Quantum Information and Computation, Vol. 12, No. 9&10 (2012) 0820–0842

ON NONBINARY QUANTUM CONVOLUTIONAL BCH CODES

GIULIANO G. LA GUARDIA
Department of Mathematics and Statistics, State University of Ponta Grossa (UEPG)
Ponta Grossa, PR 84030-900, Brazil[a]

Received August 4, 2011
Revised May 23, 2012

Several new families of nonbinary quantum convolutional Bose-Chaudhuri-Hocquenghem (BCH) codes are constructed in this paper. These code constructions are performed algebraically and not by computation search. The quantum convolutional codes constructed here have parameters better than the ones available in the literature and they have non-catastrophic encoders and encoder inverses. These new families consist of unit-memory as well as multi-memory convolutional stabilizer codes.

Keywords: convolutional codes; BCH codes; quantum codes
Communicated by: R Jozsa & B Terhal

1 Introduction

There exist many researchers and consequently several works available in the literature dealing with constructions of quantum error-correcting codes (QECC) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. As it is well known, the techniques more utilized for this purpose are the Calderbank-Shor-Steane (CSS) [1, 4], the Hermitian [1, 8] and the symplectic quantum code constructions [1, 2]. In fact, the theory of stabilizer codes has been extensively investigated throughout the last two decades.

In contrast with this subject of research one has the theory of (in general, unit-memory) quantum convolutional codes (QCC) [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Certainly the latter class of quantum codes has received less attention. To be more precise, Ollivier and Tillich [13, 14] were the first to develop the stabilizer structure for these codes. Almeida and Palazzo Jr. construct an $[(4, 1, 3)]$ quantum convolutional code [15]. Grassl and Rötteler [16, 17, 18] constructed new QCC as well as they provide algorithms to obtain non-catastrophic encoders for such class of codes. Forney, in a joint work with Guha and Grassl, constructed rate $(n-2)/n$ QCC. Wilde and Brun [22, 23] constructed entanglement-assisted quantum convolutional coding and Tan and Li [24] constructed QCC derived from LDPC codes. Recently, in [25], the authors have constructed minimal-memory encoders for quantum convolutional codes. They have introduced a new definition of quantum memory for QCC, which is more natural than the previous definitions. However, we adopt the definition of quantum (convolutional) memory presented in [19], since we have to compare the new code parameters with the ones shown in [19] because, to our best knowledge, the latter paper and

[a]gguardia@uepg.br

the present one are the only works available in the literature dealing with constructions of nonbinary QCC derived from (classical) BCH codes by means of algebraic methods.

As can be seen, there exist few works in the literature addressing the construction of QCC. Additionally, in many of them, there is no algebraic technique employed for this purpose. Because of these facts, in this paper the attention is focused on the construction of more families of QCC by means of algebraic methods. More precisely, we apply the well known method proposed by Piret [26] which was generalized by Aly *et al.* [19], which consists in the construction of classical convolutional codes derived from block codes.

The families of QCC constructed in this paper consist of codes whose parameters are better than the ones available in the literature. In other words, fixing the code length and the free distance, our codes have greater dimension than the dimension of QCC available in the literature. It is a consensus that constructions of codes with better parameters than the existing ones is a useful task; in particular, constructions of QCC with better parameters by means of algebraic methods are very difficult and useful tasks. Therefore, the first aim of the present paper is attained. A second contribution of this work is the construction of new families of multi-memory QCC, once multi-memory QCC are not much investigated in the literature. Additionally, as already mentioned, our constructions are performed algebraically and not by computation search, providing therefore (new) families of QCC and not only few specific codes.

The first construction generates quantum convolutional codes of length $n = q^4 - 1$, where $q \geq 3$ is a prime power, derived from the Hermitian construction, with parameters

- $[(n, n-4(i-2)-2, 1; 2, d_f \geq i+1)]_q$, $3 \leq i \leq q^2 - 1$;
- $[(n, n-4i-2, 1; 2j, d_f \geq i+j+2)]_q$, $1 \leq i = j$ and $2 \leq i+j \leq q^2 - 2$.

The second one generates quantum convolutional codes of length $n = q^{2m} - 1$, where $q \geq 4$ is a prime power and $m = ord_n(q^2) \geq 3$, derived from the Hermitian construction, with parameters

- $[(n, n-2m(2q^2-3)-2, 1; m, d_f \geq 2q^2+2)]_q$;
- $[(n, n-2mi-2, 1; mj, d_f \geq i+j+2)]_q$, where $1 \leq i = j \leq q^2 - 2$;
- $[(n, n-2m(i-1)-2, 1; m, d_f \geq i+2)]_q$, where $1 \leq i < q^2 - 1$;
- $[(n, n-2m(q^2-2)-2, 1; m, d_f \geq q^2+2)]_q$;
- $[(n, n-2m(i+q^2-2)-2, 1; m, d_f \geq i+q^2+2)]_q$, where $1 \leq i < q^2 - 1$;
- $[(n, n-2m(i-2)-2, 2; 2m, d_f \geq i+2)]_q$, where $3 \leq i < q^2 - 1$;
- $[(n, n-2m(i-\mu)-2, \mu; m\mu, d_f \geq i-\mu+4)]_q$, where $\mu \geq 3$ and $\mu+1 \leq i < q^2 - 1$.

Finally, the third construction proposed here generates convolutional stabilizer codes of length $n = q^m - 1$, where $q \geq 4$ is a prime power and $m = ord_n(q) \geq 3$, derived from the Euclidean construction, with parameters

- $[(n, n-2m(c-1)-2, 1; m, d_f \geq c+2)]_q$, where $2 \leq c = i+j \leq q-2$ and $i, j \geq 1$;

- $[(n, n-2mi-2, 1; mj, d_f \geq i+j+2)]_q$, where $1 \leq i = j \leq q-2$;
- $[(n, n-2m(q-2)-2, 1; m, d_f \geq q+2)]_q$;
- $[(n, n-2m(q-1), 1; m+1, d_f \geq q+3)]_q$;
- $[(n, n-2m(q-1)-2, 1; mj, d_f \geq q+j+2)]_q$, where $1 \leq j < q-1$;
- $[(n, n-2m(2q-3), 1; m, d_f \geq 2q+1)]_q$.

As can be seen above, we construct new families of unit-memory as well as multi-memory QCC, although we focus the attention on the construction of unit-memory codes, since they have parameters better than the multi-memory ones [27].

This paper is structured as follows. Section 2 presents a brief review of cyclic codes. Section 3 presents a review of convolutional codes. In Section 4, the structure of convolutional stabilizer codes is reviewed. In Section 5, we construct algebraically new families of quantum convolutional codes. In Section 6, we compare the new code parameters with the ones available in the literature. Finally, in Section 7, a summary of this paper is given.

2 Review of Cyclic codes

Notation. Throughout this paper, we assume that $q \neq 2$ is a prime power and F_q is a finite field with q elements. The code length is denoted by n and we always consider that $\gcd(q, n) = 1$. As usual, the multiplicative order of q modulo n is denoted by $m = ord_n(q)$, α denotes a primitive element of some extension field F_{q^m} (or $F_{q^{2m}}$, in the Hermitian case) and the minimal polynomial (over F_q) of an element $\alpha^j \in F_{q^m}$ is denoted by $M^{(j)}(x)$. The notation $\mathbb{C}_{[a]}$ denotes the cyclotomic coset containing a, where a is not necessarily the smallest number in the coset $\mathbb{C}_{[a]}$.

Let C be a cyclic code of length n over F_q. Then there exists only one monic polynomial $g(x)$ with minimal degree in C and $g(x)$ is a generator polynomial of C; $g(x)$ is a factor of $x^n - 1$. The dimension of C equals $n - \deg g(x)$.

Definition 1 *[36, pg. 202] Let α be a primitive n-th root of unity. A cyclic code C of length n over F_q is a BCH code with designed distance δ if, for some integer $b \geq 0$, we have*

$$g(x) = l.c.m.\{M^{(b)}(x), M^{(b+1)}(x), \ldots, M^{(b+\delta-2)}(x)\},$$

that is, $g(x)$ is the monic polynomial of smallest degree over F_q having $\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+\delta-2}$ as zeros. Therefore $c \in C$ if and only if $c(\alpha^b) = c(\alpha^{b+1}) = \ldots = c(\alpha^{b+\delta-2}) = 0$. Thus the code has a string of $\delta - 1$ consecutive powers of α as zeros. It is well known that from the BCH bound, the minimum distance of a BCH code is greater than or equal to its designed distance δ. A parity check matrix for C is given by

$$H_{\delta,b} = \begin{bmatrix} 1 & \alpha^b & \alpha^{2b} & \cdots & \alpha^{(n-1)b} \\ 1 & \alpha^{(b+1)} & \alpha^{2(b+1)} & \cdots & \alpha^{(n-1)(b+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(b+\delta-2)} & \cdots & \cdots & \alpha^{(n-1)(b+\delta-2)} \end{bmatrix},$$

where each entry is replaced by the corresponding column of m elements from F_q, where $m = ord_n(q)$. The rows of the resulting matrix over F_q are the parity checks satisfied by C.

