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NUMBER-PHASE UNCERTAINTY RELATIONS IN TERMS OF GENERALIZED ENTROPIES

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Number-phase uncertainty relations are formulated in terms of unified entropies which form a family of two-parametric extensions of the Shannon entropy. For two generalized measurements, unified-entropy uncertainty relations are given in both the state-dependent and state-independent forms. A few examples are discussed as well. Using the Pegg–Barnett formalism and the Riesz theorem, we obtain a nontrivial inequality between norm-like functionals of generated probability distributions in finite dimensions. The principal point is that we take the infinite-dimensional limit right for this inequality. Hence number-phase uncertainty relations with finite phase resolutions are expressed in terms of the unified entropies. Especially important case of multiphoton coherent states is separately considered. We also give some entropic bounds in which the corresponding integrals of probability density functions are involved.

Keywords: number-phase uncertainty, Rényi entropy, Tsallis entropy, Riesz's theorem, coherent states
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1. Introduction

Since celebrate Heisenberg's result [1] had been published, many forms of uncertainty relations were proposed [2, 3]. A new interest was inspired by recent advances in use of quantum information. Entropy plays a central role in statistical physics and information theory. Entropic measures have found use in various topics including a quantification of uncertainty in quantum measurements [4, 5]. Within the most known approach by Robertson [6], some doubts have been observed [7, 8]. An alternate way is to express the uncertainty principle by means of information-theoretic terms. Although the Shannon entropy is of great importance, many generalizations are found to be useful. Both the Rényi [9] and Tsallis entropies [10] have widely been adopted for interdisciplinary applications. The authors of the paper [11] proposed the notion of unified entropy which includes the above entropies as particular cases. Properties of unified entropies were examined in both the classical and quantum regimes [11, 12].

The first relation in terms of the Shannon entropies for the position-momentum pair was derived by Hirschman [13]. An improvement of his result has been stated in Ref. [14] (see also Ref. [15]). Mamojka [16] and Deutsch [7] initiated a discussion of the problem in general form and obtained particular results. An improvement of Deutsch's entropic bound [7] had been conjectured in Ref. [17] and proved with use of Riesz's theorem in Ref. [8]. In Ref. [18], we have discussed two-measurement entropic bounds that cannot be derived on the base of Riesz's theorem. The time-energy case [19] and tomographic processes [20] were considered within entropic approach. Cryptography applications of entropic uncertainty relations are discussed in Refs. [21, 22]. Entropic uncertainty relations have also been given for more than two measurements. Entropic bounds for $(N+1)$ complementary observables in N-dimensional Hilbert space were derived in Refs. [23, 24]. New results in this issue have been reported in Ref. [25]. For arbitrary number of binary observables, a nearly optimal relation for the collision entropy was given in Ref. [26].

The problem of constructing a Hermitian operator of phase has a long history (see the review [27] and references therein). Different ways to measure a quantum phase uncertainty are compared in Ref. [28, 29]. Both the well-defined Hermitian operator of phase and corresponding number-phase relation of Robertson type have been obtained within the Pegg–Barnett formalism [30, 31] (see also section 4.3 in Ref. [32]). Entropic relations for the number-phase pair in terms of the Shannon entropies were obtained in Refs. [33, 34, 35]. Using some ideas of Ref. [34], entropic relations of "number-phase" type have been posed for solvable quantum systems with discrete spectra [36]. Uncertainty relations for the number and annihilation operators have been considered in Refs. [37, 38].

In the present paper, we formulate number-phase uncertainty relations in terms of unified entropies. The phase operator is approached within the Pegg–Barnett formalism [30, 31]. The paper is organized as follows. In Section 2, the required preliminary material is presented. For two generalized measurements, the entropic uncertainty relations in terms of unified entropies are derived in Section 3. Entropic bounds of both state-dependent and state-independent forms are simultaneously treated. Several examples of interest are considered in Section 4. One includes the cases of complementary observables for N-level system, angle and angular momentum, and extremal unravelings of quantum channels. In Section 5, the developed method is appropriately modified. Hence number-phase uncertainty relations in terms of unified entropies are immediately derived. Incidentally, the case of canonically conjugate variables is mentioned. Section 6 concludes the paper with a summary of results.

2. Definitions and background

In this section, we introduce some terms and conventions that will be used through the text. First, we recall operators and states commonly used in quantum optics. An outline of the Pegg–Barnett formalism is given as well. Finally, the utilized entropic measures are considered.

2.1. *Operators and states. Elements of the Pegg–Barnett formalism*

It is well known that the energy eigenstates for a single field mode are the number states analogous to those for the harmonic oscillator [32]. For given mode, the annihilation and creation operators $\boldsymbol{a}$ and $\boldsymbol{a}^{\dagger}$ satisfy the commutation rule $[\boldsymbol{a},\boldsymbol{a}^{\dagger}]=\mathbb{1}$, where $\mathbb{1}$ is the identity operator. By $|n\rangle$ we denote the normalized eigenstate of the number operator $\boldsymbol{n}=\boldsymbol{a}^{\dagger}\boldsymbol{a}$, so that $\boldsymbol{n}|n\rangle=n|n\rangle$ with integer $n\geq 0$. These states form an orthonormal basis in the Hilbert space. Under the action of the annihilation and creation operators, the numbers states are transformed as $\boldsymbol{a}|n\rangle=n^{1/2}|n-1\rangle$ and $\boldsymbol{a}^{\dagger}|n\rangle=(n+1)^{1/2}|n+1\rangle$. In thermal equilibrium, the state of a system is described by the density operator corresponding to the grand canonical ensemble. For a single field mode with frequency ω in a thermal state, the density operator is diagonal in the basis $\{|n\rangle\}$ with the probabilities that follow the Bose–Einstein statistics [32]. Coherent states form another class which is especially important in quantum optics. For a complex number $z = |z|e^{i\phi}$, we write coherent state as [39]

$$|z\rangle = \exp\bigl(-|z|^2/2\bigr) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, |n\rangle \ . \tag{1}$$

The state $|z\rangle$ obeys $\boldsymbol{a}|z\rangle = z|z\rangle$ and $\langle z|\boldsymbol{a}^{\dagger} = \langle z|z^*$, i.e. coherent states are *right* eigenstates of the annihilation operator and *left* eigenstates of the creation one. Coherent states are not mutually orthogonal, but do satisfy the completeness relation $(1/\pi) \int d^2 z \, |z\rangle\langle z| = \mathbb{1}$, where the double integral is taken over the whole complex plane [39, 40]. Coherent states have many nice properties and various applications (see, e.g., a collection of papers in Ref. [40] and Ref. [41] with a focus on entangled coherent states). We only recall that the number of photons in a single-mode coherent state $|z\rangle$ has the Poisson distribution [42]. Hence, the mean number of photons and the photon number variance are both equal to $|z|^2$.

In order to represent the phase operator, we first consider operators and states in finite-dimensional state spaces. Let $\{|n\rangle\}$ be an orthonormal basis in $(N+1)$-dimensional space $\mathcal{H}_{N+1}$. The number operator is decomposed as

$$\boldsymbol{n}_{N+1} = \sum\nolimits_{n=0}^{N} n|n\rangle\langle n| \ . \tag{2}$$

We introduce normalized states

$$|\theta_m\rangle = \frac{1}{\sqrt{N+1}} \sum\nolimits_{n=0}^{N} e^{in\theta_m} |n\rangle \ , \tag{3}$$

where $\theta_m = \theta_0 + 2\pi m/(N+1)$ and $m = 0, 1, 2, \ldots, N$. It is clear that the vectors $|\theta_m\rangle$ form another orthonormal basis in $\mathcal{H}_{N+1}$. Note that the bases $\{|n\rangle\}$ and $\{|\theta_m\rangle\}$ are mutually unbiased. The phase operator is defined as

$$\boldsymbol{\theta}_{N+1} = \sum\nolimits_{m=0}^{N} \theta_m |\theta_m\rangle\langle\theta_m| \ . \tag{4}$$

Within the considered formalism, quantities of interest are first calculated for finite N and then taken in the limit $N \to \infty$. In particular, the moments of the phase operator can be obtained in this way (for details, see section 4.3 in [32]). For physical states, however, there is a more straightforward approach. Following the Pegg–Barnett formalism [30, 31], we introduce the phase probability density function by

$$P(\theta) = \lim_{N\to\infty} \frac{N+1}{2\pi} \, \langle\theta|\boldsymbol{\rho}|\theta\rangle = \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \langle m|\boldsymbol{\rho}|n\rangle \exp\bigl[i(n-m)\theta\bigr] \ . \tag{5}$$

