$ (a), 1.264×10^{-2} (b), and 1.632×10^{-2} (c), at which the strongest squeezing occurs with $\xi^2_{\rm min} = 6.56 \times 10^{-3}$ (a), 9.278×10^{-2} (b), and 0.4504 (c).

$$\langle\hat{S}_+^2\rangle \approx S\left(S-\frac{1}{2}\right)\sin^2(\theta_0)e^{2i\phi}e^{-4\beta}, \tag{15}$$

and

$$\langle\hat{S}_+(2\hat{S}_z+1)\rangle \approx 2S\left(S-\frac{1}{2}\right)\sin(\theta_0)e^{i\phi}e^{-\beta}\left(\cos\theta_0+i\tau\right), \tag{16}$$

where the argument of $\langle\hat{S}_+\rangle$ now becomes $\phi \approx \phi_0 + 2S\tau\cos(\theta_0)$, and

$$\beta = S\tau^2\sin^2\theta_0 + \gamma\tau. \tag{17}$$

Note that without the dephasing, the phase coherence reduces to $|\langle S_+\rangle| \approx S\sin(\theta_0)e^{-(\tau/\tau_d)^2}$, with the coherent time $\tau_d = \chi t_d = 1/(\sqrt{S}\sin\theta_0)$ [29]. Such a kind of exponential decay is well known as the so-called phase diffusion of a two-mode BEC. For the BEC atoms, the nonlinearity $\chi \propto \frac{1}{2}(a_{\uparrow\uparrow}+a_{\downarrow\downarrow}-2a_{\uparrow\downarrow})$ depends upon the intra- and the inter-species atom-atom scattering lengthes. When the three coupling constants are close to each other, the coherence time t_d increases dramatically due to $\chi \to 0$ [48]. If we take the phase dephasing into account (i.e., $\gamma \neq 0$), the phase diffusion process will speed up, as demonstrated recently in Ref. [39]. In what's following, we will investigate the role of the phase dephasing on the spin squeezing.

Firstly, using Eq. (15) and Eq.(16), as well as the exact result of $\langle\hat{S}_z^2\rangle$, we obtain the short-time solutions of the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ [see Eq. (B.1)-Eq.(B.2)]. Next, we focus on a time regime: $\tau < S\tau^2 \ll 1$ and $\gamma\tau \ll 1$, which allows us to expand the above results in terms of β ($\ll 1$) [4]. From Eq. (6), it is easy to find that the product of the variances $V_+V_- = [(\mathcal{C}+\mathcal{A})(\mathcal{C}-\mathcal{A})-\mathcal{B}^2]/4$. To simplify it, we expand $\mathcal{C}\pm\mathcal{A}$, $\mathcal{B}^2$, and hence V_+V_- up to the third-order of β [see Eq. (B.4)-Eq. (B.5)]. On the other hand, we can reduce the increased variance V_+ by keeping the lowest-order of β (see Appendix B). Finally, using the relation $V_- = (V_+V_-)/V_+$, we obtain analytical result of the reduced variance and also the squeezing parameter:

$$\xi^2 \approx \frac{\gamma\tau}{\beta} + \frac{1}{4S\beta\sin^2(\theta_0)} + \frac{2\beta^2}{3}\left[1+9S\sin^2(\theta_0)\cos^2(\theta_0)\right], \tag{18}$$

where β is given by Eq. (17) and θ_0 is polar angle of the initial state. In Fig. 1, we compare our analytical result of ξ^2 (red dash) with its exact solution (solid line) for different values of the dephasing rate γ. When γ is not too large, we find that Eq. (18) works well to predict the minimal value of the squeezing parameter $\xi^2_{\min} = 2V_-(\tau_{\min})/S$. This is because both the analytical and the exact results almost merge with each other around the maximal-squeezing time $\tau_{\min}$.

Based upon Eq. (17) and Eq. (18), we now analyze in detail the role of the phase dephasing on the spin squeezing. If the first term on right-hand side of Eq. (18) is comparable with the second one, we obtain $\gamma\tau \sim [4S\sin^2(\theta_0)]^{-1}$. On the other hand, we compare the two terms of Eq. (17) and get $\gamma \sim S\tau\sin^2(\theta_0)$.

Obviously, the dephasing effect can be *neglected* only if $\gamma \ll S\tau\sin^2(\theta_0)$ and $\gamma\tau \ll [4S\sin^2(\theta_0)]^{-1}$, for which Eq. (17) becomes $\beta \approx S\tau^2\sin^2(\theta_0)$ and the first term of Eq. (18) can be omitted. As a result, we obtain the analytical result of the squeezing parameter [29]:

$$\xi^2 \approx \frac{1}{4S\beta\sin^2(\theta_0)} + \frac{2\beta^2}{3}\left[1+9S\sin^2(\theta_0)\cos^2(\theta_0)\right]. \tag{19}$$

From the relation $(d\xi^2/d\tau)\big|_{\tau_{\min}} = 0$, we obtain the maximal-squeezing time:

$$\tau_{\min} = \chi t_{\min} \approx \frac{3^{1/6}[2S\sin^2(\theta_0)]^{-2/3}}{[1+9S\sin^2(\theta_0)\cos^2(\theta_0)]^{1/6}}. \tag{20}$$

