Quantum Information and Computation, Vol. 13, No. 11&12 (2013) 0986–0994

UPPER BOUNDS ON MIXING RATES

ELLIOTT H. LIEB[a]
Departments of Mathematics and Physics, Princeton University, Jadwin Hall
Princeton, NJ 08544 USA

ANNA VERSHYNINA[b]
Department of Physics, Princeton University, Jadwin Hall
Princeton, NJ 08544 USA

Received February 22, 2013
Revised April 3, 2013

We prove upper bounds on the rate, called "mixing rate", at which the von Neumann entropy of the expected density operator of a given ensemble of states changes under non-local unitary evolution. For an ensemble consisting of two states, with probabilities of p and $1-p$, we prove that the mixing rate is bounded above by $4\sqrt{p(1-p)}$ for any Hamiltonian of norm 1. For a general ensemble of states with probabilities distributed according to a random variable X and individually evolving according to any set of bounded Hamiltonians, we conjecture that the mixing rate is bounded above by a Shannon entropy of a random variable X. For this general case we prove an upper bound that is independent of the dimension of the Hilbert space on which states in the ensemble act.

Keywords: Mixing rate, Entanglement rate, von Neumann entropy, convex

Communicated by: I Cirac & J Eisert

1 Introduction

The problem addressed in this paper is, given an ensemble of states, $\mathcal{E}$, to find an the upper bound on the rate, $\Lambda(\mathcal{E})$, at which the von Neumann entropy of the expected density operator of this ensemble changes under non-local unitary evolution. The conjecture is known as 'Small Incremental Mixing' for an ensemble consisting of two states, $\mathcal{E}_2$, and it states that the mixing rate is bounded above by a binary entropy $S(p) = -p\ln p - (1-p)\ln(1-p)$, where p and $1-p$ are the probabilities of the two states in the ensemble. The problem, to our knowledge, was first introduced by Bravyi in [1].

We prove, Theorem 1, that for any ensemble consisting of two states, the mixing rate is bounded above by the following constant, which is independent of the dimension of the Hilbert space these states act on (including infinite dimension):

$$\Lambda(\mathcal{E}_2) \le 4\sqrt{p(1-p)}.$$

[a]Work partially supported by U.S. National Science Foundation grant PHY 0965859 and a grant from the Simons Foundation # 230207.
[b]Work partially supported by U.S. National Science Foundation grant PHY 0965859.

This bound has a shape similar to that of the binary entropy, which appears in the conjecture, up to a factor of 2. But, unfortunately, our $\sqrt{p}$ behavior near $p = 0$ is significantly worse than $p \ln p$.

Bravyi proved [1] the Small Incremental Mixing conjecture for a special case in which the expected density operator has at most two distinct eigenvalues, of arbitrary multiplicity. He could bound the mixing rate by 6 times the binary entropy $S(p)$. For a general case he gave the dimension-independent and p-independent bound of 2, which was the best dimension-independent bound until now. See Section 2 for the discussion of this conjecture.

In our paper, see Section 3, we generalize the problem to the ensemble of any number of states, $\mathcal{E}$, not only two, and conjecture that the upper bound should be a Shannon entropy of the random variable X, according to which the probabilities are distributed in the ensemble,

$$\Lambda(\mathcal{E}) \leq S(X).$$

We prove that the mixing rate has an upper bound independent of the dimension of the Hilbert space the states act on. See Theorem 2 for the formulation of the result and Section 4 for the proof of the upper bound.

Bravyi introduced the Small Incremental Mixing problem as a generalization of the 'Small Incremental Entanglement' conjecture, [1]. According to Bravyi, the latter conjecture was first proposed by Kitaev in a private communications to him. It bounds the rate of change of an entanglement between two parties when the system evolves under a non-local unitary evolution. The conjecture states that the upper bound is $c \ln d$, where d is a dimension of a system of either party and c is a constant independent of either dimension. See Section 2 for the discussion of this conjecture. The proof of the conjecture with the constant $c = 4$ can be found in [2, 3]. One would hope to improve the constant 4 in this proof.