Converting the summation into a Riemann–Darboux integral, we then write the moment of order ν as

$$\mathrm{tr}\bigl(\boldsymbol{\theta}_{N+1}^{\nu} \boldsymbol{\rho}\bigr) \underset{N\to\infty}{\longrightarrow} \int_{\theta_0}^{\theta_0+2\pi} \theta^{\nu} P(\theta) \, d\theta \ . \tag{6}$$

If the density operator $\boldsymbol{\rho}$ is diagonal with respect to the basis $\{|n\rangle\}$ then the right-hand side of Eq. (5) gives $P(\theta) = (2\pi)^{-1}$. In particular, this formula holds for thermal states of a single mode. Since the function $P(\theta)$ is constant here, these are states of completely unknown phase. For coherent states, the series in Eq. (5) is difficult to evaluate. However, for large values of $|z|$ a useful expression exists. Namely, for $|z| \gg 1$ the corresponding density is commonly approximated by the Gaussian distribution [32]

$$P(\theta) \simeq \left(\frac{2\,|z|^2}{\pi}\right)^{1/2} \exp\left\{-2\,|z|^2(\theta-\phi)^2\right\} =: \widetilde{P}(\theta)\ . \tag{7}$$

Since the Gaussian distribution (7) has a narrow peak at $\theta=\phi$, such states $|z\rangle$ are physical states of almost determined phase. Note that no physical states with completely determined phase exist. Since the photon number variance is very large here, the Poisson number distribution is very spreading and can also be approximated by the Gaussian distribution [32, 42]

$$\widetilde{W}(n) := \left(\frac{1}{2\pi|z|^2}\right)^{1/2} \exp\left\{-\frac{\left(n-|z|^2\right)^2}{2\,|z|^2}\right\}\ , \tag{8}$$

where the n is now treated as continuous. This distribution is normalized in the sense that $\int_{-\infty}^{+\infty}\widetilde{W}(n)\,dn = 1$. Here we must stress the following. The expressions (7) and (8) are assumed to be used as convenient approximations only for $|z| \gg 1$. In this case, the maximum of $\widetilde{W}(n)$ at $n=|z|^2$ is equal to $\left(\sqrt{2\pi}\,|z|\right)^{-1} \ll 1$. So values of this function are negligible for negative n as well. It is important for our aims that the distributions (7) and (8) can be related via the Fourier transform. Namely, we have

$$\widetilde{p}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi\kappa}\,\widetilde{w}(\kappa)\,d\kappa\ , \qquad \widetilde{w}(\kappa) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\kappa\xi}\,\widetilde{p}(\xi)\,d\xi\ , \tag{9}$$

where $\widetilde{P}(\theta)=\widetilde{p}(\xi)^2$ and $\widetilde{W}(n)=\widetilde{w}(\kappa)^2$ in terms of the variables $\xi=\theta-\phi$ and $\kappa=n-|z|^2$.

2.2. *Generalized entropies and their forms*

The concept of entropy is a key tool in information theory. In addition to the Shannon entropy, which is fundamental, other entropic measures were found to be useful. Among them, the Rényi and Tsallis entropic functionals are very important [43]. For given probability distribution $\mathsf{p}=\{p_n\}$, its Rényi α-entropy is defined as [9]

$$R_\alpha(\mathsf{p}) := \frac{1}{1-\alpha}\,\ln\left(\sum\nolimits_n p_n^\alpha\right)\ , \tag{10}$$

where $\alpha>0$ and $\alpha\neq 1$. This quantity is a non-increasing function of α [9]. For other properties, related to the parametric dependence, see Ref. [44]. The Renyi entropy of order $\alpha=2$ is also known as the collision entropy [5, 26]. The notion of Tsallis entropy is widely used in non-extensive statistical mechanics [45]. The non-extensive entropy of positive degree $\alpha\neq 1$ is defined as [10]

$$H_\alpha(\mathsf{p}) := \frac{1}{1-\alpha}\left(\sum\nolimits_n p_n^\alpha - 1\right) = -\sum\nolimits_n p_n^\alpha \ln_\alpha(p_n)\ , \tag{11}$$

where $\ln_\alpha(\xi) := \left(\xi^{1-\alpha}-1\right)/(1-\alpha)$ is the α-logarithm. Its inverse function is the α-exponential function

$$\exp_\alpha(\xi) := \Big(1+(1-\alpha)\,\xi\Big)_+^{1/(1-\alpha)}\ , \tag{12}$$

where $(\xi)_+ := \max\{0, \xi\}$. For $\alpha \to 1$, the α-logarithm and the α-exponential are respectively reduced to the usual ones. With slightly other factor, the entropic functional (11) was derived from several axioms by Havrda and Charvát [46]. The Tsallis entropy of degree $\alpha = 2$ is related to the so-called degree of certainty involved in complementarity relations for mutually unbiased observables [47]. The Rényi and Tsallis entropies are fruitful one-parametric generalizations of the Shannon entropy. In the limit $\alpha \to 1$, the entropies (10) and (11) both recover the standard Shannon entropy $S(\mathsf{p}) = -\sum_n p_n \ln p_n$. In Ref. [11], Hu and Ye proposed the class of more general extensions of the Shannon entropy. For positive $\alpha \neq 1$ and $s \neq 0$, the unified (α, s)-entropy of probability distribution $\mathsf{p} = \{p_n\}$ is defined by

$$E_\alpha^{(s)}(\mathsf{p}) := \frac{1}{(1-\alpha)\,s} \left[\Bigl(\sum\nolimits_n p_n^\alpha \Bigr)^s - 1 \right] . \tag{13}$$

This two-parametric entropic functional includes the entropy (11) as the partial case $s = 1$ and the Rényi entropy (10) as the limiting case $s \to 0$ [11]. In quantum regime, this entropic form leads to a two-parametric extension of the von Neumann entropy. The quantum unified entropies enjoy many properties similarly to the von Neumann entropy [11, 12].

The above expressions for entropies can be rewritten in terms of norm-like functionals. For $b \geq 1$, the l_b norm of vector $\mathsf{x} = \{x_n\}$ is written as [48]

$$\|\mathsf{x}\|_b := \Bigl(\sum\nolimits_n |x_n|^b \Bigr)^{1/b} . \tag{14}$$

These norms are convenient for studying uncertainty relations in the context of information processing [49]. For $b = \infty$, this norm is defined as $\|\mathsf{x}\|_\infty := \max_n |x_n|$. For probability distribution $\mathsf{p} = \{p_n\}$ and $\beta > 0$, we then introduce a norm-like β-functional

$$\|\mathsf{p}\|_\beta := \Bigl(\sum\nolimits_n p_n^\beta \Bigr)^{1/\beta} . \tag{15}$$

This expression is actually a norm for $\beta \geq 1$, but we will also use it for $\beta \in [1/2; 1)$. In terms of these functionals, the entropy (13) is rewritten as

$$E_\alpha^{(s)}(\mathsf{p}) = \frac{\|\mathsf{p}\|_\alpha^{\alpha s} - 1}{(1-\alpha)\,s} , \tag{16}$$

including $H_\alpha(\mathsf{p}) = \bigl(\|\mathsf{p}\|_\alpha^\alpha - 1\bigr)/(1-\alpha)$ for $s = 1$ and $R_\alpha(\mathsf{p}) = \alpha(1-\alpha)^{-1} \ln \|\mathsf{p}\|_\alpha$ for $s = 0$. When a probability distribution is continuous, we merely replace each sum with proper integral of the probability density function.