Note that after replacing each entry by columns with m elements from F_q one must remove any linearly dependent rows. If $n = q^m - 1$ then the BCH code is called primitive and if $b = 1$ it is called narrow-sense. In this paper we will construct convolutional stabilizer codes derived from non-narrow-sense (classical) BCH codes.

3 Review of Convolutional Codes

Constructions of (classical) convolutional codes as well as their properties have been investigated in the literature [28, 29, 30, 31, 32, 33, 34]. As usual, we utilize the module theoretic approach to describe such class of codes. For more details the reader can consult [26, 35].

Recall that an $(n, k, \gamma)_q$ *convolutional code* C of length n, dimension k and *overall constraint length* γ over F_q is a free module of rank k that is a direct summand of $F_q[D]^n$. A matrix $G = (g_{ij}) \in F_q[D]^{k\times n}$ such that $C = \text{im } G = \{\mathbf{u}G \mid \mathbf{u} \in F_q[D]^k\}$ is called a *basic generator matrix* of C (similarly, one says that G is basic if G has a polynomial right inverse [32]). The overall constraint length γ is defined as $\gamma = \sum_{i=1}^{k} \gamma_i$, where $\gamma_i = \max_{1\leq j\leq n}\{\deg g_{ij}\}$. The *memory* μ of a convolutional code is the maximal value of γ_i. A basic generator matrix of a convolutional code C is called *reduced* (or minimal, see [32, 34]) if the overall constraint length $\gamma = \sum_{i=1}^{k} \gamma_i$ has the smallest value among all basic generator matrices of C. In this case one says that γ is the *degree* of the code.

In this paper we write the generator matrix in the form $G = G_0 + G_1 D + \ldots + G_\mu D^\mu$, where $G_i \in F_q^{k\times n}$. More precisely, we construct families of unit-memory [27] as well as multi-memory QCC.

Next we recall the *free distance* of a convolutional codes. For this, consider $F_q((D))$ to be the field of Laurent series whose elements are given by $\mathbf{v}(D) = \sum_i v_i D^i$, where $v_i \in F_q$ and $v_i = 0$ for $i \leq r$, for some $r \in \mathbb{Z}$. One can associate with a convolutional code C another module of the form $C^\infty = \{\mathbf{u}(D)G \mid \mathbf{u}(D) \in F_q((D))^k\}$. Let $\mathbf{v}(D) = (v_1(D), \ldots, v_n(D)) \in F_q((D))^n$, where $v_i(D) = \sum_j v_{ij} D^j$. Then one can identify $\mathbf{v}(D)$ with an element in $F_q^n((D))$ of the form $\sum_j \mathbf{v}_j D^j$, where $\mathbf{v}_j = (v_{1j}, \ldots, v_{nj}) \in F_q^n$. The weight of $\mathbf{v}(D)$ is defined as $\text{wt}(\mathbf{v}(D)) = \sum_{\mathbb{Z}} \text{wt}(v_i)$. A generator matrix G is called *catastrophic* if there exists a $\mathbf{u}(D) \in F_q((D))^k$ of infinite Hamming weight such that $\mathbf{u}(D)G \in C^\infty$ has finite Hamming weight.

The *free distance* d_f of a convolutional code C is defined as

$$d_f = wt(C) = \min\{wt(\mathbf{v}(D)) \mid \mathbf{v}(D) \in C, \mathbf{v}(D) \neq 0\}.$$

A rate k/n convolutional code (over F_q) with memory μ, degree γ and free distance d_f is denoted by $(n, k, \gamma; \mu, d_f)_q$.

Recall that the Euclidean inner product of two n-tuples $\mathbf{u}(D) = \sum_i \mathbf{u}_i D^i$ and $\mathbf{v}(D) = \sum_j \mathbf{v}_j D^j$ in $F_q[D]^n$ is defined as $\langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i$. If C is a convolutional code then $C^\perp = \{\mathbf{u}(D) \in F_q[D]^n \mid \langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle = 0 \text{ for all } \mathbf{v}(D) \in C\}$ denotes its Euclidean dual code. Similarly, the Hermitian inner product is defined as $\langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle_h = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i^q$, where $\mathbf{u}_i, \mathbf{v}_i \in F_{q^2}^n$ and $\mathbf{v}_i^q = (v_{1i}^q, \ldots, v_{ni}^q)$, and the Hermitian dual of C is denoted by $C^{\perp_h} = \{\mathbf{u}(D) \in F_{q^2}[D]^n \mid \langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle_h = 0 \text{ for all } \mathbf{v}(D) \in C\}$.

3.1 Convolutional Codes Derived from Block Codes

In this section we recall some known results available in the literature for performing the desired construction proposed here. The following results can be found in [19].

Assume that $[n,k,d]_q$ is a block code with parity check matrix H and split the matrix H into $\mu+1$ disjoint submatrices H_i, each of which has n columns, such that

$$H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{bmatrix}, \qquad (1)$$

obtaining therefore the polynomial matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \ldots + \tilde{H}_\mu D^\mu, \qquad (2)$$

where the number of rows of $G(D)$ equals the maximal number κ of rows among the matrices H_i. The matrices $\tilde{H}_i$ are obtained from the matrices H_i by adding zero-rows at the bottom such that the matrix $\tilde{H}_i$ has κ rows in total. Then $G(D)$ generates a convolutional code.

The following result shown in [19] generalizes the well-known result by Piret [26] that constructs convolutional codes derived from block codes:

Theorem 1 *[19, Theorem 3] Let $C \subseteq F_q^n$ be an $[n,k,d]_q$ linear code with parity check matrix $H \in F_q^{(n-k)\times n}$. Assume that H is partitioned into submatrices $H_0, H_1, \ldots, H_\mu$ as in eq. (1) such that $\kappa = rkH_0$ and $rkH_i \leq \kappa$ for $1 \leq i \leq \mu$. Define the polynomial matrix $G(D)$ as in eq. (2). Then we have:*
(a) The matrix $G(D)$ is a reduced basic generator matrix.
(b) If the code C contains its Euclidean dual $C^{\perp}$, respectively its Hermitian dual $C^{\perp_h}$, then the convolutional code $V = \{\mathbf{v}(D) = \mathbf{u}(D)G(D) \mid \mathbf{u}(D) \in F_q^{n-k}[D]\}$ is contained in its dual $V^{\perp}$, respectively its Hermitian dual $V^{\perp_h}$.
(c) Let d_f and $d_f^{\perp}$ respectively denote the free distances of V and $V^{\perp}$. Let d_i be the minimum distance of the code $C_i = \{\mathbf{v} \in F_q^n \mid \mathbf{v}\tilde{H}_i^t = 0\}$, and let $d^{\perp}$ denote the minimum distance of $C^{\perp}$. Then the free distances are bounded by $\min\{d_0 + d_\mu, d\} \leq d_f^{\perp} \leq d$ and $d_f \geq d^{\perp}$.

4 Review of Quantum Convolutional Codes

We begin this section by describing the concept of quantum convolutional codes. For more details the reader can consult [14, 21, 19, 20]. The stabilizer formalism presented here can be found in [19].

The stabilizer is given by a matrix of the form

$$S(D) = (X(D) \mid Z(D)) \in F_q[D]^{(n-k)\times 2n}$$

satisfying the symplectic orthogonality condition $X(D)Z(1/D)^t - Z(D)X(1/D)^t = 0$. If C is a quantum convolutional code defined by a stabilizer matrix given above, then n is called the frame size, k the number of logical qudits per frame, and k/n the rate of C. C can be utilized to encode a sequence of blocks with k qudits in each block (that is, each element in the sequence consists of k quantum systems each of which is q-dimensional) into a sequence of blocks with n qudits.

As already mentioned, in this paper we adopt the definition of memory of a QCC according to [19], that is,

$$\mu = \max_{1 \leq i \leq n-k, 1 \leq j \leq n} \{\max(\deg X_{ij}(D), \deg Z_{ij}(D))\},$$

as already explained in the introduction. However, we acknowledge the fact that the notion of memory of QCC has been redefined recently, in order to make the concept more natural [25]. Moreover, this new definition can be extended similarly to nonbinary alphabets (in [25], the authors consider the binary alphabet).

The notation $[(n, k, \mu)]_q$ denotes a quantum convolutional code where the parameters are defined above. One can identify $S(D)$ with the generator matrix of a self-orthogonal (classical) convolutional code over F_q (Euclidean case) or F_{q^2} (Hermitian case), in order to construct convolutional stabilizer codes in a natural way. The free distance d_f and the degree γ are defined similarly as in the case of classical convolutional codes, generating the notation $[(n, k, \mu; \gamma, d_f)]_q$. The following two lemmas show how to construct convolutional stabilizer codes derived from classical convolutional codes:

Lemma 1 *[19, Proposition 1] Let C be an $(n, (n-k)/2, \gamma; \mu)_q$ convolutional code such that $C \subset C^{\perp}$. Then there exists an $[(n, k, \mu; \gamma, d_f)]_q$ convolutional stabilizer code, where $d_f = wt(C^{\perp} \backslash C)$.*

Lemma 2 *[19, Proposition 2] Let C be an $(n, (n-k)/2, \gamma; \mu)_{q^2}$ convolutional code such that $C \subset C^{\perp_h}$. Then there exists an $[(n, k, \mu; \gamma, d_f)]_q$ convolutional stabilizer code, where $d_f = wt(C^{\perp_h} \backslash C)$.*

5 Code Constructions

In this section we present the contributions of this paper. As we will see in the following, several new families of good quantum convolutional codes derived from classical Hermitian as well as Euclidean self-orthogonal BCH codes are constructed. These families consist of codes generated algebraically and not by computational search. Additionally, these new QCC have parameters better than the ones available in the literature. Moreover, they have non-catastrophic encoders and encoder inverses since the corresponding generator matrices are basic.

