Quantum Information and Computation, Vol. 12, No. 5&6 (2012) 0442–0447

OPTIMAL ESTIMATION OF QUANTUM PROCESSES USING INCOMPLETE INFORMATION: VARIATIONAL QUANTUM PROCESS TOMOGRAPHY

THIAGO O. MACIEL[a] REINALDO O. VIANNA[b]

Departamento de Física - ICEx - Universidade Federal de Minas Gerais

Av. Antônio Carlos 6627 - Belo Horizonte - MG - Brazil - 31270-90

Received March 30, 2011

Revised February 3, 2012

We develop a quantum process tomography method, which variationally reconstruct the map of a process, using noisy and incomplete information about the dynamics. The new method encompasses the most common quantum process tomography schemes. It is based on the variational quantum tomography method (VQT) proposed by Maciel *et al.* in arXiv:1001.1793[quant-ph] [1].

Keywords:

Communicated by: R Cleve & G Milburn

The characterization of a quantum system and its dynamics is a daunting challenge. The first question which arises in this scenario is what information should we possess to characterize the dynamics. To answer this question, one needs to choose a *quantum process tomography* (QPT) scheme. Each procedure demands different resources and operations.

There are four general types of QPT procedures: (i) *standard quantum process tomography* (SQPT)[2]; (ii) *ancilla-assisted process tomography* (AAPT)[3, 4, 5]; (iii) *direct characterization of quantum dynamics* (DCQD)[6, 7, 8]; (iv) *selective and efficient quantum process tomography*(SEQPT)[9, 10, 11].

In (i) the information is obtained indirectly, performing a set of *quantum state tomographies*(QST)[12, 13, 14, 15, 1, 16] of the linear independent states, which spans the Hilbert-Schmidt space of interest, after the action of the unknown map. The second scheme (ii) - also an indirect procedure - makes use of an auxiliary system. The information is then extracted by means of QST of the joint space (system and ancilla). The third one (iii) obtains the dynamical information directly - by means of *quantum error detection* (QED)[17] concepts - measuring stabilizers and normalizers. Finally, the last method (iv) - which also measures the parameters directly - consists in estimating averages over the entire Hilbert space of products of expectation values of two operators. For the special case of one-parameter quantum channels, there is also an interesting method developed by Sarovar and Milburn [18].

[a]maciel@gmail.com
[b]reinaldo@fisica.ufmg.br

In [1], we developed a variational quantum tomography method (VQT). The VQT approach can reconstruct a quantum state with high fidelity out of incomplete and noisy information. The method was successfully employed in a quantum optics experiment, where entangled qutrits were generated [16]. In this letter, we extend VQT to the tomography of quantum processes. The new method inherits all the advantages of the VQT, and opens the door to the characterization of maps in larger systems, where *myriad* of measurements could be necessary. With the variational quantum process tomography method (VQPT) we propose here, maps can be reconstructed, with high fidelity, using just a fraction of the effort employed in an Informationally Complete Measurement.

The method we derive has the particular form of a linear convex optimization problem, known as Semidefinite Program (SDP), for which efficient and stable algorithms are available [19, 20, 21]. SDP consists of minimizing a linear objective under a linear matrix inequality constraint, precisely,

$$\begin{gathered} \textit{minimize } c^\dagger x \\ \textit{subject to} \begin{cases} F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0, \end{cases} \end{gathered} \tag{1}$$

where $c \in C^m$ and the Hermitian matrices $F_i \in C^{n\times n}$ are given, and $x \in C^m$ is the vector of optimization variables. $F(x) \geq 0$ means that $F(x)$ is a positive matrix. The problem defined in Eq.1 has no local minima. When the unique minimum of this problem cannot be found analytically, one can resort to powerful algorithms that return the exact answer [20]. To solve the problem in Eq.1 could be compared to finding the eigenvalues of a Hermitian matrix. If the matrix is small enough or has very high symmetry, one can easily determine its eigenvalues on the back of an envelope, but in other cases some numerical algorithm is needed. Anyway, one never doubts that the eigenvalues of such a matrix can be determined to arbitrary precision.