Utility of entropic approach for measuring uncertainty in quantum measurement is widely discussed (see the works [5, 7, 8, 50, 51] and references therein). The only point we should note about entropic measures for a continuous distribution is that they can be used in two different forms. The first form with clear operational meaning is written for some partition of the interval, in which continuous variable does range. For the phase, we consider a partition $\{\vartheta_m\}$ of the interval $[0; 2\pi]$. The probability to find value of the phase variable in the mth bin is written as

$$r_m = \int_{\vartheta_m}^{\vartheta_{m+1}} P(\theta)\, d\theta . \tag{17}$$

For probability vector $\mathsf{r}=\{r_m\}$, the corresponding (α,s)-entropy $E_\alpha^{(s)}(\mathsf{r})$ is then defined in line with Eq. (13). As a rule, the bins are all of the same size which characterizes a resolution of measurement. Further, we can directly introduce entropic measures

$$E_\alpha^{(s)}(P)=\frac{1}{(1-\alpha)\,s}\left\{\left(\int_0^{2\pi}P(\theta)^\alpha\,d\theta\right)^s-1\right\}\qquad(s\neq 0)\ , \tag{18}$$

$$R_\alpha(P)=\frac{1}{1-\alpha}\,\ln\left(\int_0^{2\pi}P(\theta)^\alpha\,d\theta\right)\ . \tag{19}$$

For $\alpha=1$, the Shannon entropy is $S(P)=-\int_0^{2\pi}P(\theta)\ln P(\theta)\,d\theta$. The latter measure is used in number-phase uncertainty relations given in Refs. [33, 34, 35]. With the entropies (18) and (19), however, we have observed some doubts. The discrete (α,s)-entropy (13) is always positive, and vanish only when all the probabilities, except one, are zero. By $\|\mathsf{r}\|_1=1$, we actually have $\|\mathsf{r}\|_\alpha\leq 1\leq\|\mathsf{r}\|_\beta$ for $\alpha>1>\beta$. It is not the case for continuous distributions. In general, the normalization $\|P\|_1=1$ does not provide

$$\|P\|_\alpha\leq 1\leq\|P\|_\beta \tag{20}$$

for $\alpha>1>\beta$. If the condition (20) fails then the quantities (18) and (19) become negative-valued. We do not treat such functionals for expressing an uncertainty, though the Shannon entropy of the phase with negative values up to $-\infty$ was noted in the literature (see, e.g., section 2.3 of Ref. [29]). Instead, we will mainly use entropies calculated for discretized distribution with probabilities of a kind (17). The entropies (18) and (19) cannot be negative, when the probability density function obeys the condition (20). Say, this condition is always satisfied with the density such that $P(\theta)\leq 1$ for all $\theta\in[0;2\pi]$. The class of densities with sufficiently small variations is wide enough. For instance, such densities stand for thermal states and, generally, for those density matrices that are diagonal in the number eigenbasis $\{|n\rangle\}$. In Ref. [38] we have analyzed an example, in which the probability density function does not exceed one with necessity. On the other hand, coherent states $|z\rangle$ with large $|z|$, for which the formula (7) takes place, are clearly beyond the above class. Physical quantities are usually dimensional that leads to another question for entropic functionals expressed as integrals. Since the phase is dimensionless, we do not enter into details and refer to the extensive review [51].

3. Entropic uncertainty relations for two generalized measurements

In this section, entropic uncertainty relations for two generalized measurements are posed in terms of unified entropies. First, we recall an inequality between norm-like functionals of the generated probability distributions. This statement is based on Riesz's theorem, which is also recalled. Second, we find the minimum of a certain function of two variables in the proper domain. The results are then used for deriving unified-entropy uncertainty relations.

3.1. *An inequality based on Riesz's theorem*

A generalized quantum measurement is described by "positive operator-valued measure" (POVM). This is a set $\mathcal{M}=\{\mathsf{M}_i\}$ of positive semidefinite operators obeying the completeness relation $\sum_i \mathsf{M}_i = \mathbb{1}$ [52]. The standard von Neumann measurement is represented by the set of mutually orthogonal projectors. For given POVM $\mathcal{M} = \{\mathsf{M}_i\}$ and density operator $\boldsymbol{\rho}$, the probability of ith outcome is expressed as $p_i = \mathrm{tr}(\mathsf{M}_i\boldsymbol{\rho})$ [52]. The Hilbert–Schmidt inner product of operators A and B is defined as $\langle \mathsf{A}\,,\mathsf{B}\rangle_{\mathrm{hs}} := \mathrm{tr}(\mathsf{A}^{\dagger}\,\mathsf{B})$. To get entropic relations, we will use a version of Riesz's theorem (see theorem 297 in the book [53]). Consider the tuples x and y of complex numbers related by a linear transformation T as

$$y_i = \sum\nolimits_j \tau_{ij}\, x_j \ . \tag{21}$$

Let η be maximum of $|\tau_{ij}|$, i.e. $\eta := \max|\tau_{ij}|$, and let conjugate indices $a, b \in [1;\infty]$ obey $1/a + 1/b = 1$. If the matrix $\mathsf{T} = [[\tau_{ij}]]$ satisfies $\|\mathsf{y}\|_2 \le \|\mathsf{x}\|_2$ for all x and $1 < b < 2$, then

$$\|\mathsf{y}\|_a \le \eta^{(2-b)/b}\|\mathsf{x}\|_b \ . \tag{22}$$

So, Riezs's theorem provides an upper estimate on the (b, a)-norm of transformation T. For more general versions, see the book [54] and references therein. Using Eq. (22), we deduce the following [55]. Let $\mathcal{M} = \{\mathsf{M}_i\}$ and $\mathcal{N} = \{\mathsf{N}_j\}$ be two POVMs. For given $\boldsymbol{\rho}$, the corresponding probabilities are written as $p_i = \mathrm{tr}(\mathsf{M}_i\boldsymbol{\rho})$ and $q_j = \mathrm{tr}(\mathsf{N}_j\boldsymbol{\rho})$. Then we have [55]

$$\|\mathsf{p}\|_\alpha \le g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho})^{2(1-\beta)/\beta}\,\|\mathsf{q}\|_\beta \ , \tag{23}$$

where $1/\alpha + 1/\beta = 2$, $1/2 < \beta < 1$, and the function $g(\mathcal{M},\mathcal{N}|\rho)$ is determined by

$$g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho}) := \max\Big\{(p_i q_j)^{-1/2}\left|\mathrm{tr}(\mathsf{M}_i\mathsf{N}_j\boldsymbol{\rho})\right| :\ p_i \ne 0,\, q_j \ne 0\Big\} . \tag{24}$$

Note that $g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho}) = g(\mathcal{N},\mathcal{M}|\boldsymbol{\rho})$ in view of $\mathrm{tr}(\mathsf{M}_i\mathsf{N}_j\boldsymbol{\rho}) = \big\langle \mathsf{M}_i\sqrt{\boldsymbol{\rho}}\,,\mathsf{N}_j\sqrt{\boldsymbol{\rho}}\big\rangle_{\mathrm{hs}}$. To approach relations of a state-independent form, we use the inequality [55]

$$g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho}) \le \bar{f}(\mathcal{M},\mathcal{N}) := \max\Big\{\big\|\mathsf{M}_i^{1/2}\mathsf{N}_j^{1/2}\big\|_\infty :\ \mathsf{M}_i \in \mathcal{M},\, \mathsf{N}_j \in \mathcal{N}\Big\} . \tag{25}$$

Here the spectral norm $\|\mathsf{A}\|_\infty$ is put as the largest eigenvalue of operator $\sqrt{\mathsf{A}^{\dagger}\mathsf{A}} \ge \mathbf{0}$. The relation (25) shows that $g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho}) \le 1$. Combining Eqs. (23) and (25) finally gives

$$\|\mathsf{p}\|_\alpha \le \bar{f}(\mathcal{M},\mathcal{N})^{2(1-\beta)/\beta}\,\|\mathsf{q}\|_\beta \ , \tag{26}$$

under the same conditions on α and β. In Ref. [55], the inequalities (23) and (26) were used to derive entropic uncertainty relations in terms of both the Rényi and Tsallis entropies. We shall now extend the method of Ref. [55] to the case of (α, s)-entropies.

3.2. *Entropic uncertainty relations in terms of* (α, s)*-entropies*

Putting two variables $\xi = \|\mathsf{p}\|_\alpha^\alpha$ and $\zeta = \|\mathsf{q}\|_\beta^\beta$, the sum of two unified entropies reads

$$E_\alpha^{(s)}(\mathsf{p}) + E_\beta^{(t)}(\mathsf{q}) = \frac{\xi^s - 1}{(1-\alpha)\,s} + \frac{\zeta^t - 1}{(1-\beta)\,t} =: h(\xi,\zeta) \ . \tag{27}$$

Assuming $\alpha > 1 > \beta$, we clearly have $\xi \le 1$ and $\zeta \ge 1$. The function $h(\xi,\zeta)$ is zero at the point $(1, 1)$. We now add Eq. (23) in the form

$$c\,\xi^{\beta/\alpha} \le \zeta, \qquad c = g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho})^{-2(1-\beta)} \ . \tag{28}$$