Our constructions differ from the constructions given in [19] at least in two aspects: 1) in this paper we construct unit-memory and also multi-memory QCC, whereas in [19] only unit-memory QCC were constructed; 2) we make use directly of minimal polynomials in defining the BCH codes.

Our main strategy is given as follows. In order to construct families of unit-memory QCC, we construct a BCH code C with parity check matrix H (according with the notation of Subsection 3.1), a BCH code C_0 with parity check H_0 and a BCH code C_1 with parity check matrix H_1, in such a way that H splits into H_0 and H_1, that is,

$$H = \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}, \tag{3}$$

and after this procedure we apply Theorem 1. Similarly, for the construction of 2-memory QCC, we construct a BCH code C with parity check matrix H (according with the notation of Subsection 3.1), a BCH code C_0 with parity check H_0, a BCH code C_1 with parity check matrix H_1 and a BCH code C_2 with parity check matrix H_2, in such a way that H splits into H_0, H_1 and H_2, that is,

$$H = \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix}, \qquad (4)$$

and after this procedure we apply Theorem 1. The construction of multi-memory QCC is similar to these ones.

The first construction makes use of Hermitian self-orthogonal cyclic codes of length q^4-1; the second one deals with Hermitian self-orthogonal cyclic codes of length $q^{2m}-1$ and, the third one makes use of Euclidean self-orthogonal cyclic codes. Although the first and the second constructions are similar, we prefer to consider these two cases separately because the corresponding Lemmas 3 and 5 (shown [11]) used in the construction of QCC have different hypothesis.

5.1 Construction I - Codes of length q^4-1 over F_{q^2}

Here we focus on the construction of convolutional stabilizer codes of length q^4-1 over F_{q^2}. Let us now begin the construction of the new QCC. To proceed further we need some results available in [11].

Lemma 3 *Let $n = q^4-1$, where $q \geq 3$ is a prime power, and consider the (q^2-1) q^2-ary cyclotomic cosets modulo n given by*

$$\begin{aligned} &\mathbb{C}_{[q^2+1]}, \\ \mathbb{C}_{[q^2+2]} &= \{q^2+2, \quad 1+2q^2\}, \\ &\vdots \\ \mathbb{C}_{[2q^2-1]} &= \{2q^2-1, \quad 1+(q^2-1)q^2\}. \end{aligned}$$

Then the following results hold:

a) the q^2-ary coset $\mathbb{C}_{[q^2+1]}$ contains only one element;

b) each one of the other cosets contains two elements;

c) all these q^2-ary cyclotomic cosets are mutually disjoints.

Proof. See [11, Lemma 3.2.]. □

Lemma 4 *Let $n = q^4-1$ where $q \geq 3$ is a prime power. Let C be the cyclic code of length n over F_{q^2} generated by the product of the minimal polynomials*

$$M^{(q^2+1)}(x)M^{(q^2+2)}(x)\ldots M^{(q^2+j)}(x),$$

$1 \leq j \leq q^2-1$. *Then C is Hermitian self-orthogonal.*

Proof. See [11, Theorem III.1.]. □

At this point we are ready to show Theorem 2, one of the main results of this subsection.

Theorem 2 *Let $n = q^4 - 1$, where $q \geq 3$ is a prime power. Then there exist quantum convolutional codes with parameters $[(n, n-4(i-2)-2, 1; 2, d_f \geq i+1)]_q$, where $3 \leq i \leq q^2-1$.*

Proof. We know the equalities $\gcd(q, n) = 1$ and $ord_n(q) = 2$ hold. Assume first that C is the BCH code of length $n = q^4-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C = \langle M^{(q^2+1)}(x)M^{(q^2+2)}(x) \cdot \ldots\ldots \cdot M^{(q^2+i-1)}(x)M^{(q^2+i)}(x)\rangle,$$

where $3 \leq i \leq q^2 - 1$. A parity check matrix of C is obtained from the matrix

$$H_{i+1,q^2+1} = \begin{bmatrix} 1 & \alpha^{(q^2+1)} & \alpha^{2(q^2+1)} & \ldots & \alpha^{(n-1)(q^2+1)} \\ 1 & \alpha^{(q^2+2)} & \alpha^{2(q^2+2)} & \ldots & \alpha^{(n-1)(q^2+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(q^2+i-1)} & \ldots & \ldots & \alpha^{(n-1)(q^2+i-1)} \\ 1 & \alpha^{(q^2+i)} & \ldots & \ldots & \alpha^{(n-1)(q^2+i)} \end{bmatrix}$$

by expanding each entry as a column vector (in this case, containing 2 rows) over some $F_{q^2}-$basis β of F_{q^4} and then removing any linearly dependent rows. We denote this new matrix by H. From Lemma 3, C has parameters $[n, n-2(i-1)-1, d \geq i+1]_{q^2}$. Moreover, since C has dimension $n-2(i-1)-1$, H has $2(i-1)+1$ linearly independent rows.

We next consider that C_0 is the BCH code of length $n = q^4 - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_0 = \langle M^{(q^2+1)}(x)M^{(q^2+2)}(x) \cdot \ldots\ldots \cdot M^{(q^2+i-2)}(x)M^{(q^2+i-1)}(x)\rangle.$$

Analogously, C_0 has a parity check matrix derived from the matrix

$$H_{i,q^2+1} = \begin{bmatrix} 1 & \alpha^{(q^2+1)} & \alpha^{2(q^2+1)} & \ldots & \alpha^{(n-1)(q^2+1)} \\ 1 & \alpha^{(q^2+2)} & \alpha^{2(q^2+2)} & \ldots & \alpha^{(n-1)(q^2+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(q^2+i-2)} & \ldots & \ldots & \alpha^{(n-1)(q^2+i-2)} \\ 1 & \alpha^{(q^2+i-1)} & \ldots & \ldots & \alpha^{(n-1)(q^2+i-1)} \end{bmatrix}$$

by expanding each entry as a column vector (containing 2 rows) over some $F_{q^2}-$basis β of F_{q^4} and then removing any linearly dependent rows. After these operations the new matrix is denoted by H_0. Applying again Lemma 3, the code C_0 has parameters $[n, n-2(i-2)-1, d_0 \geq i]_{q^2}$. Since C_0 has dimension $n-2(i-2)-1$, H_0 has $2(i-2)+1$ linearly independent rows.

Let C_1 be the BCH code of length $n = q^4-1$ over F_{q^2}, generated by the minimal polynomial $M^{(q^2+i)}(x)$

$$C_1 = \langle M^{(q^2+i)}(x)\rangle.$$

C_1 has parameters $[n, n-2, d_1 \geq 2]_{q^2}$ and a parity check matrix of C_1 is given by expanding each entry of the matrix

$$H_{2,q^2+i} = \begin{bmatrix} 1 & \alpha^{(q^2+i)} & \alpha^{2(q^2+i)} & \ldots & \alpha^{(n-1)(q^2+i)} \end{bmatrix}$$

with respect to β. The new matrix is denoted by H_1 and since C_1 has dimension $n-2$, H_1 has 2 linearly independent rows.

We know that $\mathrm{rk}H_0 \geq \mathrm{rk}H_1$. Then we form the convolutional code V generated by the reduced basic (according to Theorem 1 Item (a)) generator matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D,$$

where $\tilde{H}_0 = H_0$ and $\tilde{H}_1$ is obtained from H_1 by adding zero-rows at the bottom such that $\tilde{H}_1$ has the number of rows of H_0 in total. By construction, V has dimension $2(i-2)+1$ and degree $\delta_V = 2$, so V has parameters $(n, 2(i-2)+1, 2; 1, d_{f^*})_{q^2}$. The Euclidean dual $V^{\perp}$ of the convolutional code V has dimension $n-2(i-2)-1$ and degree 2. Let us now compute the free distance $d_f^{\perp}$ of $V^{\perp}$. By Theorem 1 Item (c), the free distance of $V^{\perp}$ is bounded by $\min\{d_0+d_1, d\} \leq d_f^{\perp} \leq d$, where d_i is the minimum distance of the code $C_i = \{\mathbf{v} \in F_q^n \mid \mathbf{v}\tilde{H}_i^t = 0\}$. From construction one has $d \geq i+1$, $d_0 \geq i$ and $d_1 \geq 2$, so $d_f^{\perp} \geq i+1$ and $V^{\perp}$ has parameters $(n, n-2(i-2)-1, 2; \mu, d_f^{\perp} \geq i+1)_{q^2}$ for each $3 \leq i \leq q^2-1$. The codes $V^{\perp}$ and $V^{\perp_h}$ have the same degree as code (see the proof of Theorem 7 in [20]). Since wt($V^{\perp}$)=wt($V^{\perp_h}$), the convolutional code $V^{\perp_h}$ has parameters $(n, n-2(i-2)-1, 2; m^*, d_f^{\perp_h} \geq i+1)_{q^2}$. From Lemma 4 and from Theorem 1 Item (b), one has $V \subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n, n-4(i-2)-2, 1; 2, d_f \geq i+1)]_q$ convolutional stabilizer code, for each $3 \leq i \leq q^2-1$. □