A *bona fide* completely positive and non increasing trace map $\mathcal{E}$ can be generally represented as [22]

$$\mathcal{E}(\rho) = \sum_{i,j=1}^{d^2} \chi_{ij} E_i \rho E_j^\dagger, \tag{2}$$

where ρ is the system initial state and the $\{E_m\}$ form an IC-POVM (Informationally Complete Positive Operator Valued Measure), *i.e.* a complete basis in the Hilbert-Schmidt space satisfying

$$\sum_{i=1}^{d^2} E_i^\dagger E_i = \mathbf{I}. \tag{3}$$

The $\{\chi_{ij}\}$ defines the super-operator χ, which has all the information about the process. It is a Hermitian positive operator. Thus the super-operator can be thought as a $d^2 \times d^2$ density matrix in the Hilbert-Schmidt space with d^4 independent real parameters (or $d^4 - d^2$ in the trace preserving case). More precisely, for a process whose rank is r, the number of independent parameters scales as $O(r \times d^2)$ [22]. Therefore, the number of POVM elements to reconstruct the map is also of order $O(r \times d^2)$.

Now we recast both SQPT(i) and AAPT(ii) - which rely on tomography of states - using the VQT[1] methodology. We will name the output states of the unknown map as $\tilde{\varrho}^k =$

$\mathcal{E}(\rho^k) = \sum_{i,j=1}^{d^2} \chi_{ij} E_i \rho^k E_j^\dagger$. Suppose n^k elements of the POVM ($n^k < d^2$) in the k^{th} output state have been measured, namely

$$Tr(E_\lambda \rho^k) = p_\lambda^k,\ \lambda \in [1, n^k]. \tag{4}$$

Note that p_λ^k are positive numbers, the *known* frequencies. The frequencies obtained from an experiment are noisy, thus:

$$(1 - \Delta_\lambda^k) p_\lambda^k \leq Tr(\tilde{\varrho}^k E_\lambda^k) \leq (1 + \Delta_\lambda^k) p_\lambda^k,\ \lambda \in [1, n^k], \tag{5}$$

with Δ_λ^k positive and hopefully small. Let us refer to the unmeasured POVM elements as the *unknown* subset. Then we can define the *Hamiltonian*

$$H^k = \sum_{\lambda = n^k + 1}^{d^2} E_\lambda^k. \tag{6}$$

Thus we obtain a cost function over the unknown subset, namely,

$$Tr(\tilde{\varrho}^k H^k) \equiv Tr(\sum_{i,j=1}^{d^2} \tilde{\chi}_{ij} E_i \rho^k E_j^\dagger H^k).$$

This linear functional should be minimized, for we do not know the action of the map on the unknown subset.

The SQPT method demands quantum state tomography in all linearly independent states which span the Hilbert-Schmidt space. With n^k measurements in k_t different states, the variational SQPT reads:

$$\mathit{minimize}\ (\sum_k Tr(\tilde{\varrho}^k H^k) + \sum_{\lambda=1}^{n^k} \Delta_\lambda^k)$$

$$\mathit{subject\ to} \begin{cases} \tilde{\chi} \geq 0, \\ Tr(\tilde{\varrho}^k) \leq 1, \\ \Delta_\lambda^k \geq 0, \\ (1 - \Delta_\lambda^k) p_\lambda^k \leq Tr(\tilde{\varrho}^k E_\lambda^k) \leq (1 + \Delta_\lambda^k) p_\lambda^k, \\ \forall \lambda^k \in [1, n^k]\ \ \mathit{and}\ \ k = [1, k_t]. \end{cases} \tag{7}$$

Eq.7 returns a map $\tilde{\chi}$ which is the optimal approximation to the unknown process χ. Note that, at the same time, we were able to identify $\tilde{\varrho}^k$ optimally.

In Fig.1 and Fig.2, we illustrate the application of the method for the reconstruction of two-qubit processes. We use the trace distance of the reconstructed map (χ) to the ideal map (χ_{ideal}) as the figure of merit. In Fig.1, we plot the minimum number of POVM elements necessary to reconstruct a random map to a precision of $Tr|\chi - \chi_{ideal}| < 10^{-2}$. The measurement sequence of the POVM elements is the same for all processes and is arbitrary. Of course, for a particular process, there is an optimal choice of POVM elements that minimizes the number of measurements. This was investigated, in the context of state tomography, in one of our previous works [16]. Here, as we are not assuming any previous knowledge about the process, we have no reason to choose a particular sequence of POVM elements *a priori.*