Non-trivial uncertainty relations take place for the case $g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho}) < 1$, in which $c > 1$ due to $\beta < 1$. Then the curve $\zeta = c\,\xi^{\beta/\alpha}$ cuts off the down right corner of the rectangle $\bigl\{(\xi,\zeta):\ 0 \leq \xi \leq 1,\ 1 \leq \zeta < +\infty\bigr\}$ and, herewith, the point $(1,1)$ (see, e.g., Fig. A1 of Ref. [55]). So we have arrived at a task of minimizing $h(\xi,\zeta)$ in the domain

$$D := \Bigl\{(\xi,\zeta):\ 0 \leq \xi \leq 1,\ 1 \leq \zeta < +\infty,\ c\,\xi^{\beta/\alpha} \leq \zeta\Bigr\} \ . \tag{29}$$

Lemma 1 *Let strictly positive numbers α and β obey $1/\alpha+1/\beta = 2$, and let two real numbers s and t obey $st > 0$. The minimum of the function $h(\xi,\zeta)$ in the domain D is equal to*

$$\min_{D} h(\xi,\zeta) = \frac{1}{\nu}\,\ln_{\mu}\bigl(g^{-2\nu}\bigr) \ . \tag{30}$$

Here the parameters μ and ν are defined as

$$\mu := \left\{ \begin{array}{ll} \max\{\alpha,\beta\}\ , & s,t \in (0;+\infty) \\ \min\{\alpha,\beta\}\ , & s,t \in (-\infty;0) \end{array} \right\} \ , \qquad \nu := s(\mu)\ , \tag{31}$$

under the notation $s(\alpha) \equiv s$ and $s(\beta) \equiv t$.

Proof. For definiteness, we assume that $\alpha > 1 > \beta$. By $(\xi_0,1)$ with $\xi_0 = c^{-\alpha/\beta}$, we denote the point of intersection of the lines $\zeta = 1$ and $\zeta = c\,\xi^{\beta/\alpha}$. Since $h(1,1) = 0$, the desired minimum is strictly positive. In the interior of D, we have

$$\frac{\partial h}{\partial \xi} = \frac{\xi^{s-1}}{1-\alpha} < 0\ , \qquad \frac{\partial h}{\partial \zeta} = \frac{\zeta^{t-1}}{1-\beta} > 0\ , \tag{32}$$

due to $\alpha > 1 > \beta$. Hence the minimal value is reached on the boundary of the domain D. The derivatives (32) show that the function $h(\xi,\zeta)$ is decreasing in ξ and increasing in ζ. On the linear boundary segments, therefore, the minimal value is either $h(\xi_0,1)$ or $h(1,c)$. Let us consider the boundary segment on the curve $\zeta = c\,\xi^{\beta/\alpha}$. Substituting $\zeta = (\xi/\xi_0)^{\beta/\alpha}$ in the expression of $h(\xi,\zeta)$ and differentiating with respect to ξ, we obtain the derivative

$$\frac{1}{(1-\alpha)\,\xi}\ \xi^{s} + \frac{\beta}{\alpha(1-\beta)\,\xi}\left(\frac{\xi}{\xi_0}\right)^{t\beta/\alpha} = \frac{1}{(\alpha-1)\,\xi}\left(\left(\frac{\xi}{\xi_0}\right)^{t\beta/\alpha} - \xi^{s}\right) \ . \tag{33}$$

Here the equality $\beta/(1-\beta) = \alpha/(\alpha-1)$ was used due to $1/\alpha + 1/\beta = 2$. We shall now consider the following cases: (i) $s > 0$ and $t > 0$; (ii) $s < 0$ and $t < 0$.

In the case (i), the powers $t\beta/\alpha$ and s in the right-hand side of (33) are both strictly positive and

$$\xi^{s} \leq 1 \leq (\xi/\xi_0)^{t\beta/\alpha} \tag{34}$$

by $\xi_0 \leq \xi \leq 1$. So the derivative (33) is positive and the minimum is reached at the point $(\xi_0,1)$. By calculations,

$$h(\xi_0,1) = \frac{\xi_0^{s} - 1}{(1-\alpha)s} = \frac{c^{-s\alpha/\beta} - 1}{(1-\alpha)s} = \frac{g^{2s(\alpha-1)} - 1}{(1-\alpha)s} = \frac{1}{s}\,\ln_{\alpha}\bigl(g^{-2s}\bigr) \ . \tag{35}$$

If $\alpha > 1 > \beta$ then $\mu = \alpha$ and $\nu = s$ due to the definition (31). Hence the right-hand sides of (30) and (35) coincide.

In the case (ii), the powers $t\beta/\alpha$ and s in the right-hand side of (33) are both strictly negative and

$$(\xi/\xi_0)^{t\beta/\alpha} \leq 1 \leq \xi^s \tag{36}$$

by $\xi_0 \leq \xi \leq 1$. So the derivative (33) is negative and the minimum is reached at the point $(1, c)$. By calculations,

$$h(1,c) = \frac{c^t - 1}{(1-\beta)t} = \frac{g^{-2t(1-\beta)} - 1}{(1-\beta)t} = \frac{1}{t}\ \ln_\beta\bigl(g^{-2t}\bigr)\ . \tag{37}$$

If $\alpha > 1 > \beta$ then $\mu = \beta$ and $\nu = t$ by the definition (31). Hence the right-hand sides of (30) and (37) coincide. ■

Applying the result of Lemma 1, we get both the state-dependent and state-independent forms of uncertainty relations in terms of (α, s)-entropies. Note that the right-hand side of Eq. (30) tends to $-2\ln g$ in the limit $\nu \to 0$. Taking $s = 0$ and $t = 0$, we then obtain uncertainty relations in terms of the Rényi entropies. In effect, these relations can immediately be derived from Eqs. (23) and (26) by simple algebra (for details, see Refs. [50, 55]). Adding the case of Rényi's entropies, we have the following statement.

Theorem 1 *Let $\mathcal{M} = \{\mathsf{M}_i\}$ and $\mathcal{N} = \{\mathsf{N}_j\}$ be two POVMs, and let $\boldsymbol{\rho}$ be a density matrix. For $st > 0$ and $s = t = 0$, there holds*

$$E_\alpha^{(s)}(\mathsf{p}) + E_\beta^{(t)}(\mathsf{q}) \geq \frac{1}{\nu}\ \ln_\mu\Bigl\{g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho})^{-2\nu}\Bigr\}\ , \tag{38}$$

where $1/\alpha + 1/\beta = 2$, the function $g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho})$ is given by Eq. (24) and the numbers μ and ν are given by Eq. (31). The state-independent form is obtained by replacing $g(\mathcal{M},\mathcal{N}|\boldsymbol{\rho})$ with $\bar{f}(\mathcal{M},\mathcal{N})$ defined by (25).

So, we have obtained uncertainty relations for many of the unified entropies of generated probability distributions. The entropic bound (38) is valid for generalized measurements as well as for any mixed state of interest. Few comments on state-independent entropic bounds must be given here. The right-hand side of Eq. (25) has been adopted with use of Riesz's theorem. This theorem is generally applicable to any linear transformation, but only an upper estimate on its (b, a)-norm is provided in this way. As a rule, it is very difficult to find exactly values of such norms. However, these exact values are known for some important cases, including the Fourier transform in both the discrete and continuous varieties. In the former, the exact values are given by the Young-Hausdorff inequalities (see, e.g., point (2.25) of chapter XII in [56]); in the latter, the ones have been found by Beckner [14]. When the exact estimates are known, we will use Eq. (26) with the correspondingly replaced value of $\bar{f}(\mathcal{M},\mathcal{N})$. Hence stronger entropic bounds of state-independent form will be obtained. But estimates of the above kind can initially be posed only for pure states. Suppose that the relation (26) with given $\bar{f}(\mathcal{M},\mathcal{N})$ holds for all the pure state. Then it is still valid for impure states as well. The justification is based on the Minkowski inequality (for details, see the proof of proposition 3 in Ref. [50]). So, entropic uncertainty relations with improved value of $\bar{f}(\mathcal{M},\mathcal{N})$ do also hold for any mixed state. It is important because many of generalized entropies do not enjoy the concavity, including the Rényi α-entropy of order $\alpha > 1$. Before an analysis of the number-phase case, we consider several examples on base of the result (38).

4. Examples of uncertainty relations in terms of (α, s)-entropies

In this section, three interesting examples of uncertainty relations in terms of the unified entropies are presented. First, we consider a pair of complementary observables for a N-level system. Second, the angle and the angular momentum are examined. For the angle, we take discretized probability distribution with respect to some partition of range of the angle. Finally, we discuss extremal unravelings of two quantum channels.