The next theorem generates more new QCC:

Theorem 3 *Let $n = q^4-1$ where $q \geq 3$ is a prime power. Then there exist quantum convolutional codes with parameters $[(n, n-4i-2, 1; 2j, d_f \geq i+j+2)]_q$, where $1 \leq i = j$ and $2 \leq i+j \leq q^2-2$.*

Proof. Let C be the BCH code of length $n = q^4-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C = \langle M^{(q^2+1)}(x) M^{(q^2+2)}(x) \cdot \ldots\ldots \cdot M^{(q^2+i+j)}(x) M^{(q^2+i+j+1)}(x)\rangle,$$

where $1 \leq i = j$ and $2 \leq i+j \leq q^2-2$. C has a parity check matrix H. Suppose that C_0 is the BCH code of length $n = q^4-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_0 = \langle M^{(q^2+1)} \ldots M^{(q^2+i)}(x) M^{(q^2+i+1)}(x)\rangle,$$

where $1 \leq i = j$ and $2 \leq i+j \leq q^2-2$; C_0 has parity check matrix H_0. Suppose that C_1 is the BCH code of length $n = q^4-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_1 = \langle M^{(q^2+i+2)} \ldots M^{(q^2+i+j+1)}(x)\rangle,$$

where $1 \leq i = j$ and $2 \leq i+j \leq q^2-2$; C_1 has parity check matrix H_1. Applying Lemma 3 one can easily verify that C has parameters $[n, n-2(i+j)-1,\ d \geq i+j+2]_{q^2}$, C_0 has parameters $[n, n-2i-1, d_0 \geq i+2]_{q^2}$ and C_1 has parameters $[n, n-2j, d_1 \geq j+1]_{q^2}$.

The convolutional code V generated by the reduced basic generator matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D,$$

is a unit-memory convolutional code of dimension $2i+1$ and degree $\delta_V = 2j$, so V has parameters $(n, 2i+1, 2j; 1, d_{f^*})_{q^2}$. The convolutional code $V^{\perp_h}$ has parameters $(n, n-2i-1, 2j; \mu, d_f^{\perp_h} \geq i+j+2)_{q^2}$, where $i+j+2$ was found by applying Theorem 1 Item (c). From Lemma 4 and from Theorem 1 Item (b), one has $V \subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n, n-4i-2, 1; 2j, d_f \geq i+j+2)]_q$ convolutional stabilizer code, for each $1 \leq i = j$ and $2 \leq i+j \leq q^2-2$. □

Let us present an illustrative example of the construction given above:

Example 5.1 *In Theorem 2 consider that* $q = 5$ *and* $i = 8$. *Let* C *be the BCH code of length* 624 *over* F_{25}, *generated by*

$$C = \langle M^{(26)}(x)M^{(27)}(x)\ldots M^{(32)}(x)M^{(33)}(x)\rangle,$$

C_0 *be the BCH code of length* 624 *over* F_{25}, *generated by*

$$C_0 = \langle M^{(26)}(x)M^{(27)}(x)\ldots M^{(31)}(x)M^{(32)}(x)\rangle,$$

and suppose also that C_1 *is the BCH code of length* 624 *over* F_{25}, *generated by* $M^{(33)}(x)$. *Applying Theorem 2 one has an* $[(624, 598, 1; 2, d_f \geq 9)]_5$ *quantum convolutional code.*

Analogously, in Theorem 3, consider that $q = 5$ *and* $i = j = 3$. *Let* C *be the BCH code of length* 624 *over* F_{25}, *generated by*

$$C = \langle M^{(26)}(x)M^{(27)}(x)\ldots M^{(32)}(x)\rangle,$$

C_0 *be the BCH code of length* 624 *over* F_{25}, *generated by*

$$C_0 = \langle M^{(26)}M^{(27)}(x)M^{(28)}(x)M^{(29)}(x)\rangle,$$

and suppose that C_1 *is the BCH code of length* 624 *over* F_{25}, *generated by*

$$C_1 = \langle M^{(30)}\ldots M^{(32)}(x)\rangle.$$

Applying Theorem 3 one can get an $[(624, 610, 1; 6, d_f \geq 8)]_5$ *convolutional stabilizer code.*

5.2 Construction II - Hermitian BCH codes

In this subsection we apply similar technique which was developed in the previous subsection in order to obtain more convolutional stabilizer codes. Lemmas 5 and 6 are essentials for our constructions:

Lemma 5 *Suppose that* $n = q^{2m}-1$, *where* $q \geq 4$ *and* $m = ord_n(q^2) \geq 3$. *Let* $s = \sum_{i=0}^{m-1} (q^2)^i$. *Then the following hold:*

a) the q^2*-ary coset* $\mathbb{C}_{[s]}$ *has only one element;*

b) the q^2*-ary cosets* $\mathbb{C}_{[s+i]}$ *are mutually disjoints, where* $1 \leq i \leq q^2-1$;

c) the q^2*-ary cosets* $\mathbb{C}_{[s-j]}$ *are mutually disjoints, where* $1 \leq j \leq q^2-1$;

d) the q^2*-ary cosets of the forms* $\mathbb{C}_{[s+i]}$ *and* $\mathbb{C}_{[s-j]}$ *are mutually disjoints, where* $1 \leq i, j \leq q^2-1$;

e) the cosets of the form $\mathbb{C}_{[s+i]}$, *where* $1 \leq i \leq q^2 - 1$, *contain* m *elements;*

f) the cosets of the form $\mathbb{C}_{[s-j]}$ *contain* m *elements, where* $1 \leq j \leq q^2 - 1$.

Proof. See [11, Lemmas III.3, III.4. and III.5.]. □

Lemma 6 *Suppose that* $n = q^{2m} - 1$, *where* $q \geq 4$, $\gcd(q^2, n) = 1$ *and* $m = ord_n(q^2) \geq 3$. *Let* $s = \sum_{i=0}^{m-1} (q^2)^i$. *If* C *is the cyclic code generated by the product of the minimal polynomials*

$$M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+i)}(x) \cdot M^{(s-1)}(x) \ldots M^{(s-j)}(x),$$

for all $1 \leq i, j \leq q^2 - 1$, *then* C *is Hermitian self-orthogonal.*

Proof. See [11, Lemma III.6.]. □

Keeping these results in mind we are able to prove Theorems 4 and 5 and their respective corollaries:

Theorem 4 *Let* $n = q^{2m} - 1$, *where* $q \geq 4$ *is a prime power and* $m = ord_n(q^2) \geq 3$. *Then there exist quantum convolutional codes with parameters* $[(n, n - 2m(2q^2 - 3) - 2, 1; m, d_f \geq 2q^2 + 2)]_q$.

Proof. Clearly one has $\gcd(q, n) = 1$. Consider first that C is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+q^2-2)}(x)M^{(s+q^2-1)}(x) \cdot \\ \cdot M^{(s-1)}(x) \ldots M^{(s-q^2+1)}(x),$$

where $s = \sum_{i=0}^{m-1} (q^2)^i$. A parity check matrix of C is obtained from the matrix

$$H_{2q^2+2, s-q^2+1} = \begin{bmatrix} 1 & \alpha^{(s-q^2+1)} & \alpha^{2(s-q^2+1)} & \cdots & \alpha^{(n-1)(s-q^2+1)} \\ 1 & \alpha^{(s-q^2+2)} & \alpha^{2(s-q^2+2)} & \cdots & \alpha^{(n-1)(s-q^2+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(s-1)} & \cdots & \cdots & \alpha^{(n-1)(s-1)} \\ 1 & \alpha^{(s)} & \cdots & \cdots & \alpha^{(n-1)(s)} \\ 1 & \alpha^{(s+1)} & \cdots & \cdots & \alpha^{(n-1)(s+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(s+q^2-2)} & \cdots & \cdots & \alpha^{(n-1)(s+q^2-2)} \\ 1 & \alpha^{(s+q^2-1)} & \cdots & \cdots & \alpha^{(n-1)(s+q^2-1)} \end{bmatrix}$$

by expanding each entry as a column vector over some F_{q^2}−basis β of $F_{q^{2m}}$ and then removing any linearly dependent rows. We denote this new matrix by H. From Lemma 5, C has parameters $[n, n - 2m(q^2 - 1) - 1, d \geq 2q^2 + 2]_{q^2}$. Moreover, since C has dimension $n - 2m(q^2 - 1) - 1$, H has $2m(q^2 - 1) + 1$ linearly independent rows.