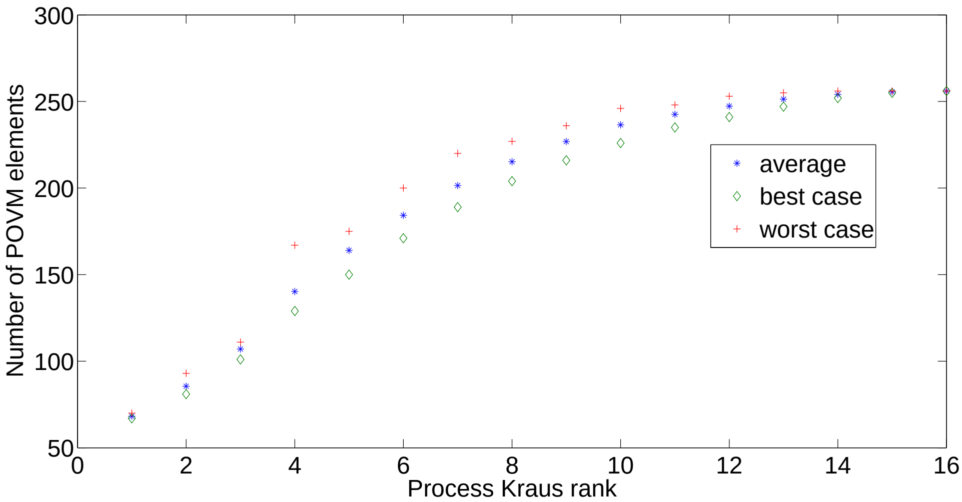

Fig. 1. Reconstruction of two-qubit processes of all ranks. We plot the number of independent POVM elements necessary to reconstruct the map against the process rank. For each rank, we randomly generated 100 processes, and showed the minimum (best case) and maximum (worst case) number of POVM elements needed for the reconstruction. It is also shown the average number of POVM elements employed for the reconstruction in each rank. We consider a map (χ) reconstructed, when its trace distance to the ideal map (χ_{ideal}) is less than 10^{-2} ($Tr|\chi - \chi_{ideal}|/2 < 10^{-2}$).

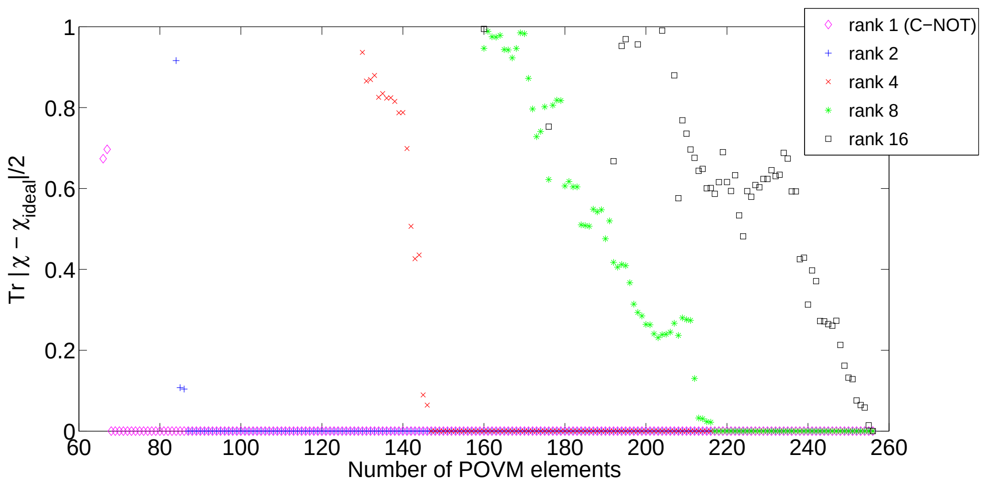

Fig. 2. Map reconstruction's convergence as a function of the number of POVM elements measured. The rank-1 map is the C-Not gate, and the other two-qubit processes of higher rank were generated randomly. The convergence is monitored by the trace distance of the reconstructed map (χ) to the ideal map (χ_{ideal}), $Tr|\chi - \chi_{ideal}|/2$.

Note that the $O(r \times d^2)$ scaling of number of POVM elements against the rank (r) of the process is clearly shown in Fig.1.

In Fig.2, we track the convergence of the reconstruction as a function of the number of measured POVM elements. As a representative of the rank-1 process we take the C-Not gate, and for the other ranks we generate random processes. The figure suggests a practical way to decide when to stop measuring the POVM elements. Take for instance the rank-4 process. It converges at about the 150^{th} POVM element, and the convergence curve is a straight line parallel to the $x-$axis after that measurement. Therefore, taking some reference map to measure the trace distance, one could stop measuring when the convergence curve stops varying, as discussed.