Example 1. Let complex amplitudes $\tilde{c}_k$ and c_l be connected by the discrete Fourier transform

$$\tilde{c}_k = \frac{1}{\sqrt{N}} \sum\nolimits_{l=1}^{N} e^{2\pi ikl/N}\, c_l \ . \tag{39}$$

The corresponding probabilities are written as $p_k = |\tilde{c}_k|^2$ and $q_l = |c_l|^2$. The transformation (39) is related to a pair of complementary observables for a N-level system [17]. It follows from $\|\tilde{\mathsf{c}}\|_2 = \|\mathsf{c}\|_2$ and (22) that

$$\|\mathsf{c}\|_a \leq \left(\frac{1}{\sqrt{N}}\right)^{(2-b)/b} \|\tilde{\mathsf{c}}\|_b \ , \qquad \|\tilde{\mathsf{c}}\|_a \leq \left(\frac{1}{\sqrt{N}}\right)^{(2-b)/b} \|\mathsf{c}\|_b \ , \tag{40}$$

where $1/a + 1/b = 1$ and $1 < b < 2$. Squaring these inequalities and putting $\alpha = a/2$ and $\beta = b/2$, we then obtain

$$\|\mathsf{q}\|_\alpha \leq \left(\frac{1}{N}\right)^{(1-\beta)/\beta} \|\mathsf{p}\|_\beta \ , \qquad \|\mathsf{p}\|_\alpha \leq \left(\frac{1}{N}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_\beta \ . \tag{41}$$

In other words, we have $\bar{f}^2 = 1/N$ here. Using (α, s)-entropies, the uncertainty relation of a state-independent form are therefore expressed as

$$E_\alpha^{(s)}(\mathsf{p}) + E_\beta^{(t)}(\mathsf{q}) \geq \frac{1}{\nu}\, \ln_\mu\!\left(N^\nu\right) \ , \tag{42}$$

where $1/\alpha + 1/\beta = 2$, $st > 0$ or $s = t = 0$, and the notation (31) is assumed. The relation (42) extends the previous relations given for both the Rényi [57] and Tsallis entropies [55].

Example 2. Let $\{\varphi_k\}$ be a partition of the angular interval. The maximal size $\Delta\varphi := \max \Delta\varphi_k$ of bins $\Delta\varphi_k = \varphi_{k+1} - \varphi_k$ is less than 2π. For given partition, the probability p_k to find the angle in kth bin is given by Eq. (17) with appropriate changes. Another probability distributions contains $q_l = |c_l|^2$. Here the coefficients c_l's are related to the decomposition of $\Psi(\varphi)$ with respect to the eigenstates of z-component of the angular momentum, namely

$$\Psi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{+\infty} c_l\, e^{il\varphi} \ . \tag{43}$$

The Young-Hausdorff inequalities are then written as (see, e.g., section 8.17 in Ref. [53])

$$\|\Psi\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|\mathsf{c}\|_b \ , \qquad \|\mathsf{c}\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|\Psi\|_b \ , \tag{44}$$

where $1/a + 1/b = 1$ and $a > 2 > b$. Applying theorem 192 of Ref. [53] for integral means, we further have

$$\left(\frac{1}{\Delta\varphi_k} \int_{\varphi_k}^{\varphi_{k+1}} d\varphi\, |\Psi(\varphi)|^2\right)^{\alpha} \left\{ \begin{array}{ll} \leq, & \alpha > 1 \\ \geq, & \alpha < 1 \end{array} \right\} \frac{1}{\Delta\varphi_k} \int_{\varphi_k}^{\varphi_{k+1}} d\varphi\, |\Psi(\varphi)|^{2\alpha} \ . \tag{45}$$

Assuming now $\alpha > 1 > \beta$, we obtain the inequalities

$$\Delta\varphi^{(1-\alpha)/\alpha}\|\mathsf{p}\|_{\alpha} \leq \|\Psi\|_{a}^{2} \ , \qquad \|\Psi\|_{b}^{2} \leq \Delta\varphi^{(1-\beta)/\beta}\|\mathsf{p}\|_{\beta} \ , \tag{46}$$

To derive (46), we respectively take the inequalities (45) for $\alpha > 1$ and $\beta < 1$, sum them with respect to k and raise the sums to the powers $1/\alpha$ and $1/\beta$. Combining Eqs. (46) with the squared Young-Hausdorff inequalities finally gives

$$\|\mathsf{p}\|_{\alpha} \leq \left(\frac{\Delta\varphi}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_{\beta} \ , \qquad \|\mathsf{q}\|_{\alpha} \leq \left(\frac{\Delta\varphi}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{p}\|_{\beta} \ , \tag{47}$$

in view of $\|\mathsf{q}\|_{\beta} = \|\mathsf{c}\|_{b}^{2}$ and $(\alpha-1)/\alpha = (1-\beta)/\beta$. In terms of (α, s)-entropies, we then write down

$$E_{\alpha}^{(s)}(\mathsf{p}) + E_{\beta}^{(t)}(\mathsf{q}) \geq \frac{1}{\nu}\, \ln_{\mu}\left\{ \left(\frac{2\pi}{\Delta\varphi}\right)^{\nu} \right\} \ , \tag{48}$$

where $1/\alpha + 1/\beta = 2$, $st > 0$ or $s = t = 0$. Particular cases of Eq. (48) in terms of the Rényi and Tsallis entropies were derived in Refs. [57] and [55], respectively.

For continuous distribution with the probability density function $W(\varphi) = |\Psi(\varphi)|^{2}$, the unified (α, s)-entropy is expressed by Eqs. (18) and (19) with $W(\varphi)$ instead of $P(\theta)$. Note again that only discretized distributions are actually related to real experiments. So, entropic inequalities with continuous distributions are rather interesting as some limiting varieties of inequalities with discrete distributions, when the size of bins tends to zero. First, we square both the inequalities of Eq. (44) and obtain

$$\|W\|_{\alpha} \leq \left(\frac{1}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_{\beta} \ , \qquad \|\mathsf{q}\|_{\alpha} \leq \left(\frac{1}{2\pi}\right)^{(1-\beta)/\beta} \|W\|_{\beta} \ , \tag{49}$$

where $1/\alpha + 1/\beta = 2$ and $\alpha > 1 > \beta$. In the Rényi case, when $s = 0$, the uncertainty relation is posed as

$$R_{\alpha}(W) + R_{\beta}(\mathsf{q}) \geq \ln(2\pi) \ , \tag{50}$$

for all α and β that obey $1/\alpha + 1/\beta = 2$. The derivation is simple. Taking the logarithm of the first inequality of Eq. (49), one gets

$$\frac{1-\alpha}{\alpha}\, \frac{\beta}{1-\beta}\, R_{\alpha}(W) \leq -\ln(2\pi) + R_{\beta}(\mathsf{q}) \ . \tag{51}$$

To obtain Eq. (50) for $\alpha > \beta$, we merely notice that the multiplier of $R_{\alpha}(W)$ is equal to (-1). In a similar manner, we resolve the case, when Rényi's entropy of the distribution q has larger order.

For $s \neq 0$, we cannot generally apply the statement of Lemma 1 in view of the following reasons. Within minimization, the restriction to the domain (29) is crucial. If one of the variables ξ and ζ takes values from both the ranges $(0;1)$ and $(1;+\infty)$ then the reasons from the proof of Lemma 1 fail. So we should restrict our consideration to those functions $W(\varphi)$ that obey $\|W\|_{\alpha} \leq \|W\|_{1} = 1$ for $\alpha > 1$ and $\|W\|_{\beta} \geq \|W\|_{1} = 1$ for $\beta < 1$. For instance, the last conditions are clearly satisfied, when $W(\varphi) \leq 1$ for all $\varphi \in [0; 2\pi]$. Combining Eq. (49) with Lemma 1, we then obtain

$$E_{\alpha}^{(s)}(W) + E_{\beta}^{(t)}(\mathsf{q}) \geq \frac{1}{\nu}\, \ln_{\mu}\left\{ (2\pi)^{\nu} \right\} \ , \tag{52}$$

where $1/\alpha+1/\beta=2$, $st>0$ or $s=t=0$. Values of (α,s)-entropies become negative, when the above conditions on norm-like functionals do not hold for given probability density function $W(\varphi)$. In such a case, entropies of the continuous distribution hardly have a physical meaning. Instead of such entropies, we then consider entropies of a discretized angular distribution and obtain Eq. (48).