Substituting Eq. (20) back to Eq. (19), we further obtain the smallest value of ξ^2:

$$\xi^2_{\min} \approx \frac{3}{4}\left\{\frac{2\left[1+9S\sin^2(\theta_0)\cos^2(\theta_0)\right]}{3S^2\sin^4(\theta_0)}\right\}^{1/3}. \tag{21}$$

For the initial CSS with $\theta_0 = \pi/2$, Eq. (21) shows $\xi^2_{\min} \approx \frac{1}{2}(\frac{2S}{3})^{-2/3}$, which is the best squeezing that the one-axis twisting scheme can reach [4, 29]. Considering a large enough particle-number with $S = N/2 = 10^3$, we can obtain $\tau_{\min} \approx 7.565 \times 10^{-3}$ and $\xi^2_{\min} \approx 6.552 \times 10^{-3}$, fitting very well with numerical simulations [see Fig. 1(a)]. From Eq. (20), we find that the time scales as $\tau_{\min} \propto S^{-2/3}$ for $\theta_0 = \pi/2$ [4], and $\tau_{\min} \propto S^{-5/6}$ for $\theta_0 \neq \pi/2$ [29]. Substituting the power rules into the condition $\gamma\tau \ll [4S\sin^2(\theta_0)]^{-1}$, we make a conclusion that our analytical results, Eq. (20) and Eq. (21), are valid for the dephasing rate $\gamma \ll S^{-1/3}$ $(\theta_0 = \pi/2)$, or $\gamma \ll S^{-1/6}$ $(\theta_0 \neq \pi/2)$. As shown by the dash lines of Fig. 2, one can find that Eq. (21) coincides with the exact results when $\ln(\gamma)/\ln(S) < -0.5$.

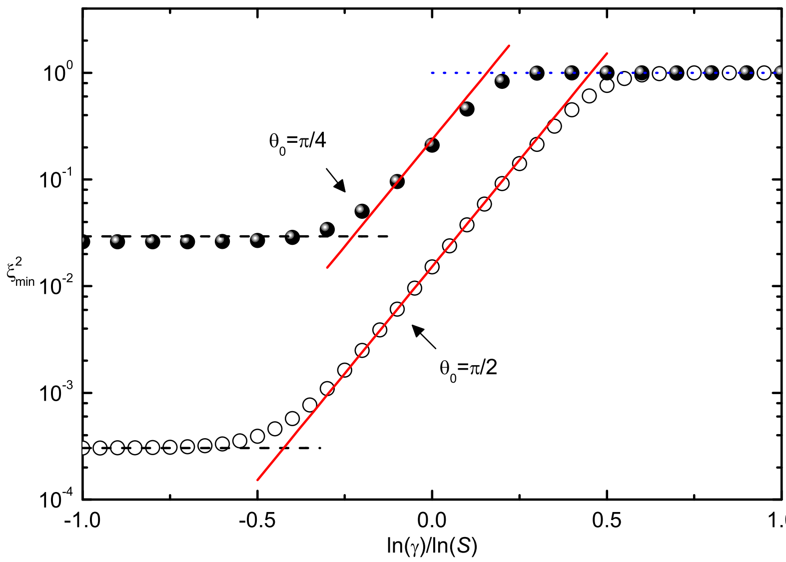

Fig. 2. (color online) The minimal value of the squeezing parameter $\xi^2_{\min}$ as a function of $\ln(\gamma)/\ln(S)$ for $\theta_0 = \pi/2$ (open circles) and $\theta_0 = \pi/4$ (balls). The dash and the red solid lines are obtained from our analytical results, Eq. (21) and Eq. (24). The blue dotted line corresponds to the case $\xi^2 \approx 1$ (almost no squeezing) as $\ln(\gamma)/\ln(S) > 0.5$ for $\theta_0 = \pi/2$ and $\ln(\gamma)/\ln(S) > 0.25$ for $\theta_0 = \pi/4$. Other parameters: $\chi = 1$, $S = N/2 = 10^5$ and $\phi_0 = 0$.

To proceed, let us consider the case $\gamma < S\tau\sin^2(\theta_0)$, but $\gamma\tau > [4S\sin^2(\theta_0)]^{-1}$, for which the first term of Eq. (18) becomes important in a comparison with the second one so we get

$$\xi^2 \approx \frac{\gamma\tau}{\beta} + \frac{2\beta^2}{3}\left[1+9S\sin^2(\theta_0)\cos^2(\theta_0)\right], \tag{22}$$

where $\beta \approx S\tau^2 \sin^2(\theta_0)$. Minimizing ξ^2 with respect to τ, we obtain

$$\tau_{\min} \approx \frac{(3\gamma)^{1/5}\left[8S^3 \sin^6(\theta_0)\right]^{-1/5}}{[1+9S\sin^2(\theta_0)\cos^2(\theta_0)]^{1/5}}, \tag{23}$$

and

$$\xi^2_{\min} \approx \frac{5}{4}\left\{\frac{8\gamma^4[1+9S\sin^2(\theta_0)\cos^2(\theta_0)]}{3S^2\sin^4(\theta_0)}\right\}^{1/5}. \tag{24}$$

From Eq. (23), we find that the strongest squeezing appears at $\tau \propto \gamma^{1/5}S^{-3/5}$ for $\theta_0 = \pi/2$, and $\tau \propto \gamma^{1/5}S^{-4/5}$ for $\theta_0 \neq \pi/2$. Using the conditions $\gamma\tau > [4S\sin^2(\theta_0)]^{-1}$ and $\gamma < S\tau\sin^2(\theta_0)$, it is easy to find that Eq. (24) works quite well for a relatively weak dephasing rate with $S^{-1/3} < \gamma < S^{1/2}$ ($\theta_0 = \pi/2$), or $S^{-1/6} < \gamma < S^{1/4}$ ($\theta_0 \neq \pi/2$).