We next consider C_0 be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_0 = \langle M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+q^2-2)}(x) \cdot M^{(s-1)}(x) \ldots M^{(s-q^2+1)}(x) \rangle.$$

Analogously, C_0 has a parity check matrix derived from the matrix

$$H_{2q^2,s-q^2+1} = \begin{bmatrix} 1 & \alpha^{(s-q^2+1)} & \alpha^{2(s-q^2+1)} & \ldots & \alpha^{(n-1)(s-q^2+1)} \\ 1 & \alpha^{(s-q^2+2)} & \alpha^{2(s-q^2+2)} & \ldots & \alpha^{(n-1)(s-q^2+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(s-1)} & \ldots & \ldots & \alpha^{(n-1)(s-1)} \\ 1 & \alpha^{(s)} & \ldots & \ldots & \alpha^{(n-1)(s)} \\ 1 & \alpha^{(s+1)} & \ldots & \ldots & \alpha^{(n-1)(s+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{(s+q^2-2)} & \ldots & \ldots & \alpha^{(n-1)(s+q^2-2)} \end{bmatrix}$$

by expanding each entry as a column vector over some $F_{q^2}-$basis β of $F_{q^{2m}}$ and then removing any linearly dependent rows. After performing these operations the obtained matrix is denoted by H_0. Applying again Lemma 5, the code C_0 has parameters $[n, n-m(2q^2-3)-1, d_0 \geq 2q^2]_{q^2}$. Since C_0 has dimension $n-m(2q^2-3)-1$, H_0 has $m(2q^2-3)+1$ linearly independent rows.

Next, assume that C_1 is the BCH code of length $n = q^{2m}-1$ over F_{q^2}, generated by the minimal polynomial $M^{(q^2+i)}(x)$

$$C_1 = \langle M^{(s+q^2-1)}(x)\rangle.$$

From Lemma 5 C_1 has parameters $[n, n-m, d_1 \geq 2]_{q^2}$. A parity check matrix of C_1 is given by expanding each entry of the matrix

$$H_{2,s+q^2-1} = \begin{bmatrix} 1 & \alpha^{(s+q^2-1)} & \ldots & \ldots & \alpha^{(n-1)(s+q^2-1)} \end{bmatrix}$$

with respect to β. The new matrix is denoted by H_1 and since C_1 has dimension $n-m$, H_1 has m linearly independent rows.

We know that $\mathrm{rk}H_0 \geq \mathrm{rk}H_1$. The convolutional code V generated by the reduced basic (according to Theorem 1, Item (a)) generator matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D,$$

where $\tilde{H}_0 = H_0$ and $\tilde{H}_1$ is obtained from H_1 by adding zero-rows at the bottom such that $\tilde{H}_1$ has the number of rows of H_0 in total. By construction, V has parameters $(n, m(2q^2-3)+1, m; 1, d_{f^*})_{q^2}$. The Hermitian dual $V^{\perp_h}$ of the convolutional code V has dimension $n-m(2q^2-3)-1$ and degree m.

From construction one has $d \geq 2q^2+2$, $d_0 \geq 2q^2$ and $d_1 \geq 2$; so, by Theorem 1 Item (c), the free distance of $V^{\perp_h}$ satisfies $d_f^{\perp} \geq 2q^2+2$. Thus $V^{\perp_h}$ has parameters $(n, n-m(2q^2-3)-1, m; \mu, d_f^{\perp_h} \geq 2q^2+2)_{q^2}$. From Lemma 6 and by Theorem 1 Item (b), one has $V \subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n, n-2m(2q^2-3)-2, 1; m, d_f \geq 2q^2+2)]_q$ QCC. □

Theorem 5 also generates good QCC:

Theorem 5 *Let $n = q^{2m}-1$, where $q \geq 4$ is a prime power and $m = ord_n(q^2) \geq 3$. Then there exist quantum convolutional codes with parameters $[(n, n-2mi-2, 1; mj, d_f \geq i+j+2)]_q$, for each $1 \leq i = j \leq q^2-2$.*

Proof. Assume the same notation used in the proof of Theorem 4. We know that $\gcd(q,n)=1$. Consider first that C is the BCH code of length $n=q^{2m}-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C=\langle M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i)}(x)\cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x)\rangle,$$

where $s=\sum_{i=0}^{m-1}(q^2)^i$, $1\leq i=j\leq q^2-2$. C has a parity check matrix H.

Let C_0 be the BCH code of length $n=q^{2m}-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_0=\langle M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i)}(x)\rangle,$$

where $1\leq i=j\leq q^2-2$. C_0 has parity check matrix H_0.

Next, suppose that C_1 is the BCH code of length $n=q^{2m}-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$C_1=\langle M^{(s-1)}(x)\ldots M^{(s-j)}(x)\rangle,$$

where $1\leq i=j\leq q^2-2$. C_1 has parity check matrix H_1. Applying Lemma 5 one can easily verify that C has parameters $[n,n-m(i+j)-1,\ d\geq i+j+2]_{q^2}$, C_0 has parameters $[n,n-mi-1,d_0\geq i+2]_{q^2}$ and C_1 has parameters $[n,n-mj,d_1\geq j+1]_{q^2}$.

The convolutional code V generated by the reduced basic generator matrix

$$G(D)=\tilde{H}_0+\tilde{H}_1D,$$

is a unit-memory convolutional code of dimension $mi+1$ and degree $\delta_V=mj$, so V has parameters $(n,mi+1,mj;1,d_{f^*})_{q^2}$. The convolutional code $V^{\perp_h}$ has parameters $(n,n-mi-1,mj;\mu,d_f^{\perp_h}\geq i+j+2)_{q^2}$, where $i+j+2$ was found by applying Theorem 1 Item (c). From Lemma 4 and from Theorem 1 Item (b), one has $V\subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n,n-2mi-2,1;mj,d_f\geq i+j+2)]_q$ QCC, for each $1\leq i=j\leq q^2-2$. □

Corollary 1 *Let $n=q^{2m}-1$, where $q\geq 4$ is a prime power and $m=\ ord_n(q^2)\geq 3$. Then there exist convolutional stabilizer codes with parameters*

a) $[(n,n-2m(i-1)-2,1;m,d_f\geq i+2)]_q$, *for each* $1\leq i<q^2-1$*;*

b) $[(n,n-2m(q^2-2)-2,1;m,d_f\geq q^2+2)]_q$*;*

c) $[(n,n-2m(j+q^2-2)-2,1;m,d_f\geq j+q^2+2)]_q$, *for each* $1\leq j<q^2-1$.

Proof.

a) Let C be the BCH code of length $n=q^{2m}-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-1)}(x)M^{(s+i)}(x),$$

where $1\leq i<q^2-1$; C has parameters $[n,n-mi-1,d\geq i+2]_{q^2}$, $1\leq i<q^2-1$, and parity check matrix H. Let C_0 be the BCH code of length $n=q^{2m}-1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-1)}(x),$$

where $1 \leq i < q^2 - 1$; C_0 has parameters $[n, n - m(i-1) - 1, d_0 \geq i+1]_{q^2}$, where $1 \leq i < q^2 - 1$, and parity check matrix H_0. Next, consider that C_1 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the minimal polynomial

$$M^{(s+i)}(x);$$

C_1 has parameters $[n, n - m, d_1 \geq 2]_{q^2}$ and parity check matrix H_1. The convolutional code V generated by the reduced basic generator matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D$$

has parameters $(n, m(i-1)+1, m; 1, d_{f^*})_{q^2}$. The convolutional code $V^{\perp_h}$ has parameters $(n, n-m(i-1)-1, m; \mu, d_f^{\perp_h} \geq i+2)_{q^2}$. From Lemma 6 and by Theorem 1 Item (b), one has $V \subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n, n - 2m(i-1) - 2, 1; m, d_f \geq i+2)]_q$ QCC, for each $1 \leq i < q^2 - 1$.

b) Let C be the BCH code of length $n = q^{2m} - 1$ over F_{q^2} generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x) \cdot \ldots \cdot M^{(s+q^2-2)}(x)M^{(s+q^2-1)}(x);$$

C has parameters $[n, n - m(q^2-1) - 1, d \geq q^2 + 2]_{q^2}$ and parity check matrix H. Consider that C_0 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2} generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+q^2-2)}(x);$$

C_0 has parameters $[n, n - m(q^2-2) - 1, d_0 \geq q^2]_{q^2}$ and and parity check matrix H_0. Now let C_1 be the BCH code of length $n = q^{2m} - 1$ over F_{q^2} with parity check H_1, generated by the minimal polynomial

$$M^{(s+q^2-1)}(x).$$

Proceeding similarly as in the proof of Item a), one has an $[(n, n - 2m(q^2-2) - 2, 1; m, d_f \geq q^2 + 2)]_q$ QCC.

c) Let C be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s-1)}(x) \ldots M^{(s-[j-1])}(x)M^{(s-j)}(x) \cdot \\ \cdot M^{(s+1)}(x) \ldots M^{(s+q^2-1)}(x),$$

where $1 \leq j < q^2 - 1$, with parameters $[n, n - m(j + q^2 - 1) - 1, d \geq j + q^2 + 2]_{q^2}$. Let C_0 be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s-1)}(x) \ldots M^{(s-[j-1])}(x) \cdot M^{(s+1)}(x) \ldots M^{(s+q^2-1)}(x),$$

where $1 \leq j < q^2 - 1$, with parameters $[n, n - m(j + q^2 - 2) - 1, d_0 \geq j + q^2 + 1]_{q^2}$. Suppose also that C_1 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the minimal polynomial

$$M^{(s-j)}(x),$$

where $1 \leq j < q^2 - 1$, with parameters $[n, n - m, d_1 \geq 2]_{q^2}$. Proceeding similarly as in the proofs above one can get an $[(n, n - 2m(j + q^2 - 2) - 2, 1; m, d_f \geq j + q^2 + 2)]_q$ QCC for each $1 \leq j < q^2 - 1$.