The question of bounding a mixing rate by a binary entropy for an ensemble of two states is still open. In the special case discussed by Bravyi [1], one would hope to improve the constant 6 in front of the binary entropy. A conjecture of bounding a mixing rate by a Shannon entropy for a general ensemble is open as well.

The paper is organized as follows. In Section 2 we discuss original Small Incremental Mixing problem (for the ensemble consisting of two states), Small Incremental Entangling problem, the relation between the two and the progress on both problems. In Section 3 we generalize Small Incremental Mixing to a general ensemble consisting of any number of states, pose a new conjecture and provide our main result on the upper bound of the mixing rate, Theorem 2. Section 4 contains the proof of Theorem 2.

2 Preliminaries

Let $\mathcal{H}$ denote a D-dimensional Hilbert space (which could be infinite dimensional). Let $\mathcal{E}_2 = \{(p, \rho_1), (1-p, \rho_2)\}$ be a probabilistic ensemble of two states acting on $\mathcal{H}$. The expected density operator of this ensemble is a convex combination $\rho = p\rho_1 + (1-p)\rho_2$. For any Hamiltonian H (self-adjoint operator on $\mathcal{H}$) we can define a time dependent state

$$\rho(t) = p\rho_1 + (1-p)e^{-iHt}\rho_2 e^{iHt}.$$

That is H acts locally on ρ_2, but not on ρ_1.

The von Neumann entropy of this state is

$$S(\rho(t)) = -\mathrm{Tr}\Big(\rho(t)\ln\rho(t)\Big).$$

From the basic properties of von Neumann entropy, the following holds.

Small Total Mixing. (Binary case) *For any fixed ensemble $\mathcal{E}_2$, the entropy of a state $\rho(t)$ at any time t satisfies*

$$\overline{S}(\mathcal{E}_2) \le S(\rho(t)) \le \overline{S}(\mathcal{E}_2) + S(p),$$

where $\overline{S}(\mathcal{E}_2) = pS(\rho_1) + (1-p)S(\rho_2)$ is the average entropy of the ensemble and $S(p) = -p\ln p - (1-p)\ln(1-p)$ is a binary entropy.

The inequality is proved in Chapter 3 for a general ensemble of any number of states, see (4).

The analogue of the small total mixing for infinitely small times is formulated in terms of a mixing rate.

A ***mixing rate*** is defined as

$$\Lambda(\mathcal{E}_2, H) = \frac{dS(\rho(t))}{dt}\bigg|_{t=0}.$$

Conjecture 1. (Bravyi [1]) Small Incremental Mixing. *For any ensemble $\mathcal{E}_2 = \{(p,\rho_1),(1-p,\rho_2)\}$, the maximum mixing rate is bounded above by a binary entropy.*

$$\begin{aligned}\Lambda(\mathcal{E}_2) &:= \max\{|\Lambda(\mathcal{E},H)| : \|H\| = 1\}\\ &\le S(p) = -p\ln p - (1-p)\ln(1-p).\end{aligned}$$

Some useful formulas for the mixing rate are

$$\begin{aligned}\Lambda(\mathcal{E}_2, H) &= -ip\mathrm{Tr}([\rho_1, \ln\rho]H)\\ &= i(1-p)\mathrm{Tr}([\rho_2, \ln\rho]H),\end{aligned} \tag{1}$$

and

$$\begin{aligned}\Lambda(\mathcal{E}_2) &= p\mathrm{Tr}|[\rho_1, \ln\rho]|\\ &= p\|[\rho_1, \ln\rho]\|_1,\end{aligned}$$

here the maximum is achieved for $H = 1 - 2R$, with R being a projector on the negative eigenspace of $i[\rho_1, \ln\rho]$. The norm $\|\cdot\|_1$ is a trace-norm.

Bravyi [1] proved that $\Lambda(\mathcal{E}_2) \le 6S(p)$, where ρ has at most two distinct eigenvalues of arbitrary multiplicity.