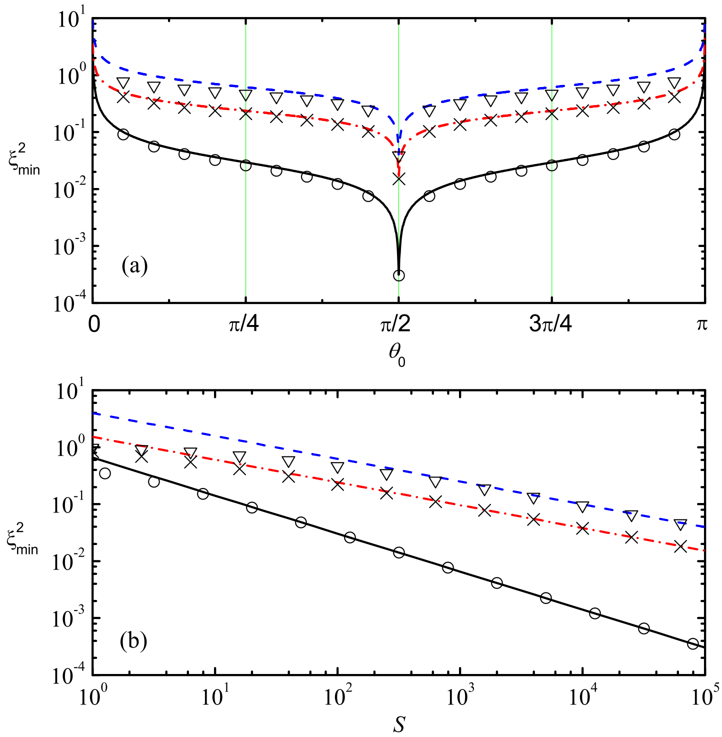

Fig. 3. (Color online) The minimal value of the squeezing parameter $\xi^2_{\min}$ against θ_0 for $S = 10^5$ (a), and S for $\theta_0 = \pi/2$ (b). From top to bottom in each figure, the dephasing rates are taken as $\gamma = 3.3$ (triangles), 1 (crosses), and 0 (open circles). The solid line is given by Eq. (21). The dashed and dashed-dot curves are predicted by analytical result, Eq. (24). Other parameters are $\chi = 1$ and $\phi_0 = 0$.

As shown in Fig. 1(b), for the case $S = 10^3$ and $\gamma = 1$, Eq. (23) and Eq. (24) predict $\tau_{\min} \approx 1.303 \times 10^{-2}$ and $\xi^2_{\min} \approx 9.596 \times 10^{-2}$, respectively. Both of them fit with the exact numerical results $\tau_{\min} = 1.264 \times 10^{-2}$ and $\xi^2_{\min} = 9.278 \times 10^{-2}$. As the dephasing rate increases up to 10, our analytical results give $\tau_{\min} \approx 2.064 \times 10^{-2}$ and $\xi^2_{\min} \approx 0.605$, which are in order-of-magnitude agreement with the exact results [see Fig. 1(c)]. Moreover, we find that the red solid line of Fig. 2, given by Eq. (24) for $\theta_0 = \pi/4$, shows clearly that $\xi^2_{\rm min}$ increases rapidly with γ in the regime $-0.2 < \ln(\gamma)/\ln(S) < 0.25$.

Our analytical results, Eq. (21) and Eq. (24), can not work to predict the spin squeezing when γ becomes large. For instance, let us consider $\gamma\tau > [4S\sin^2(\theta_0)]^{-1}$ and $\gamma > S\tau\sin^2(\theta_0)$. In this case, Eq. (22) reduces to

$$\xi^2 \approx 1 + \frac{2\beta^2}{3}\left[1 + 9S\sin^2(\theta_0)\cos^2(\theta_0)\right], \tag{25}$$

with $\beta \approx \gamma\tau$. Actually, when $\gamma > S^{1/2}$ ($\theta_0 = \pi/2$), or $\gamma > S^{1/4}$ ($\theta_0 \neq \pi/2$), the squeezing effect is week due to $\xi^2 \approx 1$, as shown by the blue dotted line of Fig. 2.

For a fixed S ($= N/2$), we find from Fig. 3(a) that the achievable squeezing depends upon the polar angle of the initial state; the initial CSS with $\theta_0 = \pi/2$ is the optimal one to minimize ξ^2 even in the presence of phase dephasing. In Fig. 3(b), we focus on the optimal case $\theta_0 = \pi/2$ and investigate the dependence of $\xi^2_{\rm min}$ on S for the dephasing rates $\gamma = 0$ (open circles), 1 (red crosses), and 3.3 (blue triangles). The three curves, given by our analytical results, show good agreement with the exact numerical simulations for large enough particle number (say $S = N/2 > 10^2$). More interestingly, it is also found that the slop of the solid curve is different with that of other two lines. This is because our analytical result for the case $\gamma = 0$, Eq. (21), predicts $\xi^2_{\rm min} \propto S^{-2/3}$ [4]; while for a small but nonzero γ, Eq. (24) gives $\xi^2_{\rm min} \propto S^{-2/5}$. Such a power rule has been also obtained by Takeuchi *et al.* [43]. However, their scheme is based upon a double-pass Faraday interaction between atoms and far-off-resonant light. In addition, the starting point of their work, though quite similar with Eq. (5), is derived by an approximated commutation relation of the light-field Stokes operators [43].

