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STEADY-STATE CORRELATIONS OF TWO ATOMS INTERACTING WITH A RESERVOIR

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We consider two two-level atoms fixed at different positions, driven by a resonant monochromatic laser field, and interacting collectively with the quantum electromagnetic field. A Born-Markov-secular master equation is used to describe the dynamics of the two atoms and the steady-state is obtained analytically for a configuration of the atoms. The steady-state populations of the energy levels of the free atoms, entanglement, quantum and geometric discords and degree of mixedness are calculated analytically as a function of the laser field intensity and the distance between the two atoms. It is found that there is a possibility of considerable steady-state entanglement and left/right quantum discord and that these can be controlled either by increasing/decreasing the intensity of the laser field or by increasing/decreasing the distance between atoms. It is shown that the system of two atoms can be prepared in a separable mixed state with non-zero quantum discord that turns into an X-state for high laser field intensities. The behavior and relationships between the different correlations are studied and several limiting cases are investigated.

Keywords: Entanglement, quantum discord, open systems, two-level atoms
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1 Introduction

For the past two decades there has been an ever growing interest in quantifying and characterizing the correlations in the states of quantum systems. This was initially provoked by the realization that quantum correlations known as entanglement could be harnessed as a resource to bring significant advantage for computing and information processing [1]. Now, one of the main interests is to identify which correlations are responsible for these advantages. Moreover, it has been recognized that quantifying correlations in quantum systems is a difficult task and, in general, several measures are needed to capture all of their subtleties.

In order to understand correlations, one generally starts with bipartite systems. Consider two quantum systems $\mathcal{A}$ and $\mathcal{B}$. The state of the composite system $\mathcal{A}+\mathcal{B}$ can be described by a density operator $\rho_{\mathcal{AB}}$ which contains both classical and quantum correlations. A widely accepted measure of the total correlations in $\rho_{\mathcal{AB}}$ is the *quantum mutual information*. It is defined as

$$I(\rho_{\mathcal{AB}}) \;=\; S(\rho_{\mathcal{A}})+S(\rho_{\mathcal{B}})-S(\rho_{\mathcal{AB}})\;, \tag{1}$$

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where $\rho_{\mathcal{A}}$ ($\rho_{\mathcal{B}}$) is the density operator of $\mathcal{A}$ ($\mathcal{B}$) and $S(\rho) = -\mathrm{Tr}[\rho \log_2(\rho)]$ is the von Neumann entropy of ρ (the trace is taken in the state space where ρ is a density operator). It has been found that $I(\rho_{\mathcal{AB}})$ measures the asymptotically minimal amount of local noise one has to add to turn $\rho_{\mathcal{AB}}$ into a product state [2]. Also, $I(\rho_{\mathcal{AB}})$ is the maximum amount of information that Alice can send secretly to Bob using $\rho_{\mathcal{AB}}$ as a one-time pad [3].

Once the total correlations in $\rho_{\mathcal{AB}}$ have been quantified, it is natural to ask whether these can be clearly divided into a classical and quantum part. Several axioms have been proposed as requirements of a measure of the classical correlations $C^{cl}(\rho_{\mathcal{AB}})$ in $\rho_{\mathcal{AB}}$ [4]. These consist in being zero for product states, being invariant under local unitary transformations and non-increasing under local operations, and being equal to $S(\rho_{\mathcal{A}}) = S(\rho_{\mathcal{B}})$ for pure states $\rho_{\mathcal{AB}}$. Also, it has been pointed out that a measure of the classical correlations should quantify the correlation between $\mathcal{A}$ and $\mathcal{B}$ instead of a property of only one of them. Taking these axioms as a basis, the following measures for the classical correlations in $\rho_{\mathcal{AB}}$ have been proposed [4]:

$$C^{cl}_{\mathcal{B}}(\rho_{\mathcal{AB}}) = S(\rho_{\mathcal{A}}) - min_{\{B_i\}} \sum_i p_i S(\rho^i_{\mathcal{A}}) \ , \tag{2}$$

$$C^{cl}_{\mathcal{A}}(\rho_{\mathcal{AB}}) = S(\rho_{\mathcal{B}}) - min_{\{A_j\}} \sum_j p'_j S(\rho^j_{\mathcal{B}}) \ , \tag{3}$$

where the minimum is taken over all sets $\{B_i\}$ ($\{A_j\}$) of one-dimensional projectors of $\mathcal{B}$ ($\mathcal{A}$) that sum up to the identity and that constitute measurements performed only on $\mathcal{B}$ ($\mathcal{A}$). Here we have restricted to one-dimensional projective measurements, but the original reference [4] uses POVM. Also,