Our result for an ensemble of two states is the following theorem.

Theorem 1. (Binary case) *For any binary ensemble $\mathcal{E}_2 = \{(p,\rho_1),(1-p,\rho_2)\}$, the maximum mixing rate is bounded above*

$$\Lambda(\mathcal{E}_2) \le 4\sqrt{p(1-p)}.$$

The proof of this theorem is given in Section 3 for a more general case, when the ensemble consists of any number of states, see Theorem 2.

Although we do not pursue this direction, we mention the following question posed by Audenaert and Kittaneh [4].

Problem 1. (Audenaert, Kittaneh [4]) *Let A and B be arbitrary positive semi-definite $D\times D$ matrices with $a=TrA$ and $b=TrB$. For what functions $f:\mathbb{R}\to\mathbb{R}$ does there exist a constant c, independent of d, A and B, such that*

$$\|[B,f(A+B)]\|_1\le c(F(a+b)-F(a)-F(b)),$$

where $F(x)=\int_0^x f(y)dy$?

Note that if $A=p\rho_1$, $B=(1-p)\rho_2$ and $f(x)=\ln x$, Problem 1 becomes Conjecture 2.

As mentioned in the introduction, the Small Incremental Mixing problem is a generalization of the Small Incremental Entangling conjecture. To formulate the later conjecture, we suppose that two parties, say Alice and Bob, have control over systems A and B. Both systems evolve according to a non-local Hamiltonian H_{AB}. In time entanglement between A and B can be generated. In ancilla-assisted entangling both parties have access to additional subsystems, called local ancillas, i.e. Alice is in control of two systems A and a and Bob is in control of B and b. Alice and Bob start with a pure state $\rho(0)=|\Psi\rangle\langle\Psi|$ on the system $aABb$.

A time dependent joint state of Alice and Bob is

$$\rho(t)=U^*(t)|\Psi\rangle\langle\Psi|U(t),$$

where $U(t)=I_a\otimes e^{iH_{AB}t}\otimes I_b$ is a unitary transformation. The joint state of Alice and Bob is pure at any time.

One of the ways to describe the entanglement between Alice and Bob is to calculate the entanglement entropy

$$E(\rho(t)):=S(\rho_{aA}(t))=-\mathrm{Tr}\rho_{aA}(t)\ln\rho_{aA}(t),$$

where $\rho_{aA}(t)=\mathrm{Tr}_{Bb}\rho(t)$ is a state that Alice has after time t. Since the joint state is pure, the entanglement entropy is also can be calculated from the state that Bob has $E(\rho(t))=S(\rho_{Bb}(t))$.

Small Total Entangling. *The total change of the entanglement $E(\rho(t))$ is at most $2\ln d$, where $d=\min\{\dim(A),\dim(B)\}$.* See [8] for the proof.

A problem of bounding the infinitesimal change of the entanglement is formulated using the entangling rate.

The ***entangling rate*** is defined by

$$\Gamma(\Psi,H)=\left.\frac{dE(\rho(t)}{dt}\right|_{t=0}.$$

After calculating the derivative, the entangling rate can be expressed as

$$\begin{aligned}\Gamma(\Psi,H)&=-i\mathrm{Tr}\Big(H_{AB}[\rho_{aAB},\ln(\rho_{aA})\otimes I_B]\Big)\\&=-i\mathrm{Tr}\Big(H_{AB}[\rho_{aAB},\ln(\rho_{aA}\otimes\frac{I_B}{\dim(B)})]\Big).\end{aligned}\tag{2}$$

Similarly, $\Gamma(\Psi, H) = -i\mathrm{Tr}\Big(H_{AB}[\rho_{ABb}, I_A \otimes \ln(\rho_{Bb})]\Big)$

Conjecture 2. (Bravyi [1]) Small Incremental Entangling. *There is a universal constant c such that for all dimensions of ancillas a, b and for all states $|\Psi\rangle$, the following holds*

$$\Gamma(\Psi, H) \leq c\|H\| \ln d,$$

where $d = \min\{\dim(A), \dim(B)\}$.