In the AAPT method, one adds an ancillary system with the same dimension of the main one. Then the quantum process takes place in half subspace, and finally a quantum state tomography is performed in the whole space, ancilla plus main system. The output state now reads $\tilde{\varrho} = (\mathbf{I} \otimes \mathcal{E})(\rho) = \sum_{i,j=1}^{d^2} \tilde{\chi}_{ij} (\mathbf{I} \otimes E_i) \rho^k (\mathbf{I} \otimes E_j)^\dagger$. With n measurements performed in this scheme, AAPT can be recast as

$$minimize \ (Tr(\tilde{\varrho} H) + \sum_{\lambda=1}^{n} \Delta_\lambda)$$

$$subject\ to \begin{cases} \tilde{\chi} \geq 0, \\ Tr(\tilde{\varrho}) \leq 1, \\ \Delta_\lambda \geq 0, \\ (1-\Delta_\lambda)p_\lambda \leq Tr(\tilde{\varrho} E_\lambda) \leq (1+\Delta_\lambda)p_\lambda, \\ \forall \lambda \in [1, n]. \end{cases} \tag{8}$$

In conclusion, we have introduced a new method to perform quantum process tomography, which reconstructs a map, with high fidelity, using noisy and incomplete information. The method is linear and convex, and its unique solution can be obtained very efficiently. It opens the door to the characterization of the dynamics of larger quantum systems, avoiding the need of very large informationally complete measurements.

Acknowledgments

Financial support by the Brazilian agencies FAPEMIG, and INCT-IQ (National Institute of Science and Technology for Quantum Information).

References

1. T.O. Maciel, A.T. Cesário, R. O. Vianna, Int. J. Mod. Phys. C **22**, 1361 (2011); arXiv:1001.1793v[quant-ph].
2. I. L. Chuang and M. A. Nielsen, J. Mod. Opt., **44**, 2455 (1997). quant-ph/9610001.
3. G.M. D'Ariano, and P. Lo Presti, Phys. Rev. Lett., **86** 4195 (2001)
4. D.W. Leung, Ph.D. thesis Stanford University, 2000; J. Math. Phys. **44**, 528 (2003).
5. J.B. Altepeter, D. Branning, E. Jeffrey, T.C. Wei, P.G. Kwiat, R.T. Thew, J.L. O'Brien, M.A. Nielsen, and A.G. White, Phys. Rev. Lett. **90**, 193601 (2003).
6. M. Mohseni, and D.A. Lidar, Phys. Rev. Lett. **97**, 170501 (2006).
7. M. Mohseni, and D.A. Lidar, Phys. Rev. A **75**, 062331 (2007).
8. M. Mohseni, A.T. Rezakhani, J.T. Barreiro, P.G. Kwiat, and A. Aspuru-Guzik, Phys. Rev. A **81**, 032102 (2010).
9. A. Bendersky, F. Pastawski, and J.P. Paz, Phys. Rev. Lett. **100**, 190403 (2008).

10. A. Bendersky, F. Pastawski, and J.P. Paz, Phys. Rev. A **80**, 032116 (2009).
11. C.T. Schmiegelow, M.A. Larotonda, and J.P. Paz, Phys. Rev. Lett. **104**, 123601 (2010).
12. G.M. D'Ariano, D.F. Magnani, P. Perinotti, Phys. Lett. A, **373**, 111-115 (2009).
13. G.M. D'Ariano, P. Perinotti, Phys. Rev. Lett., **98**, 020403 (2007).
14. J. Rehacek, Z. Hradil, E. Knill, A.I. Lvovsky, Phys. Rev. A, **75**, 042108 (2007).
15. T.O. Maciel, and R.O. Vianna, Phys. Rev. A, **80**, 032325 (2009).
16. G. Lima, E. S. Gómez, A. Vargas, R. O. Vianna, and C. Saavedra, Phys. Rev. A **82**, 012302 (2010).
17. M.A. Nielsen, and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
18. M. Sarovar, and G.J. Milburn, J. Phys. A: Math. Gen. **39**, 8487 (2006).
19. S. Boyd, L. Vandenberghe, *Convex Optimization*, (Cambridge University Press, Cambridge, 2000).
20. J.F. Sturm, Optim. Methods and Software **11**, 625-653 (1999); SEDUMI: `http://fewcal.kub.nl/sturm/softwares/sedumi.html`.
21. J. Löfberg, YALMIP : A Toolbox for Modeling and Optimization in MATLAB, Proceedings of the CACSD Conference, 2004, Taipei, Taiwan, (unpublished) `http://control.ee.ethz.ch/~joloef/yalmip.php`.
22. I. Bengtson and K. Życzkowski, *Geometry of Quantum States (An introduction to QUANTUM ENTANGLEMENT)* (Cambridge University Press, Cambridge, 2006).