Example 3. Changes of states in quantum theory are generally represented by linear maps [58]. Maps of such a kind must be completely positive and, when describe deterministic processes, trace-preserving as well. Each completely positive map Φ can be written in the operator-sum representation. For each operator X on the input Hilbert space, we have

$$\Phi(\mathsf{X})=\sum\nolimits_j \mathsf{A}_j\,\mathsf{X}\,\mathsf{A}_j^{\dagger}\ , \tag{53}$$

where Kraus operators A_j map the input Hilbert space to the output one [48, 58]. The preservation of the trace implies that $\mathrm{tr}\big(\Phi(\mathsf{X})\big)=\mathrm{tr}(\mathsf{X})$ for all X, whence

$$\sum\nolimits_j \mathsf{A}_j^{\dagger}\,\mathsf{A}_j=\mathbb{1}\ . \tag{54}$$

Trace-preserving completely positive maps are usually called "quantum channels" [58]. For given quantum channel, much many operator-sum representations exist. It the sets $\mathcal{A}=\{\mathsf{A}_j\}$ and $\mathcal{B}=\{\mathsf{B}_i\}$ represent the same quantum channel then

$$\mathsf{B}_i=\sum\nolimits_j \mathsf{A}_j\,u_{ji}\ , \tag{55}$$

where the matrix $\mathsf{U}=[[u_{ij}]]$ is unitary [48, 58]. Each set $\mathcal{A}=\{\mathsf{A}_i\}$ that does obey (53) will be named an "unraveling" of the channel Φ. This terminology is due to Carmichael [59] who used this word for a representation of the master equation. Applications of the master equation and its unravelings in quantum control are reviewed in Ref. [60]. It is of interest that there exist so-called "extremal unravelings" [61, 62]. In Ref. [63], we examine those unravelings of a channel that are extremal with respect to (α,s)-entropies. For given density matrix ρ and unraveling $\mathcal{A}=\{\mathsf{A}_j\}$, we define the matrix

$$\Pi(\mathcal{A}|\boldsymbol{\rho}):=\big[\big[\langle\mathsf{A}_i\sqrt{\boldsymbol{\rho}}\,,\mathsf{A}_j\sqrt{\boldsymbol{\rho}}\rangle_{\mathrm{hs}}\big]\big]=\big[\big[\mathrm{tr}(\mathsf{A}_i^{\dagger}\mathsf{A}_j\boldsymbol{\rho})\big]\big]\ . \tag{56}$$

The matrix $\Pi(\mathcal{A}|\boldsymbol{\rho})$ is Hermitian, its diagonal element $p_i=\mathrm{tr}(\mathsf{A}_i^{\dagger}\mathsf{A}_i\boldsymbol{\rho})$ gives the ith effect probability. Then the entropy $E_{\alpha}^{(s)}(\mathcal{A}|\rho)$ is defined by Eq. (13) with these probabilities. Note that matrices of the form (56) were used by Lindblad [64] to introduce the entropic quantity, which is known in quantum information as the entropy exchange. If $\boldsymbol{\rho}$ is taken to be completely mixed, then the entropy exchange coincides with the map entropy of Φ [65]. The latter is another characteristic of the channel related to its Jamiołkowski–Choi representation. Due to Eq. (55), the matrices $\Pi(\mathcal{A}|\rho)$ and $\Pi(\mathcal{B}|\rho)$ are unitarily similar [55], namely

$$\Pi(\mathcal{B}|\boldsymbol{\rho})=\mathsf{U}^{\dagger}\,\Pi(\mathcal{A}|\boldsymbol{\rho})\,\mathsf{U}\ . \tag{57}$$

So, for given unraveling $\mathcal{A}=\{\mathsf{A}_i\}$ we build $\Pi(\mathcal{A}|\rho)$ and diagonalize it as $\mathsf{V}^{\dagger}\,\Pi(\mathcal{A}|\boldsymbol{\rho})\,\mathsf{V}=\mathrm{diag}(\lambda_1,\lambda_2,\ldots)$. Using this unitary matrix $\mathsf{V}=[[v_{ij}]]$, we then define a specific unraveling $\mathcal{A}_{\rho}^{(ex)}$ such that

$$\mathsf{A}_i^{(ex)}:=\sum\nolimits_j \mathsf{A}_j\,v_{ji}\ . \tag{58}$$

In Ref. [63], we have shown the extremality of $\mathcal{A}_\rho^{(ex)}=\{\mathsf{A}_i^{(ex)}\}$ with respect to almost all of the (α,s)-entropies. That is, each unraveling $\mathcal{A}$ of the channel Φ obeys

$$E_\alpha^{(s)}(\mathcal{A}|\boldsymbol{\rho})\geq E_\alpha^{(s)}(\mathcal{A}_\rho^{(ex)}|\boldsymbol{\rho})\ , \tag{59}$$

where $\alpha>0$ and $s\neq 0$. In the case of Rényi's α-entropy, when $s=0$, the unraveling with Kraus operators (58) enjoys the extremality property (59) only for $0<\alpha<1$ [55]. In the case $\alpha=1$ we deal with the Shannon entropy, for which the property (59) was considered in Refs. [61, 62]. We do not consider here unravelings that are extremal with respect to Rényi's entropies of order $\alpha>1$. Such unravelings differ from the unraveling with elements (58) and may also depend on α in general. In view of the lower bound (59), we properly pose entropic uncertainty relations for extremal unravelings of two quantum channels. Suppose that the input density matrix $\boldsymbol{\rho}$ is given. Let $\mathcal{A}_\rho^{(ex)}$ and $\mathcal{B}_\rho^{(ex)}$ be extremal unravelings of the quantum channels $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$, respectively. It follows from Eq. (38) that

$$E_\alpha^{(s)}(\mathcal{A}_\rho^{(ex)}|\boldsymbol{\rho})+E_\beta^{(t)}(\mathcal{B}_\rho^{(ex)}|\boldsymbol{\rho})\geq\frac{1}{\nu}\ln_\mu\Big\{g\big(\mathcal{A}_\rho^{(ex)},\mathcal{B}_\rho^{(ex)}|\boldsymbol{\rho}\big)^{-2\nu}\Big\}\ , \tag{60}$$

where $1/\alpha+1/\beta=2$, $st>0$, the μ and ν are put by Eq. (31). For the particular case of Tsallis' entropies, this relation was derived in Ref. [55]. The uncertainty relation (60) remains valid for the Shannon entropies, when $\alpha=\beta=1$. Except for the latter, either of orders α and β of the two Rényi entropies is larger than one. Hence, the analog of Eq. (60) with Rényi's entropies will include an unknown extremal unraveling, which cannot be found from Eq. (58). So we refrain from presenting such relations here.

5. Entropic formulation of number-phase uncertainty relations

In this section, number-phase uncertainty relations in terms of (α,s)-entropies are derived within the Pegg–Barnett formalism. Since the limit is involved here, we should consider Eq. (23) again and take $N\to\infty$ in its state-independent variety. As a result, a similar inequality with the continuous distribution with respect to θ is obtained. Using this inequality, we derive several number-phase uncertainty relations of a state-independent form in terms of (α,s)-entropies. Most of them are written for discrete phase distributions. The case, which allows relations with continuous phase distribution, is considered as well.

5.1. *Inequalities and entropic relations of "number-phase" type for finite* N

Let $|\psi_{N+1}\rangle\in\mathcal{H}_{N+1}$ be a normalized vector. We can decompose it with respect to the bases $\{|n\rangle\}$ and $\{|\theta_m\rangle\}$. Using the corresponding resolutions of the identity, one gives

$$\mathbb{1}|\psi_{N+1}\rangle=\sum\nolimits_{n=0}^{N}x_n|n\rangle=\sum\nolimits_{m=0}^{N}y_m|\theta_m\rangle\ . \tag{61}$$

The complex numbers $x_n=\langle n|\psi_{N+1}\rangle$ and $y_m=\langle\theta_m|\psi_{N+1}\rangle$ form the two $(N+1)$-tuples x and y, respectively. These tuples satisfy $\|\mathsf{x}\|_2^2=\|\mathsf{y}\|_2^2=\langle\psi_{N+1}|\psi_{N+1}\rangle=1$. So, we can apply the Riesz theorem. Due to the orthonormality, we at once obtain

$$x_n=\sum_{m=0}^{N}\langle n|\theta_m\rangle\,y_m=\sum_{m=0}^{N}\frac{e^{in\theta_m}}{\sqrt{N+1}}\,y_m\ ,\qquad y_m=\sum_{n=0}^{N}\langle\theta_m|n\rangle\,x_n=\sum_{n=0}^{N}\frac{e^{-in\theta_m}}{\sqrt{N+1}}\,x_n\ . \tag{62}$$