This problem was studied by many authors. For the case when A and B are quibits, Childs et al [5] give upper bounds for the entangling rate and show that they are independent of the ancillas a and b. Wang and Sanders [6] proved that $\Gamma(H) := \max_\Psi \Gamma(\Psi, H) \leq \beta\|H\|$, where $\beta \approx 1.9123$, for an uncorrelated Hamiltonian $H = H_A \otimes H_B$, when $H_{A(B)} = H^{-1}_{A(B)}$. Child et al [7] also proved an upper bound $\Gamma(H) = \beta\frac{1}{4}\Delta_A\Delta_B \leq \beta\|H\|$ for the ancilla-assisted case and for an arbitrary uncorrelated bipartite Hamiltonian $H = H_A \otimes H_B$, where Δ_A (Δ_B) is the difference between the largest and smallest eigenvalues of H_A (H_B). For an arbitrary bipartite Hamiltonian Bennet et al [8] proved that $\Gamma(H) \leq cd^4\|H\|$, where c does not depend on a or b. Finally, the conjecture was proved by Hutter and Wehner [2], [3], where they showed that for any initial state $\rho(0)$ (pure or mixed) and any Hamiltonian H, $\Gamma(\rho(0), H) \leq 4\|H\| \ln d$.

Bravyi [1] proved that Small Incremental Mixing with a constant c in front of the Shannon entropy implies Small Incremental Entangling with a constant $4c$, by choosing particular ensemble of states: $\mathcal{E}_2 = \{((1 - \dim(B)^{-2}), \mu_{aAB}), \dim(B)^{-2}, \rho_{aAB})\}$. Here without loss of generality it was assumed that $B \leq A$ and μ_{aAB} is a state such that the expected density operator of the ensemble is of the form appearing in (2)

$$\rho_{aA} \otimes \frac{I_B}{B} = \Big(1 - \dim(B)^{-2}\Big)\mu_{aAB} + \dim(B)^{-2}\rho_{aAB}. \tag{3}$$

In Lemma 1 [1] it was proved that such a state μ_{aAB} exists. Applying (1) to this ensemble, we have that $\Lambda(\mathcal{E}_2, H) = \dim(B)^{-2}\Gamma(\Psi, H)$, which shows that Small Incremental Mixing implies Small Incremental Entangling.

Using Bravyi's proof, our bound of $4\sqrt{p(1-p)}$ for the Small Incremental Mixing problem leads to a bound of $4d\|H\|$ in Small Incremental Entangling.

3 Small Incremental Mixing for an ensemble consisting of any number of states

We generalize the Small Incremental Mixing problem to an ensemble of any number of states in the following way.

Let X be a random variable with probability density $p_X(x)$, i.e. the probability that the realization x occurs is $p_X(x)$, where the realization x belongs to a set $\mathcal{X}$. Consider a probabilistic ensemble of states $\mathcal{E} = \{p_X(x), \rho_x\}_{x\in\mathcal{X}}$, i.e. ρ_x is a density matrix on a Hilbert space $\mathcal{H}$ of arbitrary dimension (including infinite dimension), which occurs with probability $p_X(x)$, where $\sum_x p_X(x) = 1$.

The expected density operator of the ensemble $\mathcal{E}$ is convex combination of density matrices $\rho = \sum_x p_X(x)\rho_x$. For any collection of Hamiltonians $\mathbf{H} = \{H_x\}_{x\in\mathcal{X}}$ define a time-dependent state

$$\rho(t) = \sum_{x\in\mathcal{X}} p_X(x) e^{-iH_x t}\rho_x e^{iH_x t}.$$

Note that one of the states could always be left invariant, i.e. one of the Hamiltonians could always be taken as an identity I, but to simplify the notation we shall write a time evolution for all states.

The von Neumann entropy $S(\rho(t))$ of this state satisfies the following property.