□

Until now we only have constructed unit-memory convolutional stabilizer codes, since these codes have parameters better than the corresponding multi-memory ones [27]. However, the technique utilized here can also be applied to generate multi-memory convolutional codes. These constructions are possible due to Lemma 5, since such lemma provides the exact parameters of the corresponding classical block codes utilized in the proposed construction. Let us now present the constructions of families of multi-memory QCC.

Theorem 6 *(2-memory QCC) Let $n = q^{2m} - 1$, where $q \geq 4$ is a prime power and $m = ord_n(q^2) \geq 3$. Then there exist convolutional stabilizer codes with parameters $[(n, n - 2m(i - 2) - 2, 2; 2m, d_f \geq i + 2)]_q$, for each $3 \leq i < q^2 - 1$.*

Proof. Let C be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-2)}(x)M^{(s+i-1)}(x)M^{(s+i)}(x),$$

where $3 \leq i < q^2 - 1$. We know from Lemma 5 that C has parameters $[n, n - mi - 1, d \geq i + 2]_{q^2}$, where $3 \leq i < q^2 - 1$. The parity check matrix of C is the matrix H. Consider next that C_0 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-2)}(x),$$

where $3 \leq i < q^2 - 1$. From Lemma 5, C_0 has parameters $[n, n - m(i - 2) - 1,\ d_0 \geq i]_{q^2}$, $3 \leq i < q^2 - 1$. The parity check matrix of C_0 is H_0. Assume also that C_1 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the minimal polynomial

$$M^{(s+i-1)}(x)$$

and let C_2 be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the minimal polynomial

$$M^{(s+i)}(x),$$

where $3 \leq i < q^2 - 1$. Again, from Lemma 5, the codes C_1 and C_2 have parameters $[n, n - m, d_1 \geq 2]_{q^2}$ and $[n, n - m, d_2 \geq 2]_{q^2}$, respectively. The parity check matrices of C_1 and C_2 are, respectively, H_1 and H_2.

The convolutional code V generated by the reduced basic generator matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2$$

has parameters $(n, m(i-2)+1, 2m; 2, d_{f^*})_{q^2}$. Note that $\gamma = 2m$ because, from Lemma 5, since the q^2-ary coset $\mathbb{C}_{[s+i]}$ contains m elements it follows that H_2 have m linearly independent rows, and each of the first m linearly independent rows of $\tilde{H}_2$ has degree 2.

We know that $\text{rk}H_0 \geq \text{rk}H_1$ and $\text{rk}H_0 \geq \text{rk}H_2$. The convolutional code $V^{\perp_h}$ has parameters $(n, n-m(i-2)-1, 2m; \mu, d_f^{\perp_h} \geq i+2)_{q^2}$, where we compute the free distance $d_f^{\perp_h}$ by applying Theorem 1 (Item (c)). From Lemma 6 and by Theorem 1 Item (b), one has $V \subset V^{\perp_h}$. Applying Lemma 2, there exists an $[(n, n-2m(i-2)-2, 2; 2m, d_f \geq i+2)]_q$ QCC, for each $3 \leq i < q^2 - 1$. □

Theorem 6 can be generalized as follows:

Theorem 7 *(multi-memory QCC) Let $n = q^{2m} - 1$, where $q \geq 4$ is a prime power and $m = ord_n(q^2) \geq 3$. Then there exist convolutional stabilizer codes with parameters $[(n, n-2m(i-\mu)-2, \mu; m\mu, d_f \geq i-\mu+4)]_q$ QCC, where $\mu \geq 3$ and $\mu+1 \leq i < q^2-1$.*

Proof. Let C be the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i)}(x),$$

where $\mu \geq 3$ and $\mu+1 \leq i < q^2-1$. From Lemma 5, C has parameters $[n, n-mi-1, d \geq i+2]_{q^2}$, where $\mu+1 \leq i < q^2-1$. The parity check matrix of C is the matrix H. Consider next that C_0 is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-\mu)}(x),$$

where $\mu+1 \leq i < q^2-1$. From Lemma 5, C_0 has parameters $[n, n-m(i-\mu)-1, d_0 \geq i-\mu+2]_{q^2}$, $\mu+1 \leq i < q^2-1$. The parity check matrix of C_0 is H_0. Assume also that C_j, for $j = 1, \ldots, \mu$, is the BCH code of length $n = q^{2m} - 1$ over F_{q^2}, generated by the minimal polynomial

$$M^{(s+i-\mu+j)}(x),$$

where $\mu+1 \leq i < q^2-1$. The code C_j, for all $j = 1, \ldots, \mu$, has parameters $[n, n-m, d_j \geq 2]_{q^2}$. The parity check matrix of C_j is H_j for $j = 1, \ldots, \mu$. The convolutional code V generated by

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D + \ldots + \tilde{H}_\mu D^\mu$$

has parameters $(n, m(i-\mu)+1, m\mu; \mu, d_{f^*})_{q^2}$. We know that $\text{rk}H_0 \geq \text{rk}H_j$ for $j = 1, \ldots, \mu$. Moreover, $V^{\perp_h}$ has parameters $(n, n-m(i-\mu)-1, m\mu; \mu^*, d_f^{\perp_h} \geq i-\mu+4)_{q^2}$, Therefore, applying Lemma 2, there exists an $[(n, n-2m(i-\mu)-2, \mu; m\mu, d_f \geq i-\mu+4)]_q$ QCC, for each $\mu \geq 3$ and $\mu+1 \leq i < q^2-1$. □

Remark Note that (see Theorem 7) we also can consider that C is generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i)}(x) \cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x),$$

for all $1 \leq i, j \leq q^2 - 1$, because from Lemma 6, C is Hermitian self-orthogonal. After this we choose suitable range for μ (greater than displayed in Theorem 7), generating therefore more new QCC. Consequently, also in this case, the proposed construction method holds. We observe that we considered (in Theorem 7) C generated by $M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i)}(x)$ to simplify the task.

5.3 Construction III - Euclidean BCH codes

As in the previous subsection, we will construct good convolutional stabilizer codes derived from classical ones. Although the dual code is considered with respect to the Euclidean inner product, the technique applied here is similar to that of the previous subsections. Because of this fact we do not describe the totality of the proofs given below. Let us recall some results proved in [11]:

Lemma 7 *Suppose that* $n = q^m - 1$*, where* $q \geq 4$ *and* $m = \text{ord}_n(q) \geq 3$*. Let* $s = \sum_{i=0}^{m-1} q^i$*. Then the following hold:*

a) The q-ary coset $\mathbb{C}_{[s]}$ *has only one element;*

b) Each one of the q-ary cosets $\mathbb{C}_{[s+i]}$ *are mutually disjoints, where* $1 \leq i \leq q-1$*;*

c) Each one of the q-ary cosets $\mathbb{C}_{[s-j]}$ *are mutually disjoints, where* $1 \leq j \leq q-1$*;*

d) The q-ary cosets of the forms $\mathbb{C}_{[s+i]}$ *and* $\mathbb{C}_{[s-j]}$ *are mutually disjoints, where* $1 \leq i, j \leq q-1$*;*

e) The cosets of the form $\mathbb{C}_{[s+i]}$*, where* $1 \leq i \leq q-1$*, have* m *elements;*

f) The cosets of the form $\mathbb{C}_{[s-j]}$ *have* m *elements, where* $1 \leq j \leq q-1$*.*

Proof. See [11, Lemmas III.7., III.8. and III.9.]. □

Lemma 8 *Suppose that* $n = q^m - 1$*,* $q \geq 4$*,* $m = \text{ord}_n(q) \geq 3$ *and* $s = \sum_{i=0}^{m-1} q^i$*. If* C *is the cyclic code generated by the product of the minimal polynomials*

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+j)}(x) \cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x),$$

where $1 \leq j \leq q-1$*, then* C *is Euclidean self-orthogonal.*

Proof. See [11, Lemma III.10.]. □

Next we utilize Lemmas 7 and 8 in order to obtain more good quantum convolutional codes, as we will see in the sequel.