In both the cases, all elements of the transformation matrix T by size $(N+1)\times(N+1)$ have the same absolute value equal to $(N+1)^{-1/2}$. Substituting $\eta=(N+1)^{-1/2}$ into Eq. (22), for $1/a+1/b=1$ and $1<b<2$ there holds

$$\|\mathsf{x}\|_a \leq \left(\frac{1}{\sqrt{N+1}}\right)^{(2-b)/b} \|\mathsf{y}\|_b \ , \qquad \|\mathsf{y}\|_a \leq \left(\frac{1}{\sqrt{N+1}}\right)^{(2-b)/b} \|\mathsf{x}\|_b \ . \tag{63}$$

In terms of probabilities $p_m=|y_m|^2=|\langle\theta_m|\psi_{N+1}\rangle|^2$ and $q_n=|x_n|^2=|\langle n|\psi_{N+1}\rangle|^2$, the squared relations (63) are rewritten as

$$\|\mathsf{q}\|_\alpha \leq \left(\frac{1}{N+1}\right)^{(1-\beta)/\beta} \|\mathsf{p}\|_\beta \ , \qquad \|\mathsf{p}\|_\alpha \leq \left(\frac{1}{N+1}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_\beta \ , \tag{64}$$

where $1/\alpha+1/\beta=2$ and $\alpha>1>\beta$. Due to Lemma 1, the inequalities (64) lead to the bound (42) with $(N+1)$ instead of N. So, this is a unified-entropy uncertainty relation for two mutually unbiased bases. To obtain an uncertainty relation for the phase and number operators, we must consider the limit $N\to\infty$. However, the right-hand side of (42) is divergent in this limit. One of ways to formulate a meaningful statement was proposed in Ref. [34]. This approach allows to obtain uncertainty relations in terms of the Shannon entropies [34, 36]. For generalized entropies, however, other methods should be used.

5.2. *Relations between norm-like functionals in the limit* $N\to\infty$

The key idea of our approach is to take the desired limit directly in relations between norms. So, we shall now rewrite the inequalities (63) in an appropriate way. Below we will omit any subscript for those objects that stand for taking $N\to\infty$. To given $(N+1)$-tuple $\mathsf{x}=\{x_n\}$, we assign the function $F_{N+1}(\theta)$ of variable θ, namely

$$F_{N+1}(\theta) := \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N} e^{-in\theta}\, x_n \ . \tag{65}$$

When $N\to\infty$, the infinite series of Eq. (65) defines some 2π-periodic function $F(\theta)$. In view of the relation

$$\frac{1}{2\pi}\int_0^{2\pi} \exp[i(m-n)\theta]\, d\theta = \delta_{mn} \ , \tag{66}$$

there holds

$$\|F_{N+1}\|_2^2 = \int_0^{2\pi} |F_{N+1}(\theta)|^2\, d\theta = \sum\nolimits_{n=0}^{N} |x_n|^2 = 1 \ . \tag{67}$$

By virtue of Eq. (62), the elements of the tuple y are then represented as

$$y_m = \langle\theta_m|\psi_{N+1}\rangle = \sqrt{\frac{2\pi}{N+1}}\, F_{N+1}(\theta_m) \ . \tag{68}$$

The right-hand side of the first inequality in Eq. (63) is therefore rewritten as

$$\left(\frac{1}{\sqrt{N+1}}\right)^{(2-b)/b} \sqrt{\frac{2\pi}{N+1}} \left(\frac{N+1}{2\pi}\right)^{1/b} \left(\sum_{m=0}^{N} |F_{N+1}(\theta_m)|^b \Delta\theta\right)^{1/b} =$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \left(\sum_{m=0}^{N} |F_{N+1}(\theta_m)|^b \Delta\theta\right)^{1/b} \ , \tag{69}$$

where $\Delta\theta = 2\pi/(N+1)$. The set $\{\theta_0, \theta_1, \ldots, \theta_N\}$ with $\theta_N = \theta_0 + 2\pi N/(N+1)$ is a partition of the half-open interval $[\theta_0; \theta_0 + 2\pi)$. For brevity, we take $\theta_0 = 0$. In the limit $N \to \infty$, the sum in the right-hand side of Eq. (69) is clearly converted into a Riemann–Darboux integral of periodic function $F(\theta)$. Hence the first inequality in Eq. (63) becomes

$$\|\mathsf{x}\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \left(\int_0^{2\pi} |F(\theta)|^b \, d\theta\right)^{1/b} = \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|F\|_b \ . \tag{70}$$

In a similar manner, the second inequality in Eq. (63) is merely converted into

$$\left(\int_0^{2\pi} |F(\theta)|^a \, d\theta\right)^{1/a} = \|F\|_a \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{(2-b)/b} \|\mathsf{x}\|_b \ . \tag{71}$$

Since $\|F_{N+1}\|_2 = 1$ by construction, one enjoys $\|F\|_2 = 1$ as well. For pure state $|\psi\rangle = \sum_{n=0}^{\infty} x_n |n\rangle$ (the state (61) in the limit $N \to \infty$), the formulas (5) and (68) give

$$P(\theta) = \lim_{N\to\infty} \frac{N+1}{2\pi} \, |\langle\theta_m|\psi_{N+1}\rangle|^2 = |F(\theta)|^2 \ . \tag{72}$$

The equality $\|F\|_2 = 1$ leads to $\int_0^{2\pi} P(\theta) \, d\theta = 1$. In terms of parameters $\alpha = a/2$ and $\beta = b/2$, we also have $\|F\|_a^2 = \|P\|_\alpha$ and $\|F\|_b^2 = \|P\|_\beta$. Squaring Eqs. (70) and (71), we transform them into the desired relations between norm-like functionals, namely

$$\|\mathsf{q}\|_\alpha \leq \left(\frac{1}{2\pi}\right)^{(1-\beta)/\beta} \|P\|_\beta \ , \qquad \|P\|_\alpha \leq \left(\frac{1}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_\beta \ . \tag{73}$$

These inequalities are analogous to the inequalities (64). We shall now use them for deriving number-phase uncertainty relations in terms of unified entropies.

5.3. *Number-phase uncertainty relations for arbitrary states*

Using the method of Refs. [55, 38], one can obtain an uncertainty relation with some discretization of probability density function $P(\theta)$. For a partition $\{\vartheta_m\}$ of the interval $[0; 2\pi]$, we introduce probabilities by Eq. (17). Due to theorem 192 of the book [53] for integral means, we have

$$\frac{r_m^\alpha}{\Delta\vartheta_m^\alpha} = \left(\frac{1}{\Delta\vartheta_m} \int_{\vartheta_m}^{\vartheta_{m+1}} P(\theta) \, d\theta\right)^\alpha \left\{ \begin{array}{ll} \leq, & \alpha > 1 \\ \geq, & \alpha < 1 \end{array} \right\} \frac{1}{\Delta\vartheta_m} \int_{\vartheta_m}^{\vartheta_{m+1}} P(\theta)^\alpha \, d\theta \ , \tag{74}$$

where $\Delta\vartheta_m = \vartheta_{m+1} - \vartheta_m$. Summing Eq. (74) with respect to m and putting $\Delta\vartheta = \max \Delta\vartheta_m$, we get

$$\Delta\vartheta^{1-\alpha} \, \|\mathsf{r}\|_\alpha^\alpha = \Delta\vartheta^{1-\alpha} \sum\nolimits_m r_m^\alpha \left\{ \begin{array}{ll} \leq, & \alpha > 1 \\ \geq, & \alpha < 1 \end{array} \right\} \int_0^{2\pi} P(\theta)^\alpha \, d\theta = \|P\|_\alpha^\alpha \ . \tag{75}$$

Using the last relation, we convert the inequalities (73) to the form

$$\|\mathsf{q}\|_\alpha \leq \left(\frac{\Delta\vartheta}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{r}\|_\beta \ , \qquad \|\mathsf{r}\|_\alpha \leq \left(\frac{\Delta\vartheta}{2\pi}\right)^{(1-\beta)/\beta} \|\mathsf{q}\|_\beta \ , \tag{76}$$

where $1/\alpha + 1/\beta = 2$ and $\alpha > 1 > \beta$. Combining the statement of Lemma 1 with Eq. (76), the uncertainty relations are written in the following way.

Theorem 2 *Let $\{\vartheta_m\}$ be a partition of the interval $[0;2\pi]$. The unified entropies of the corresponding probability distribution $\mathsf{r}=\{r_m\}$, defined by Eq. (17), and the number probability distribution $\mathsf{q}=\{q_n\}$ satisfy*

$$E_\alpha^{(s)}(\mathsf{r})+E_\beta^{(t)}(\mathsf{q})\geq\frac{1}{\nu}\,\ln_\mu\left\{\left(\frac{2\pi}{\Delta\vartheta}\right)^\nu\right\}\ . \tag{77}$$

Here $st>0$ or $s=t=0$, positive α and β obey $1/\alpha+1/\beta=2$, the parameters μ and ν are defined by Eq. (31).