5 Conclusion

In summary, we have investigated the role of phase diffusion on spin squeezing of the one-axis twisting model. Our results show that the spin squeezing depends upon the initial state $|\theta_0, 0\rangle$ ($= e^{-i\theta_0\hat{S}_y}|S, S\rangle$). The optimal initial state corresponds to the polar angle $\theta_0 = \pi/2$, even in the presence of phase dephasing. If the dephasing rate $\gamma = \Gamma_p/\chi \ll S^{-1/3}$, the dephasing effect is negligible and the ideal one-axis twisting is restored. The strongest squeezing scales as $\xi^2_{\rm min} \propto S^{-2/3}$. For a moderate dephasing rate (i.e., $S^{-1/3} < \gamma < S^{1/2}$), the achievable squeezing obeys the power rule $\xi^2_{\rm min} \propto S^{-2/5}$, which is slightly worse than the ideal case. When the dephasing rate $\gamma > S^{1/2}$, we show that the squeezing becomes very weak due to $\xi^2 \sim 1$.

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References

1. C. M. Caves (1981), *Quantum-mechanical noise in an interferometer*, Phys. Rev. D, 23, pp. 1693-1708.
2. B. Yurke, S. L. McCall, and S. R. Klauder (1986), *SU(2) and SU(1,1) interferometers*, Phys. Rev. A, 33, pp. 4033-4054.
3. D. J. Wineland, J. J. Bollinger, W. M. Itano, and F. L. Moore (1992), *Spin squeezing and reduced quantum noise in spectroscopy*, Phys. Rev. A, 46, pp. R6797-R6800; D. J. Wineland, J. J. Bollinger, W. M. Itano, and F. L. Moore (1994), *Squeezed atomic states and projection noise in spectroscopy*, Phys. Rev. A, 50, pp. 67-88.
4. M. Kitagawa and M. Ueda (1993), *Squeezed spin states*, Phys. Rev. A, 47, pp. 5138-5143.
5. J. Ma, X. Wang, C. P. Sun, and F. Nori (2011), *Quantum spin squeezing*, Phys. Reports, 509, pp. 89-166.
6. K. Mølmer and A. Sørensen (1999), *Multiparticle Entanglement of Hot Trapped Ions*, Phys. Rev. Lett., 82, pp. 1835-1838.
7. L. You (2003), *Creating Maximally Entangled Atomic States in a Bose-Einstein Condensate*, Phys. Rev. Lett., 90, pp. 030402.
8. L. Pezzé and A. Smerzi (2009), *Entanglement, Nonlinear Dynamics, and the Heisenberg Limit*, Phys. Rev. Lett., 102, pp. 100401.
9. Y. C. Liu, Z. F. Xu, G. R. Jin, and L. You (2011), *Spin Squeezing: Transforming One-Axis Twisting into Two-Axis Twisting*, Phys. Rev. Lett., 107, pp. 013601.
10. G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls (1997), *Quantum dynamics of an atomic Bose-Einstein condensate in a double-well potential*, Phys. Rev. A, 55, pp. 4318-4324.
11. J. I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller (1998), *Quantum superposition states of Bose-Einstein condensates*, Phys. Rev. A, 57, pp. 1208-1218.
12. A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller (2001), *Many-particle entanglement with BoseCEinstein condensates*, Nature, 409, pp. 63-66; U. V. Poulsen and K. Mølmer(2001), *Positive-P simulations of spin squeezing in a two-component Bose condensate*, Phys. Rev. A, 64, pp. 013616.
13. S. Raghavan, H. Pu, P. Meystre, and N. P. Bigelow (2001), *Generation of arbitrary Dicke states in spinor Bose-Einstein condensates*, Opt. Commu., 188, pp. 149-154.
14. C. K. Law, H. T. Ng, and P. T. Leung (2001), *Coherent control of spin squeezing*, Phys. Rev. A, 63, pp. 055601.
15. G. R Jin and S. W. Kim (2007), *Storage of Spin Squeezing in a Two-Component Bose-Einstein Condensate*, Phys. Rev. Lett., 99, pp. 170405; G. R Jin and S. W. Kim (2007), *Spin squeezing and maximal-squeezing time*, Phys. Rev. A, 76, pp. 043621.
16. G. R Jin and C. K. Law (2008), *Relationship between spin squeezing and single-particle coherence in two-component Bose-Einstein condensates with Josephson coupling*, Phys. Rev. A, 78, pp. 063620; G. R. Jin, X. W. Wang, D. Li, and Y. W. Lu (2010), *Atom-number squeezing and bipartite entanglement of two-component Bose-Einstein condensates: analytical results*, J. Phys. B, 43, pp. 045301.
17. J. Grond, J. Schmiedmayer, and U. Hohenester (2009), *Optimizing number squeezing when splitting a mesoscopic condensate*, Phys. Rev. A, 79, pp. 021603(R).
18. P. Bouyer and M. A. Kasevich (1997), *Heisenberg-limited spectroscopy with degenerate Bose-Einstein gases*, Phys. Rev. A, 56, pp. R1083-R1086; C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, M. A. Kasevich (2001), Science, 291, pp. 2386-2389; W. Li, A. K. Tuchman, H. C. Chien, and M. A. Kasevich (2007), *Extended Coherence Time with Atom-Number Squeezed States*, Phys. Rev. Lett., 98, pp. 040402.
19. M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch (2002), *Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms*, Nature, 415, pp. 39-44.
20. J. Sebby-Strabley, B. L. Brown, M. Anderlini, P. J. Lee, W. D. Phillips, and J. V. Porto (2007), *Preparing and Probing Atomic Number States with an Atom Interferometer*, Phys. Rev. Lett., 98, pp. 200405.
21. M. Rodríguez, S. R. Clark, and D. Jaksch (2007), *Generation of twin Fock states via transition from a two-component Mott insulator to a superfluid*, Phys. Rev. A, 75, pp. 011601(R).