Theorem 8 *Let* $n = q^m - 1$*, where* $q \geq 4$ *and* $m = \text{ord}_n(q) \geq 3$*. Then there exist quantum convolutional codes with parameters* $[(n, n-2m(c-1)-2, 1; m, d_f \geq c+2)]_q$*, where* $2 \leq c = i+j \leq q-2$ *and* $i, j \geq 1$*.*

Proof. Assume that C is the BCH code of length $n = q^m - 1$ over F_q with parity check matrix H, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-1)}(x)M^{(s+i)}(x) \cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x),$$

where $2 \leq i+j = c \leq q-2$ and $i, j \geq 1$. Assume that C_0 is the BCH code of length $n = q^m - 1$ over F_q, with parity check matrix H_0, generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+i-1)}(x) \cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x),$$

where $2 \leq i + j = c \leq q - 2$ and $i, j \geq 1$. Let C_1 be the BCH code of length $n = q^m - 1$ over F_q, with parity check matrix H_1, generated by the minimal polynomial $M^{(s+i)}(x)$. Then there exists an $[(n, n - 2m(c-1) - 2, 1; m, d_f \geq c + 2)]_q$ QCC, where $2 \leq c = i + j \leq q - 2$ and $i, j \geq 1$. □

Theorem 9 *Let $n = q^m - 1$, where $q \geq 4$ is a prime power and $m = ord_n(q) \geq 3$. Then there exist quantum convolutional codes with parameters $[(n, n - 2mi - 2, 1; mj, d_f \geq i + j + 2)]_q$ convolutional stabilizer code, for each $1 \leq i = j \leq q - 2$.*

Proof. Consider first that C is the BCH code of length $n = q^m - 1$ over F_q generated by the product of the minimal polynomials

$$C = \langle M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+i)}(x) \cdot M^{(s-1)}(x) \ldots M^{(s-j)}(x)\rangle,$$

where $s = \sum_{i=0}^{m-1} q^i$, $1 \leq i = j \leq q - 2$. Let C_0 be the BCH code of length $n = q^m - 1$ over F_q generated by the product of the minimal polynomials

$$C_0 = \langle M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+i)}(x)\rangle,$$

where $1 \leq i = j \leq q - 2$, and suppose that C_1 is the BCH code of length $n = q^m - 1$ over F_q generated by the product of the minimal polynomials

$$C_1 = \langle M^{(s-1)}(x) \ldots M^{(s-j)}(x)\rangle,$$

where $1 \leq i = j \leq q - 2$. Then there exists an $[(n, n - 2mi - 2, 1; mj, d_f \geq i + j + 2)]_q$ QCC, for each $1 \leq i = j \leq q - 2$. □

Theorem 10 *Suppose that $n = q^m - 1$, where $q \geq 4$ and $m = ord_n(q) \geq 3$. Then there exist quantum convolutional codes with parameters*

a) $[(n, n - 2m(q-2) - 2, 1; m, d_f \geq q + 2)]_q$;

b) $[(n, n - 2m(q-1), 1; m + 1, d_f \geq q + 3)]_q$;

c) $[(n, n - 2m(q-1) - 2, 1; mj, d_f \geq q + j + 2)]_q$, *for each* $1 \leq j < q - 1$;

d) $[(n, n - 2m(2q-3), 1; m, d_f \geq 2q + 1)]_q$.

Proof. In all cases below, the codes are BCH of length $n = q^m - 1$ over F_q.

a) Consider that C is the BCH code generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+q-1)}(x),$$

C_0 is the BCH code generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x) \ldots M^{(s+q-2)}(x)$$

and C_1 is the BCH code generated by the minimal polynomial

$$M^{(s+q-1)}(x).$$

Applying Lemma 1, there exists an $[(n, n - 2m(q-2) - 2, 1; m, d_f \geq q + 2)]_q$ .QCC

b) Consider that C is the BCH code generated by the product of the minimal polynomials

$$M^{(s-1)}(x)M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+q-1)}(x),$$

C_0 is the BCH code generated by the product of the minimal polynomials

$$M^{(s+1)}(x)\ldots M^{(s+q-1)}(x)$$

and C_1 is the BCH code generated by the product of the minimal polynomials

$$M^{(s-1)}(x)M^{(s)}(x).$$

Then the code $V^{\perp}$ has parameters $(n, n-m(q-1), m+1; \mu, d_f^{\perp} \geq q+3)_q$ and there exists an $[(n, n-2m(q-1), 1; m+1, d_f \geq q+3)]_q$ QCC.

c) Let C be the BCH code generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+q-1)}(x)\cdot M^{(s-1)}(x)\ldots M^{(s-j)}(x),$$

C_0 be the BCH code generated by the product of the minimal polynomials

$$M^{(s)}(x)M^{(s+1)}(x)\ldots M^{(s+q-1)}(x)$$

and C_1 be the BCH code generated by the product of the minimal polynomials

$$M^{(s-1)}(x)\ldots M^{(s-j)}(x).$$

Therefore, there exists an $[(n, n-2m(q-1)-2, 1; mj, d_f \geq q+j+2)]_q$ QCC for each $1 \leq j < q-1$.

d) If C is the BCH code generated by the product of the minimal polynomials

$$\begin{gathered} M^{(s+1)}(x)\ldots M^{(s+q-1)}(x)\cdot \\ \cdot M^{(s-1)}(x)\ldots M^{(s-[q-2])}(x)M^{(s-[q-1])}(x), \end{gathered}$$

C_0 is the BCH code generated by the product of the minimal polynomials

$$M^{(s+1)}(x)\ldots M^{(s+q-1)}(x)\cdot M^{(s-1)}(x)\ldots M^{(s-[q-2])}(x)$$

and C_1 is the BCH code generated by the product of the minimal polynomial

$$M^{(s-[q-1])}(x),$$

one can obtain an $[(n, n-2m(2q-3), 1; m, d_f \geq 2q+1)]_q$ QCC.

□

6 Code Comparison

In this section we compare the parameters of the new quantum convolutional codes with the ones available in the literature. Since the unique families of quantum convolutional BCH codes available in the literature are the ones shown in [19], the parameters $[(n, k, \mu; \gamma, d_f)]_q$ refer to as the parameters of convolutional stabilizer codes constructed in such paper.

The new code parameters shown in Table 1 are derived from Construction I (see Theorem 3), Construction II (see Theorem 5) and from Construction III (see Theorem 9), respectively.

The criterion adopted to compare the codes is as follows: if the codes have the same code length and the same lower bound for the free distance, the code with greater dimension is better than the other. For example, the new $[(624, 598, 1; 12, d_f \geq 14)]_5$ convolutional stabilizer code is better than the $[(624, 592, 1; \gamma, d_f \geq 14)]_5$ code shown in [19] since these two codes have same code length (624) and same lower bound for the free distance (14), but the new code has greater dimension (598) than the dimension (592) of the $[(624, 592, 1; \gamma, d_f \geq 14)]_5$ code. According to the established criterion, it can be seen in Table 1 that the new code parameters are better than the ones shown in [19]. Additionally, all the parameters of the new convolutional stabilizer codes are determined.

7 Summary

In this paper we have constructed several new families of unit-memory as well as multi-memory quantum convolutional BCH codes. These families consist of codes whose parameters are better than the ones available in the literature. Moreover, our constructions are performed algebraically and not by exhaustively computational search.

Acknowledgment

We would like to thank the anonymous referees for their valuable comments and suggestions that improve significantly the quality and the presentation of this paper. We also would like to thank one of the referees for drawing our attention to [25]. This research has been partially supported by the Brazilian Agencies CAPES and CNPq.

References

1. A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane. Quantum error correction via codes over $GF(4)$. *IEEE Trans. Inform. Theory*, 44(4):1369–1387, July 1998.
2. A. Steane. Enlargement of Calderbank-Shor-Steane quantum codes. *IEEE Trans. Inform. Theory*, 45(7):2492–2495, November 1999.
3. G. D. Cohen, S. B. Encheva, and S. Litsyn. On binary constructions of quantum codes. *IEEE Trans. Inform. Theory*, 45(7):2495–2498, July 1999.
4. M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
5. A. Ashikhmin and E. Knill. Non-binary quantum stabilizer codes. *IEEE Trans. Inform. Theory*, 47(7):3065–3072, November 2001.
6. M. Grassl, T. Beth, and M. Rötteler. On optimal quantum codes. *Int. J. Quantum Inform.*, 2(1):757–766, 2004.
7. H. Chen, S. Ling, and C. P. Xing. Quantum codes from concatenated algebraic geometric codes. *IEEE. Trans. Inform. Theory*, 51(8):2915–2920, August 2005.