Small Total Mixing (General case). *For any fixed ensemble $\mathcal{E}$, the entropy of a state $\rho(t)$ at any time t satisfies*

$$\overline{S}(\mathcal{E}) \leq S(\rho(t)) \leq \overline{S}(\mathcal{E}) + S(X), \tag{4}$$

where $\overline{S}(\mathcal{E}) = \sum_x p_X(x) S(\rho_x)$ is the average entropy of an ensemble $\mathcal{E}$ and $S(X) = -\sum_x p_X(x) \ln p_X(x)$ is a Shannon entropy of a classical random variable X.

The lower bound follows from concavity property of the von Neumann entropy and the invariance of the entropy under unitary transformation

$$\begin{aligned}\overline{S}(\mathcal{E}) = \sum_x p_X(x) S(\rho_x) &= \sum_x p_X(x) S(e^{-iH_x t}\rho_x e^{iH_x t}) \\ &\leq S\Big(\sum_{x\in\mathcal{X}} p_X(x) e^{-iH_x t}\rho_x e^{iH_x t}\Big) = S(\rho(t)).\end{aligned}$$

To see the upper bound, form a classical-quantum state

$$\rho^{XA}(t) = \sum_x p_X(x)|x\rangle\langle x|^X \otimes \rho_x(t),$$

that acts on a tensor product of a classical space X and a quantum system A represented by a Hilbert space $\mathcal{H}$. Here, $\rho_x(t) = e^{-iH_x t}\rho_x e^{iH_x t}$. The entropy of the classical-quantum state is

$$S(\rho^{XA}(t)) = S(X) + \overline{S}(\mathcal{E}).$$

For a classical-quantum state ρ^{XA} the relative entropy $S(X|A) := S(\rho^{XA}) - S(\rho^A) \geq 0$ is always non-negative, therefore $S(\rho(t)) \leq S(\rho^{XA}(t))$. This proves the Small Total Mixing property.

The analogue of the small total mixing for infinitely small times is formulated in terms of a mixing rate.

A ***mixing rate*** is defined similarly to the binary case as

$$\Lambda(\mathcal{E}, \mathbf{H}) = \frac{dS(\rho(t))}{dt}\bigg|_{t=0}. \tag{5}$$

Conjecture 3. Small Incremental Mixing. *For any ensemble $\mathcal{E} = \{(p_X(x), \rho_x)\}_{x\in\mathcal{X}}$, the maximum mixing rate is bounded above by a Shannon entropy.*

$$\begin{aligned}\Lambda(\mathcal{E}) :&= \max\{|\Lambda(\mathcal{E}, \mathbf{H})| : -I \leq H_x \leq I, x \in \mathcal{X}\} \\ &\leq S(X) = -\sum_x p_X(x) \ln p_X(x).\end{aligned} \tag{6}$$

A mixing rate can be written explicitly by calculating a derivative of the entropy $\frac{dS}{dt}(\rho(t)) = -\text{Tr}\Big(\frac{d\rho(t)}{dt}\ln\rho(t)\Big)$, at $t=0$

$$\begin{aligned}\Lambda(\mathcal{E},\mathbf{H}) &= -i\sum_x p_X(x)\text{Tr}([H_x,\rho_x]\ln\rho) \\ &= -i\sum_x p_X(x)\text{Tr}(H_x[\rho_x,\ln\rho]).\end{aligned} \tag{7}$$

Note that in the definition of the mixing rate (6) the maximum is taken over all Hamiltonians H_x such that $-I \le H_x \le I$.

For any Hermitian operator A with $\text{Tr}(A) = 0$,

$$\max\{\text{Tr}(HA) : -I \le H \le I\} = 2\max\{\text{Tr}(HA) : 0 \le H \le I\}.$$

This property can be easily observed by expressing $H = 2R - I$, where $0 \le R \le I$.

Therefore the maximum in (6) can be taken over the non-negative Hamiltonians bounded above by identity operator.

$$\Lambda(\mathcal{E}) = 2\max\{|\Lambda(\mathcal{E},\mathbf{H})| : 0 \le H_x \le I, x\in\mathcal{X}\}. \tag{8}$$

Note that, similarly to the binary case, the maximum is achieved for the Hamiltonians H_x being a projector onto a positive eigenspace of $i[\rho_x,\ln\rho]$.