22. J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler (2008), *Squeezing and entanglement in a BoseCEinstein condensate*, Nature, 455, pp. 1216-1219 .
23. C.-S. Chuu, F. Schreck, T. P. Meyrath, J. L. Hanssen, G. N. Price, and M. G. Raizen (2005), *Direct Observation of Sub-Poissonian Number Statistics in a Degenerate Bose Gas*, Phys. Rev. Lett., 95, pp. 260403.
24. G.-B. Jo, Y. Shin, S. Will, T. A. Pasquini, M. Saba, W. Ketterle, and D. E. Pritchard (2007), *Long Phase Coherence Time and Number Squeezing of Two Bose-Einstein Condensates on an Atom Chip*, Phys. Rev. Lett., 98, pp. 030407.
25. C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler (2010), *Nonlinear atom interferometer surpasses classical precision limit*, Nature, 464, pp. 1165-1169.
26. M. F. Riedel, P. Bohi, Y. Li, T. W. Hansch, A. Sinatra, and P. Treutlein (2010),*Atom-chip-based generation of entanglement for quantum metrology*, Nature, 464, pp. 1170-1173.
27. K. Maussang, G. E. Marti, T. Schneider, P. Treutlein, Y. Li, A. Sinatra, R. Long, J. Estève, and J. Reiche (2010), *Enhanced and Reduced Atom Number Fluctuations in a BEC Splitter*, Phys. Rev. Lett., 105, pp. 080403.
28. Y. Li, P. Treutlein, J. Reichel, and A. Sinatra (2009), *Spin squeezing in a bimodal condensate: spatial dynamics and particle losses*, Eur. Phys. J. B., 68, pp. 365-381; Y. Li, Y. Castin, and A. Sinatra (2008), *Optimum Spin Squeezing in Bose-Einstein Condensates with Particle Losses*, Phys. Rev. Lett., 100, pp. 210401.
29. G. R. Jin, Y. C. Liu, and W. M. Liu (2009), *Spin squeezing in a generalized one-axis twisting model*, New J. Phys., 11, pp. 073049.
30. C. W. Gardiner and P. Zoller (2000), *Quantum Noise*, Springer (Berlin).
31. D. F. Walls and G. J. Milburn (2008), *Quantum Optics*, Springer-Verlag (Berlin Heidelberg).
32. S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac (1997), *Improvement of Frequency Standards with Quantum Entanglement*, Phys. Rev. Lett., 79, pp. 3865-3868.
33. D. Ulam-Orgikh and M. Kitagawa (2001), *Spin squeezing and decoherence limit in Ramsey spectroscopy*, Phys. Rev. A, 64, pp. 052106.
34. A. M. Rey, L. Jiang, and M. D. Lukin (2007), *Quantum-limited measurements of atomic scattering properties*, Phys. Rev. A, 76, pp. 053617.
35. S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves (2008), *Quantum-limited metrology with product states*, Phys. Rev. A, 77, pp. 012317.
36. Y. Khodorkovsky, G. Kurizki, and A. Vardi (2009), *Decoherence and entanglement in a bosonic Josephson junction: Bose-enhanced quantum Zeno control of phase diffusion*, Phys. Rev. A, 80, pp. 023609.
37. Y. C. Liu, G. R. Jin, and L. You (2010), *Quantum-limited metrology in the presence of collisional dephasing*, Phys. Rev. A, 82, pp. 045601.
38. G. R. Jin, Y. C. Liu, and L. You (2011), *Optimal phase sensitivity of atomic Ramsey interferometers with coherent spin states*, Front. of Phys., 6, pp. 251-257.
39. A. Widera, S. Trotzky, P. Cheinet, S. Fölling, F. Gerbier, I. Bloch, V. Gritsev, M. D. Lukin, and E. Demler (2008), *Quantum Spin Dynamics of Mode-Squeezed Luttinger Liquids in Two-Component Atomic Gases*, Phys. Rev. Lett., 100, pp. 140401.
40. R. R. Puri and G. S. Agarwal (1992), *Exact density matrix for the degenerate-Raman-coupled model in the presence of collisions*, Phys. Rev. A, 45, pp. 5073-5077.
41. T. W. Chen and P. T. Leung (2003), *Effect of collision dephasing on atomic evolutions in a high-Q cavity*, Phys. Rev. A, 67, pp. 055802.
42. S. M. Radcliffe (1971), *Some properties of coherent spin states*, S. Phys. A, 4, pp. 313; F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas (1972), *Atomic Coherent States in Quantum Optics*, Phys. Rev. A, 6, pp. 2211-2237; W.-M. Zhang, D. H. Feng, and R. Gilmore (1990), *Coherent States: Theory and Some Applications*, Rev. Mod. Phys., 62, pp. 867-928.
43. M. Takeuchi, S. Ichihara1, T. Takano1, M. Kumakura, T. Yabuzaki, and Y. Takahashi (2005), *Spin Squeezing via One-Axis Twisting with Coherent Light*, Phys. Rev. Lett., 94, pp. 023003.