Table 1. Code Comparisons

The new codes	Codes shown in [19]
$[(n, n-4i-2, 1; 2j, d_f \geq i+j+2)]_q$	$[(n, k, \mu; \gamma, d_f)]_q$
$[(624, 614, 1; 4, d_f \geq 6)]_5$	$[(624, 612, 1; \gamma, d_f \geq 6)]_5$
$[(624, 610, 1; 6, d_f \geq 8)]_5$	$[(624, 608, 1; \gamma, d_f \geq 8)]_5$
$[(624, 606, 1; 8, d_f \geq 10)]_5$	$[(624, 604, 1; \gamma, d_f \geq 9)]_5$
$[(624, 602, 1; 10, d_f \geq 12)]_5$	$[(624, 596, 1; \gamma, d_f \geq 12)]_5$
$[(624, 598, 1; 12, d_f \geq 14)]_5$	$[(624, 592, 1; \gamma, d_f \geq 14)]_5$
$[(n, n-2mi-2, 1; mj, d_f \geq i+j+2)]_q$	$[(n, k, \mu; \gamma, d_f)]_q$
$[(4095, 4081, 1; 6, d_f \geq 6)]_4$	$[(4095, 4077, 1; \gamma, d_f \geq 6)]_4$
$[(4095, 4075, 1; 9, d_f \geq 8)]_4$	$[(4095, 4071, 1; \gamma, d_f \geq 8)]_4$
$[(4095, 4069, 1; 12, d_f \geq 10)]_4$	$[(4095, 4065, 1; \gamma, d_f \geq 9)]_4$
$[(4095, 4063, 1; 15, d_f \geq 12)]_4$	$[(4095, 4053, 1; \gamma, d_f \geq 12)]_4$
$[(4095, 4057, 1; 18, d_f \geq 14)]_4$	$[(4095, 4047, 1; \gamma, d_f \geq 14)]_4$
$[(4095, 4051, 1; 21, d_f \geq 16)]_4$	$[(4095, 4041, 1; \gamma, d_f \geq 15)]_q$
$[(4095, 4045, 1; 24, d_f \geq 18)]_4$	$[(4095, 4029, 1; \gamma, d_f \geq 18)]_4$
$[(4095, 4039, 1; 27, d_f \geq 20)]_4$	$[(4095, 4023, 1; \gamma, d_f \geq 20)]_4$
$[(n, n-2mi-2, 1; mj, d_f \geq i+j+2)]_q$	$[(n, k, \mu; \gamma, d_f)]_q$
$[(64, 49, 1; 6, d_f \geq 6)]_4$	$[(64, 45, 1; \gamma, d_f \geq 6)]_4$
$[(124, 110, 1; 6, d_f \geq 6)]_5$	$[(124, 106, 1; \gamma, d_f \geq 6)]_5$
$[(124, 104, 1; 9, d_f \geq 8)]_5$	$[(124, 100, 1; \gamma, d_f \geq 8)]_5$
$[(342, 328, 1; 6, d_f \geq 6)]_7$	$[(328, 324, 1; \gamma, d_f \geq 6)]_7$
$[(342, 322, 1; 9, d_f \geq 8)]_7$	$[(328, 318, 1; \gamma, d_f \geq 8)]_7$
$[(342, 316, 1; 12, d_f \geq 10)]_7$	$[(328, 312, 1; \gamma, d_f \geq 9)]_7$
$[(342, 310, 1; 15, d_f \geq 12)]_7$	$[(328, 306, 1; \gamma, d_f \geq 12)]_7$
$[(728, 714, 1; 6, d_f \geq 6)]_9$	$[(728, 710, 1; \gamma, d_f \geq 6)]_9$
$[(728, 708, 1; 9, d_f \geq 8)]_9$	$[(728, 704, 1; \gamma, d_f \geq 8)]_9$
$[(728, 702, 1; 12, d_f \geq 10)]_9$	$[(728, 698, 1; \gamma, d_f \geq 9)]_9$
$[(728, 696, 1; 15, d_f \geq 12)]_9$	$[(728, 686, 1; \gamma, d_f \geq 12)]_9$
$[(728, 690, 1; 18, d_f \geq 14)]_9$	$[(728, 680, 1; \gamma, d_f \geq 14)]_9$
$[(728, 684, 1; 21, d_f \geq 16)]_9$	$[(728, 680, 1; \gamma, d_f \geq 15)]_9$
$[(2400, 2382, 1; 8, d_f \geq 6)]_7$	$[(2400, 2376, 1; \gamma, d_f \geq 6)]_7$
$[(2400, 2374, 1; 12, d_f \geq 8)]_7$	$[(2400, 2368, 1; \gamma, d_f \geq 8)]_7$
$[(2400, 2366, 1; 16, d_f \geq 10)]_7$	$[(2400, 2360, 1; \gamma, d_f \geq 9)]_7$
$[(2400, 2358, 1; 20, d_f \geq 12)]_7$	$[(2400, 2352, 1; \gamma, d_f \geq 12)]_7$

8. A. Ketkar, A. Klappenecker, S. Kumar, and P. K. Sarvepalli. Nonbinary stabilizer codes over finite fields. *IEEE Trans. Inform. Theory*, 52(11):4892–4914, November 2006.
9. S. A. Aly, A. Klappenecker, and P. K. Sarvepalli. On quantum and classical BCH codes. *IEEE Trans. Inform. Theory*, 53(3):1183–1188, March 2007.
10. M. Hamada. Concatenated quantum codes constructible in polynomial time: efficient decoding and error correction. *IEEE Trans. Inform. Theory*, 54(12):5689–5704, December 2008.
11. G. G. La Guardia. Constructions of new families of nonbinary quantum codes. *Phys. Rev. A*, 80(4):042331(1–11), October 2009.
12. G. G. La Guardia. New quantum MDS codes. *IEEE Trans. Inform. Theory*, 57(8):5551–5554, August 2011.
13. H. Ollivier and J.-P. Tillich. Description of a quantum convolutional code. *Phys. Rev. Lett.*, 91(17):17790214, 2003.
14. H. Ollivier and J.-P. Tillich. Quantum convolutional codes: fundamentals. e-print arXiv:quant-ph/0401134.
15. A. C. A. de Almeida and R. Palazzo Jr.. A concatenated $[(4, 1, 3)]$ quantum convolutional code. In *Proc. IEEE Inform. Theory Workshop (ITW)*, pp. 28–33, 2004.
16. M. Grassl and M. Rötteler. Quantum block and convolutional codes from self-orthogonal product codes. In *Proc. Int. Symp. Inform. Theory (ISIT)*, pp. 1018–1022, 2005.
17. M. Grassl and M. Rötteler. Non-catastrophic encoders and encoder inverses for quantum convolutional codes. In *Proc. Int. Symp. Inform. Theory (ISIT)*, pp. 1109-1113, 2006.
18. M. Grassl and M. Rötteler. Constructions of quantum convolutional codes. e-print arXiv:quant-ph/0703182.
19. S. A. Aly, M. Grassl, A. Klappenecker, M. Rötteler, P. K. Sarvepalli. Quantum convolutional BCH codes. e-print arXiv:quant-ph/0703113.
20. S. A. Aly, A. Klappenecker, P. K. Sarvepalli. Quantum convolutional codes derived from Reed-Solomon and Reed-Muller codes. e-print arXiv:quant-ph/0701037.
21. G. D. Forney Jr., M. Grassl, S. Guha. Convolutional and tail-biting quantum error-correcting codes. *IEEE Trans. Inform. Theory*, 53(3):865-880, March 2007.
22. M. M. Wilde and T. A. Brun. Unified quantum convolutional coding. In *Proc. Int. Symp. Inf. Theory (ISIT)*, pp. 359–363, 2008.
23. M. M. Wilde and T. A. Brun. Entanglement-assisted quantum convolutional coding. *Phys. Rev. A*, 81(4):042333, April 2010.
24. P. Tan and J. Li. Efficient quantum stabilizer codes: LDPC and LDPC-convolutional constructions. *IEEE Trans. Inform. Theory*, 56(1):476–491, January 2010.
25. M. Houshmand, S. Hosseini-Khayat, and M. M. Wilde. Minimal-memory, non-catastrophic, polynomial depth quantum convolutional encoders. e-print arXiv:quant-ph/1105.0649.
26. P. Piret. *Convolutional Codes: An Algebraic Approach.* Cambridge, Massachusetts: The MIT Press, 1988.
27. L. N. Lee. Short unit-memory byte-oriented binary convolutional codes having maximum free distance. *IEEE Trans. Inform. Theory*, 22:349–352, May 1976.
28. G. D. Forney Jr.. Convolutional codes I: algebraic structure. *IEEE Trans. Inform. Theory*, 16(6):720–738, November 1970.
29. J. Rosenthal and R. Smarandache. Maximum distance separable convolutional codes. *Applicable Algebra in Eng. Comm. Comput.*, 10:15–32, 1998.
30. J. Rosenthal and E. York. BCH convolutional codes. *IEEE Trans. Inform. Theory*, 45(6):1833-1844, 1999.
31. K. J. Hole. On classes of convolutional codes that are not asymptotically catastrophic. *IEEE Trans. Inform. Theory*, 46(2):663–669, March 2000.
32. R. Smarandache, H. G.-Luerssen, J. Rosenthal. Constructions of MDS-convolutional codes. *IEEE Trans. Inform. Theory*, 47(5):2045–2049, July 2001.
33. H. G.-Luerssen and W. Schmale. On doubly-cyclic convolutional codes. *Applicable Algebra in Eng. Comm. Comput.*, 17(2):151–170, 2006.

34. H. G.-Luerssen and F-L Tsang. A matrix ring description for cyclic convolutional codes. *Advances in Math. Communications*, 2(1):55–81, 2008.
35. R. Johannesson and K. S. Zigangirov. *Fundamentals of Convolutional Coding.* Digital and Mobile Communication, Wiley-IEEE Press, 1999.
36. F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes.* North-Holland, 1977.