Bravyi's proof of the Small Incremental Mixing problem for an ensemble of two states and for a state ρ with binary spectrum can be easily generalized to the case of an ensemble containing any number of states with ρ still restricted to having a binary spectrum. Also the constant 6 in front of the Shannon entropy $S(X)$ remains.

In Section 3 we prove the following theorem, showing that the maximum mixing rate is bounded above by a constant independent of the dimension D of the Hilbert space $\mathcal{H}$ that states act on.

Theorem 2. (General case) *For a fixed ensemble $\mathcal{E} = \{p_X(x),\rho_x\}_{x\in\mathcal{X}}$ the maximum mixing rate (6) is bounded above*

$$\Lambda(\mathcal{E}) \le 2\sum_x\sum_{y\neq x}\sqrt{p_X(x)p_X(y)} = 2\Big(\sum_x\sqrt{p_X(x)}\Big)^2 - 2.$$

For a binary ensemble $\mathcal{E} = \{(p,\rho_1),((1-p),\rho_2)\}$ Theorem 2 gives the upper bound of $4\sqrt{p(1-p)}$, as claimed in Theorem 1.

4 Upper bound on the mixing rate

In this section we prove Theorem 2.

In eq. (7), express the logarithm $\log\rho$ by the formula

$$\ln x = \int_0^\infty\Big(\frac{1}{1+t} - \frac{1}{x+t}\Big)dt.$$

We may assume that $0 \leq H_x \leq I$ for every $x \in \mathcal{X}$, as noted before (8), to calculate the mixing rate. Then, for reasons explained below,

$$\begin{aligned}
\Lambda(\mathcal{E}, \mathbf{H}) &= -i \sum_x p_X(x) \mathrm{Tr}(H_x[\rho_x, \ln \rho]) \\
&= -i \sum_x p_X(x) \int_0^\infty \mathrm{Tr}\Big(H_x[\rho_x, \frac{1}{\rho+t}]\Big) dt \\
&= i \sum_x p_X(x) \int_0^\infty \mathrm{Tr}\Big(H_x \frac{1}{\rho+t}[\rho_x, \rho+t]\frac{1}{\rho+t}\Big) dt \\
&= i \sum_x \sum_{y\neq x} p_X(x)p_X(y) \int_0^\infty \mathrm{Tr}\Big(H_x \frac{1}{\rho+t}[\rho_x, \rho_y]\frac{1}{\rho+t}\Big) dt \\
&= i \sum_x \sum_{y\neq x} p_X(x)p_X(y) \int_0^\infty \mathrm{Tr}\Big([\rho_x, \rho_y]\frac{1}{\rho+t} H_x \frac{1}{\rho+t}\Big) dt.
\end{aligned}$$

Here in the third equality we used that $[A, B^{-1}] = B^{-1}[A, B]B^{-1}$. In the fourth equality we wrote ρ as a convex combination of states ρ_y and eliminated the commuting terms. In the last equality the cyclicity of the trace is used.

For any $0 \leq H \leq I$, we have $0 \leq (\rho+t)^{-1}H(\rho+t)^{-1} \leq (\rho+t)^{-2}$.

Therefore, continuing our calculations, with $K_x := \int_0^\infty (\rho+t)^{-1}H_x(\rho+t)^{-1}dt \leq \rho^{-1}$, $x \in \mathcal{X}$