44. M. G. Genoni, S. Olivares, and M. G. A. Paris (2011), *Optical phase estimation in the presence of phase-diffusion*, (2011) Phys. Rev. Lett. 106, pp. 153603; M. G. Genoni, S. Olivares, D. Brivio, S. Cialdi, D. Cipriani, A. Santamato, S. Vezzoli, and M. G. A. Paris (2012), *Optical interferometry in the presence of large phase diffusion*, Phys. Rev. A., 85, pp. 043817.
45. G. Ferrini, D. Spehner, A. Minguzzi, and F. W. J. Hekking (2011), *Effect of phase noise on useful quantum correlations in Bose Josephson junctions*, Phys. Rev. A., 84, pp. 043628.
46. A. Sinatra, J.-C. Dornstetter, and Y. Castin (2012), *Spin squeezing in Bose-Einstein condensates: Limits imposed by decoherence and non-zero temperature*, Front. Phys., 7, pp. 86-97.
47. M. Ueda, T. Wakabayashi, and M. Kuwata-Gonokami (1996), *Synchronous Collapses and Revivals of Atomic Dipole Fluctuations and Photon Fano Factor beyond the Standard Quantum Limit*, Phys. Rev. Lett., 76, pp. 2045-2048; H. Saito and M. Ueda (1999), *Squeezed few-photon states of the field generated from squeezed atoms*, Phys. Rev. A, 59, pp. 3959-3974; H. Saito and M. Ueda (1997), *Quantum-Controlled Few-Photon State Generated by Squeezed Atoms*, Phys. Rev. Lett., 79, pp. 3869-3872.
48. A. Sinatra and Y. Castin (2000), *Binary mixtures of Bose-Einstein condensates: Phase dynamics and spatial dynamics*, Eur. Phys. J. D, 8, pp. 319-332.

Appendix A Exact solutions of the squeezing parameter

As Eq. (6), the reduced and the increased variances depend upon the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ (for details, see Ref. [29]):

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\left\{\sin^2(\theta)\left[S(S+1)-3\langle\hat{S}_z^2\rangle\right]-[1+\cos^2(\theta)]\,\mathrm{Re}\left[\langle\hat{S}_+^2\rangle e^{-2i\phi}\right]\right.\\ &\quad\left.+\sin(2\theta)\,\mathrm{Re}\left[\langle\hat{S}_+(2\hat{S}_z+1)\rangle e^{-i\phi}\right]\right\}, \qquad (A.1)\\ \mathcal{B} &= -\cos(\theta)\,\mathrm{Im}\left[\langle\hat{S}_+^2\rangle e^{-2i\phi}\right]+\sin(\theta)\,\mathrm{Im}\left[\langle\hat{S}_+(2\hat{S}_z+1)\rangle e^{-i\phi}\right], \qquad (A.2)\\ \mathcal{C} &= S(S+1)-\langle\hat{S}_z^2\rangle-\mathrm{Re}\left[\langle\hat{S}_+^2\rangle e^{-2i\phi}\right]-\mathcal{A}, \qquad (A.3)\end{aligned}$$

where the angles θ and ϕ are determined by the mean spin $\langle\mathbf{S}\rangle = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$, with

$$\begin{aligned}\sin(\theta) &= \frac{|\langle\hat{S}_+\rangle|}{|\langle\hat{\mathbf{S}}\rangle|},\quad \cos(\theta)=\frac{\langle\hat{S}_z\rangle}{|\langle\hat{\mathbf{S}}\rangle|}, \qquad (A.4)\\ \cos(\phi) &= \frac{\langle\hat{S}_x\rangle}{|\langle\hat{S}_+\rangle|},\quad \sin(\phi)=\frac{\langle\hat{S}_y\rangle}{|\langle\hat{S}_+\rangle|}. \qquad (A.5)\end{aligned}$$

Here, $\langle\hat{S}_x\rangle = \mathrm{Re}\langle\hat{S}_+\rangle$ and $\langle\hat{S}_y\rangle = \mathrm{Im}\langle\hat{S}_+\rangle$, as mentioned above. Note that the above formulae are valid for any SU(2) system. Moreover, one can find that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ depend only on five expectation values: $\langle\hat{S}_z\rangle$, $\langle\hat{S}_+\rangle$, $\langle\hat{S}_z^2\rangle$, $\langle\hat{S}_+^2\rangle$, and $\langle\hat{S}_+(2\hat{S}_z+1)\rangle$. Although the argument ϕ appears in the coefficients, we do *not* need to know its explicit expression. Instead, only $\cos(\phi)$ and $\sin(\phi)$ are needed in real calculations of the coefficients and hence the variances $V_\pm$.