Since all the bins are strictly less than 2π, this bound on the sum of two entropies is nontrivial. For the particular cases $s=t=0$ and $s=t=1$, the above uncertainty relation is reduced to the entropic bounds

$$R_\alpha(\mathsf{r})+R_\beta(\mathsf{q})\geq\ln\bigl(2\pi/\Delta\vartheta\bigr)\ ,\qquad H_\alpha(\mathsf{r})+H_\beta(\mathsf{q})\geq\ln_\mu\bigl(2\pi/\Delta\vartheta\bigr)\ . \tag{78}$$

These bounds are number-phase relations in terms of the Rényi and Tsallis entropies, respectively. Entropic inequalities, involving the continuous distribution $P(\theta)$, are also of interest. For the Rényi case, number-phase uncertainty relations are posed similarly to Eq. (50).

Theorem 3 *Under the condition $1/\alpha+1/\beta=2$, the Rényi entropies of the phase probability density $P(\theta)$ and the number probability distribution $\mathsf{q}=\{q_n\}$ satisfy*

$$R_\alpha(P)+R_\beta(\mathsf{q})\geq\ln(2\pi)\ . \tag{79}$$

For $\alpha=\beta=1$, the inequality (79) gives the lower bound $\ln(2\pi)$ on the sum of the corresponding Shannon entropies. This bound was presented in Refs. [34, 35]. Some doubts with the entropies of continuous distributions were mentioned above. Nevertheless, if we restrict our consideration to probability density functions, obeying $P(\theta)\leq 1$ for all $\theta\in[0;2\pi]$, then the entropic functionals (18) and (19) are positive-valued. Hence the entropic uncertainty relations are written as

$$E_\alpha^{(s)}(P)+E_\beta^{(t)}(\mathsf{q})\geq\frac{1}{\nu}\,\ln_\mu\bigl\{(2\pi)^\nu\bigr\}\ . \tag{80}$$

As above, $st>0$, the parameters $\alpha>0$ and $\beta>0$ satisfy $1/\alpha+1/\beta=2$, the parameters μ and ν are defined by Eq. (31).

5.4. *Number-phase uncertainty relations for multiphoton coherent states*

Consider coherent states $|z\rangle$ with the mean photon number $|z|^2\gg 1$. As usual, we approximate the corresponding phase and number distributions by the Gaussian functions (7) and (8). For the Fourier transform, the exact values of (b,a)-norms were found by Beckner [14]. In the notation (9), the relations between norms are such that

$$\|\widetilde{w}\|_a\leq c_b\,\|\widetilde{p}\|_b\ ,\qquad \|\widetilde{p}\|_a\leq c_b\,\|\widetilde{w}\|_b\ ,\qquad c_b^2=\left(\frac{b}{2\pi}\right)^{1/b}\left(\frac{2\pi}{a}\right)^{1/a}\ , \tag{81}$$

where $1/a+1/b=1$ and $a>2>b$. Using $\|\widetilde{w}\|_a^2=\|\widetilde{W}\|_\alpha$ and $\|\widetilde{p}\|_b^2=\|\widetilde{P}\|_\beta$ and squaring Eq. (81), we obtain

$$\|\widetilde{W}\|_\alpha\leq C_\beta\,\|\widetilde{P}\|_\beta\ ,\qquad \|\widetilde{P}\|_\alpha\leq C_\beta\,\|\widetilde{W}\|_\beta\ ,\qquad C_\beta=\left(\frac{\beta}{\pi}\right)^{1/(2\beta)}\left(\frac{\pi}{\alpha}\right)^{1/(2\alpha)}\ . \tag{82}$$

As above, we put $\alpha = a/2$ and $\beta = b/2$ so that $1/\alpha + 1/\beta = 2$ and $\alpha > 1 > \beta$. Let us introduce the probabilities $\widetilde{r}_m$ by Eq. (17) with $\widetilde{P}(\theta)$ and

$$\widetilde{q}_n = \int_n^{n+1} \widetilde{W}(y)\, dy \ , \tag{83}$$

for integer n. Similarly to Eq. (76), we derive from Eq. (82) that

$$\|\tilde{\mathsf{q}}\|_\alpha \leq \left(\frac{\Delta\vartheta}{\pi}\right)^{(1-\beta)/\beta} \frac{\beta^{1/(2\beta)}}{\alpha^{1/(2\alpha)}}\, \|\tilde{\mathsf{r}}\|_\beta \ , \qquad \|\tilde{\mathsf{r}}\|_\alpha \leq \left(\frac{\Delta\vartheta}{\pi}\right)^{(1-\beta)/\beta} \frac{\beta^{1/(2\beta)}}{\alpha^{1/(2\alpha)}}\, \|\tilde{\mathsf{q}}\|_\beta \ . \tag{84}$$

Combining these inequalities with the statement of Lemma 1 finally yields the entropic uncertainty relation

$$E_\alpha^{(s)}(\tilde{\mathsf{r}}) + E_\beta^{(t)}(\tilde{\mathsf{q}}) \geq \frac{1}{\nu}\, \ln_\mu \left\{ \left(\frac{\varkappa\pi}{\Delta\vartheta}\right)^{\nu} \right\} \ . \tag{85}$$

Here $st > 0$ or $s = t = 0$, the parameters α and β obey $1/\alpha + 1/\beta = 2$, and square of the factor $\varkappa$ is put by

$$\varkappa^2 = \alpha^{1/(\alpha-1)}\beta^{1/(\beta-1)} = \exp_\alpha(1/\alpha)\, \exp_\beta(1/\beta) \ . \tag{86}$$

Under the condition $1/\alpha + 1/\beta = 2$, the right-hand side of Eq. (86) monotonically increases with $\beta \in (1/2; 1)$. The factor increases from $\varkappa = 2$ for $\beta = 1/2$ right up to $\varkappa = e$ for $\beta = 1$. For the former, the multiphoton-state bound (85) concurs with the general bound (77). The distinction between the two bounds is maximal, when both the α and β tend to one. In this case, the relation in terms of the Shannon entropies reads

$$S(\tilde{\mathsf{r}}) + S(\tilde{\mathsf{q}}) \geq \ln\left(\frac{e\pi}{\Delta\vartheta}\right) \ . \tag{87}$$

Similar to Eqs. (50) and (79), one also gets an uncertainty relation in terms of Rényi's entropies in the form

$$R_\alpha(\widetilde{P}) + R_\beta(\widetilde{W}) \geq \ln(\varkappa\pi) \ , \tag{88}$$

where $1/\alpha + 1/\beta = 2$ and integrals of the probability density functions are involved. For the position and momentum, the same entropic bound was derived in Ref. [57]. For the Shannon entropies, the relation (88) gives the lower bound $\ln(e\pi)$ previously presented in Refs. [33, 34, 35]. Note that the relation (85) with finite resolutions can easily be reformulated for the case of canonically conjugate position and momentum. We refrain from presenting the details here.

6. Conclusions

We have considered a formulation of number-phase uncertainty relations in terms of unified entropies, which include both the Rényi and Tsallis ones as particular cases. For two generalized measurements, unified-entropy uncertainty relations of state-dependent as well as state-independent form are derived. Using Riesz's theorem, nontrivial inequalities between norm-like functionals of two generated probability distributions were obtained. The proposed method combines this constraint with a task of minimizing a certain function of two variables in the corresponding domain of their acceptable values. Hence we have stated unified-entropy uncertainty relations which are an extension of previous bounds in terms of the Rényi and Tsallis entropies. We also gave examples of two complementary observables in a N-level system, angle and angular momentum, and extremal unravelings of two quantum channels.

To obtain number-phase uncertainty relations, the infinite-dimensional limit should be treated properly. Using the Pegg–Barnett formalism and the Riesz theorem, we derived an inequality for functionals of the corresponding probability distributions in finite dimensions. Then we have taken the limit $N \to \infty$ right in this inequality. Hence unified-entropy uncertainty relations for the number-phase pair are naturally established. As it was shown, entropic bounds can be improved for the case of coherent states with large mean number of photons. The derived uncertainty relations are mainly formulated for phase distribution discretized with respect to finite experimental resolutions. Some entropic bounds with the continuous phase distribution were given as well. A lot of the presented results is a proper extension of entropic uncertainty relations previously reported in the literature.

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