$$\begin{aligned}
|\Lambda(\mathcal{E}, \mathbf{H})| &= \Big|\sum_x \sum_{y\neq x} p_X(x)p_X(y) \mathrm{Tr}\Big([\rho_x, \rho_y] \int_0^\infty \frac{1}{\rho+t} H_x \frac{1}{\rho+t} dt\Big)\Big| \\
&\leq \sum_x \sum_{y\neq x} p_X(x)p_X(y) \Big(|\mathrm{Tr}\rho_x\rho_y K_x| + |\mathrm{Tr}\rho_y\rho_x K_x|\Big) \\
&\leq \sum_x \sum_{y\neq x} p_X(x)p_X(y) \Big(\mathrm{Tr}|\sqrt{K_x}\rho_x\rho_y\sqrt{K_x}| + \mathrm{Tr}|\sqrt{K_x}\rho_y\rho_x\sqrt{K_x}|\Big) \\
&\leq 2 \sum_x \sum_{y\neq x} p_X(x)p_X(y) \sqrt{\mathrm{Tr}(\rho_x^2 K_x)\mathrm{Tr}(\rho_y^2 K_x)} \\
&\leq 2 \sum_x \sum_{y\neq x} \sqrt{p_X(x)p_X(y)} \sqrt{\mathrm{Tr}\Big(\rho_x (p_X(x)\rho_x)\rho^{-1}\Big)\mathrm{Tr}\Big(\rho_y (p_X(y)\rho_y)\rho^{-1}\Big)} \\
&\leq 2 \sum_x \sum_{y\neq x} \sqrt{p_X(x)p_X(y)} \\
&= 2\Big(\sum_x \sqrt{p_X(x)}\Big)^2 - 2
\end{aligned}$$

In the first inequality we put the absolute value inside the sums, wrote the commutator $[\rho_x, \rho_y] = \rho_x\rho_y - \rho_y\rho_x$ and used the triangular property of the absolute value. The second inequality follows from the cyclicity of the trace and by moving the absolute value inside the trace. The third inequality follows from Cauchy-Schwartz inequality for traces:

$$\mathrm{Tr}|AB| \leq \sqrt{\mathrm{Tr}(A^*A)\mathrm{Tr}(B^*B)}.$$

The fourth inequality follows from the upper bound on $K_x \leq \rho^{-1}$. The fifth inequality is obtained from the definition of ρ as a convex combination of non-negative density operators ρ_x, therefore $\rho \geq p_X(x)\rho_x$ for any $x \in \mathcal{X}$, and $\mathrm{Tr}\rho_x = 1$, $x \in \mathcal{X}$.

From (8) we obtain an upper bound for the mixing rate

$$\Lambda(\mathcal{E}) = 2 \max_{0 \leq H_x \leq I} \{|\Lambda(\mathcal{E}, \mathbf{H})|\} \leq 2\Big(\sum_x \sqrt{p_X(x)}\Big)^2 - 2.$$

Acknowledgments

We are grateful to Frank Verstraete for making us aware of the mixing rate problem and for his encouragement.

References

1. S. Bravyi (2007), "Upper bounds on entangling rates of bipartite Hamiltonians", Phys. Rev. A, **76**, 052319.
2. A. Hutter and S. Wehner (2012), "Almost all quantum states have low entropy rates for any coupling to the environment", Phys. Rev. Lett., 108, 070501.
3. A. Hutter (2011), "Understanding Equipartition and Thermalization from Decoupling", Master's Thesis, ETH Zurich.
4. K.M.R. Audenaert and F. Kittaneh (2012), "Problems and Conjectures in Matrix and Operator Inequalities", arXiv:1201.5232.
5. A. M. Childs, D. W. Leung, F. Verstraete and G. Vidal (2003), "Asymptotic entanglement capacity of the Ising and anisotropic Heisenberg interactions", Quant. Inf. Comp., 3, pp. 97.
6. X. Wang and B. Sanders (2003), "Entanglement capability of self-inverse Hamiltonian evolution", Phys. Rev. A, 68, 014301.
7. A. M. Childs, D. W. Leung and G. Vidal (2004), "Reversible simulation of bipartite Hamiltonians", IEEE Trans. Inf. Theory, vol 50, 6, pp. 1189.
8. C. H. Bennet, A. W. Harrow, D. W. Leung and J. A. Smolin (2003), "On the capacities of bipartite Hamiltonians and unitary gates", IEEE Trans. Inf. Theory, vol 49, no 8, pp. 1895.