Starting with an initial CSS $|\theta_0, \phi_0\rangle$, the OAT model can be solved exactly. For instance, we calculate the expectation value $\langle\hat{S}_+^l\rangle \equiv \mathrm{Tr}(\hat{\rho}\hat{S}_+^l)$, with an integer $l = 1, 2, \cdots, [S]$. Here, $[S]$ denotes the greatest integer of any real number S. Using Eq. (5), we obtain

$$\langle\hat{S}_+^l\rangle = \sum_{m=-S}^{S-l}\rho_{m,m+l}(0)X_{-m}X_{-m-1}\cdots X_{-m-l+1}e^{i(2ml+l^2)\tau}e^{-\gamma l^2\tau} = e^{-\gamma l^2\tau}\langle\hat{S}_+^l\rangle_0, \quad (A.6)$$

where $X_m = \sqrt{(S+m)(S-m+1)}$, and $\rho_{m,n}(0) = c_m c_n^*$ represent the density-matrix elements of the initial CSS, with c_m given by Eq. (4). In addition, we have introduced

$$\begin{aligned}
\langle \hat{S}_+^l \rangle_0 &\equiv \sum_{m=-S}^{S-l} \rho_{m,m+l}(0) e^{i(2ml+l^2)\tau} X_{-m} X_{-m-1} \cdots X_{-m-l+1} \\
&= \sum_{m=-S}^{S-l} \frac{(2S)!}{(S+m)!(S-m-l)!} e^{il\phi_0} e^{i(2ml+l^2)\tau} \left(\cos\frac{\theta_0}{2}\right)^{2S+2m+l} \left(\sin\frac{\theta_0}{2}\right)^{2S-2m-l} \\
&= \frac{(2S)!}{2^l(2S-l)!} \sin^l(\theta_0)\, e^{il\phi_0} \sum_m \binom{2S-l}{S+m} \left(e^{il\tau}\cos^2\frac{\theta_0}{2}\right)^{S+m} \left(e^{-il\tau}\sin^2\frac{\theta_0}{2}\right)^{S-m-l} \\
&= \frac{(2S)!}{2^l(2S-l)!} \sin^l(\theta_0) e^{il\phi_0} [R(l\tau)]^{2S-l}, \qquad \text{(A.7)}
\end{aligned}$$

where $R(l\tau) = \cos(l\tau) + i\cos(\theta_0)\sin(l\tau)$, and we have used the binomial formula:

$$\sum_{m=-S}^{S-l} \binom{2S-l}{S+m} a^{S+m} b^{S-m-l} = \sum_{n=0}^{2S-l} \binom{2S-l}{n} a^n b^{2S-n-l} = (a+b)^{2S-l}. \qquad \text{(A.8)}$$

With the help of Eq. (A.6) and Eq. (A.7), we can obtain the exact solutions of $\langle \hat{S}_+ \rangle$ and $\langle \hat{S}_+^2 \rangle$, given by Eq. (8) and Eq. (12), respectively.

We note that without the dephasing, the spin system evolves under governed by the OAT Hamiltonian $\hat{H} = \chi \hat{S}_z^2$, so we have $d\langle \hat{S}_+ \rangle_0 / d\tau = -i\langle [\hat{S}_+, \hat{H}] \rangle_0 / \chi = i\langle \hat{S}_+(2\hat{S}_z + 1) \rangle_0$, where the subscript 0 denotes the expectation values in the absence of phase dephasing (i.e., $\gamma = 0$), and $\langle \hat{S}_+ \rangle_0$ has been given in Eq. (A.7) with $l = 1$. Therefore, we obtain

$$\langle \hat{S}_+(2\hat{S}_z + 1) \rangle = e^{-\gamma\tau} \langle \hat{S}_+(2\hat{S}_z + 1) \rangle_0 = -ie^{-\gamma\tau} \frac{d\langle \hat{S}_+ \rangle_0}{d\tau}, \qquad \text{(A.9)}$$

which gives Eq. (13). Finally, we calculate the population imbalance and its variance:

$$\begin{aligned}
\langle \hat{S}_z \rangle &= \sum_{m=-S+1}^{S} (S+m)\rho_{m,m}(0) - \sum_{m=-S}^{S} S\rho_{m,m}(0) \\
&= 2S\cos^2\left(\frac{\theta_0}{2}\right) \sum_m \binom{2S-1}{S+m-1} \left(\cos^2\frac{\theta_0}{2}\right)^{S+m-1} \left(\sin^2\frac{\theta_0}{2}\right)^{S-m} - S \\
&= 2S\cos^2\left(\frac{\theta_0}{2}\right) - S = S\cos(\theta_0), \qquad \text{(A.10)}
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{S}_z^2 \rangle &= \sum_{m=-S}^{S} S^2 \rho_{m,m}(0) - \sum_{m=-S+1}^{S-1} (S+m)(S-m)\rho_{m,m}(0) \\
&= S^2 - S\left(S - \frac{1}{2}\right)\sin^2(\theta_0) \sum_m \binom{2S-2}{S+m-1} \left(\cos^2\frac{\theta_0}{2}\right)^{S+m-1} \left(\sin^2\frac{\theta_0}{2}\right)^{S-m-1} \\
&= S^2 - S\left(S - \frac{1}{2}\right)\sin^2(\theta_0) = \frac{S}{2} + S\left(S - \frac{1}{2}\right)\cos^2(\theta_0), \qquad \text{(A.11)}
\end{aligned}$$

where we have used the normalization condition $\sum_m \rho_{m,m}(0) = 1$. So far we have solved all the quantities that relevant to get the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, with which we can calculate exactly the variances $V_\pm$, and hence the squeezing parameter ξ^2.

Appendix B Short-time solutions of the squeezing parameter

To obtain analytical results of the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we have to make further approximations [29], $\sin(\theta) = |\langle \hat{S}_+ \rangle|/|\langle \hat{\mathbf{S}} \rangle| \approx \sin(\theta_0)$ and $\cos(\theta) = \langle \hat{S}_z \rangle/|\langle \hat{\mathbf{S}} \rangle| \approx \cos(\theta_0)$, where θ_0 is polar angle of the initial CSS. Substituting Eq. (15) and Eq.(16), as well as Eq.(A.11) into Eq. (A.1), we obtain the short-time solution of the coefficient

$$\begin{aligned}
\mathcal{A} &\approx \frac{\sin^2\theta_0}{2}\left\{\left[S(S+1) - 3\left(\frac{S}{2} + S\left(S-\frac{1}{2}\right)\cos^2\theta_0\right)\right]\right. \\
&\quad \left. -S\left(S-\frac{1}{2}\right)\left(1+\cos^2\theta_0\right)e^{-4\beta} + 4S\left(S-\frac{1}{2}\right)e^{-\beta}\cos^2\theta_0\right\} \\
&= S\left(S-\frac{1}{2}\right)\frac{\sin^2\theta_0}{2}\left\{1 - e^{-4\beta} - \left(3 + e^{-4\beta} - 4e^{-\beta}\right)\cos^2\theta_0\right\}. \qquad \text{(B.1)}
\end{aligned}$$

Hereafter, we assume $S(S-1/2) \approx S^2$ for large enough particle number N ($= 2S > 100$). Similarly, we have $\mathcal{B} \approx 2S^2\tau\sin^2(\theta_0)e^{-\beta}$, and

$$\mathcal{C} \approx \frac{S^2\sin^2\theta_0}{2}\left\{1 - e^{-4\beta} + (3 + e^{-4\beta} - 4e^{-\beta})\cos^2\theta_0\right\} + S. \qquad \text{(B.2)}$$

Following Kitagawa and Ueda [4], we focus on a time regime $\tau < S\tau^2 \ll 1$ and $\gamma\tau \ll 1$, which allows us to expand the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in terms of β because of $\beta \ll 1$. Firstly, we calculate the product of the variances V_+V_-, which is given by Eq. (6),

$$V_+V_- = \frac{1}{4}\left[(\mathcal{C}+\mathcal{A})(\mathcal{C}-\mathcal{A}) - \mathcal{B}^2\right]. \qquad \text{(B.3)}$$

To calculate it, we expand $\mathcal{C} \pm \mathcal{A}$ and $\mathcal{B}^2$ up to the third-order of β (also $\gamma\tau$) and obtain

$$\begin{aligned}
&\mathcal{B}^2 \approx 4S^3\sin^2(\theta_0)\left(\beta - \gamma\tau\right)\left(1 - 2\beta + 2\beta^2\right), \\
&\mathcal{C} + \mathcal{A} \approx 4S^2\sin^2(\theta_0)\beta\left(1 - 2\beta + \frac{8}{3}\beta^2\right) + S, \qquad \text{(B.4)} \\
&\mathcal{C} - \mathcal{A} \approx 6S^2\sin^2(\theta_0)\cos^2(\theta_0)\beta^2\left(1 - \frac{5}{3}\beta\right) + S.
\end{aligned}$$

Keeping the terms up to $O[(S\beta)^3]$, we obtain

$$\begin{aligned}
V_+V_- &\approx \frac{S^2}{4}\left\{1 + 4S\sin^2(\theta_0)\left[\frac{2}{3}\beta^3 + 6S\sin^2(\theta_0)\cos^2(\theta_0)\beta^3\right.\right. \\
&\quad \left.\left. +\gamma\tau + \frac{3}{2}\cos^2(\theta_0)\beta^2\left(1 - \frac{5}{3}\beta\right)\right]\right\}. \qquad \text{(B.5)}
\end{aligned}$$

For brevity, we will omit the last term $3\cos^2(\theta_0)\beta^2(\cdots)/2$, though its contribution may be larger than that of the term $2\beta^3/3$.

Next, we calculate the increased variance V_+. From Eq. (B.1)-Eq. (B.2), we note that the leading terms of the coefficients $\mathcal{A} \propto S^2\beta$ ($\propto S^3\tau^2$) and $\mathcal{B} \propto S^2\tau$. In the time scale with $S^2\tau > 1$, it is easy to find that $\mathcal{A} > \mathcal{B}$, so the increased variance can be simplified as

$$V_+ \approx \frac{1}{2}(\mathcal{C} + \mathcal{A}) \approx 2S^2 \sin^2(\theta_0)\beta. \tag{B.6}$$

where we only keep the lowest-order of β in the last step [see also Eq. (B.4)]. Finally, using $V_- = (V_+V_-)/V_+$, we obtain analytical result of the reduced variance

$$V_- \approx \frac{S}{2}\left\{\frac{1}{4S\sin^2(\theta_0)\beta} + \frac{\gamma\tau}{\beta} + \frac{2}{3}\beta^2\left[1 + \frac{9S}{4}\sin^2(2\theta_0)\right]\right\}, \tag{B.7}$$

which gives the analytical result of the squeezing parameter, *i.e.*, Eq. (18).