$$\rho^i_{\mathcal{A}} = \frac{1}{p_i}\mathrm{Tr}_{\mathcal{B}}\left[(I\otimes B_i)\rho_{\mathcal{AB}}(I\otimes B_i^\dagger)\right] \quad \text{with } p_i = \mathrm{Tr}_{\mathcal{AB}}\left[\ (I\otimes B_i)\rho_{\mathcal{AB}}(I\otimes B_i^\dagger)\ \right] \ , \tag{4}$$

is the density operator of $\mathcal{A}$ after obtaining the result associated with B_i in a measurement of $\mathcal{B}$ (similar equations hold for $\rho^j_{\mathcal{B}}$ and p'_j). These measures satisfy the axioms mentioned above and are equal to zero if and only if $\rho_{\mathcal{AB}} = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$. Hence, it is in accordance with the accepted idea that only product states are devoid of correlations. Nevertheless, these measures for the classical correlations are dependent on which system is measured, that is, $C^{cl}_{\mathcal{A}}(\rho_{\mathcal{AB}}) \neq C^{cl}_{\mathcal{B}}(\rho_{\mathcal{AB}})$ in general. Thus, they depend on the properties of each subsystem.

Let us now turn to quantum correlations. Quantum states $\rho_{\mathcal{AB}}$ are normally divided into separable or entangled. Let us remember that $\rho_{\mathcal{AB}}$ is separable if it can be expressed in the form

$$\rho_{\mathcal{AB}} = \sum_j p_j \rho_{\mathcal{A},j} \otimes \rho_{\mathcal{B},j} \tag{5}$$

with $\rho_{\mathcal{A},j}$ ($\rho_{\mathcal{B},j}$) density operators of $\mathcal{A}$ ($\mathcal{B}$), and $p_j \in [0,1]$ such that $\sum_j p_j = 1$. If $\rho_{\mathcal{AB}}$ is not a separable state, then it is an entangled one. It was thought that entanglement embodied all the quantum correlations in $\rho_{\mathcal{AB}}$, and that separable states were purely classical. Nevertheless, it has been realized that entanglement is not the only aspect of quantum correlations, since some separable states may still present non-classical correlations [5]-[9]. One way to measure the non-classicality of the correlations in $\rho_{\mathcal{AB}}$ is to use the quantum discord [5]. This quantity is defined to be the difference between the quantum mutual information (1) and the correlations (2):

$$D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}}) = I(\rho_{\mathcal{AB}}) - C^{cl}_{\mathcal{A}}(\rho_{\mathcal{AB}}) \ ,$$

$$D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}}) \; = \; I(\rho_{\mathcal{AB}}) - C^{cl}_{\mathcal{B}}(\rho_{\mathcal{AB}}) \; , \tag{6}$$

$D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}})$ ($D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}})$) is usually referred to as the left (right) quantum discord. It has been shown that $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}})$, $D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}})$ are always non-negative [5]. In fact, $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}}) = 0$ if and only if

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes \rho_k \tag{7}$$

where $\{|\psi_k\rangle\}$ is an orthonormal basis of $\mathcal{A}$, ρ_k are density operators of $\mathcal{B}$, and p_k are non-negative numbers such that $\sum_k p_k = 1$ [5]. A similar result holds for $D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}}) = 0$.

In general $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}}) \neq D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}})$. The quantum discord $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}})$ can be interpreted to be a measure of the information of $\mathcal{B}$ contained in the correlations between $\mathcal{A}$ and $\mathcal{B}$ in $\rho_{\mathcal{AB}}$ that cannot be extracted by performing (one-dimensional projective) measurements only on $\mathcal{A}$ [5]. Hence, if $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}})$ is large, a lot of information of $\mathcal{B}$ is destroyed by any measurement on $\mathcal{A}$ alone; if $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}})$ is small, almost all the information about $\mathcal{B}$ contained in the correlations between $\mathcal{A}$ and $\mathcal{B}$ in $\rho_{\mathcal{AB}}$ can be obtained by measurements only on $\mathcal{A}$. Moreover, if $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}}) = 0$, then $I(\rho_{\mathcal{AB}}) = C^{cl}_{\mathcal{A}}(\rho_{\mathcal{AB}})$ and there is a non-selective measurement (namely that defined by the $\{ \, |\psi_k\rangle \, \}$ in (7)) such that the state of $\mathcal{A}+\mathcal{B}$ after the measurement coincides with that before the measurement, that is, there is a non-selective measurement that does not perturb the state of the system. It follows that if classical information is understood as correlations that can be obtained without perturbing the state of the system, then $C^{cl}_{\mathcal{A}}(\rho_{\mathcal{AB}})$ is a measure of the classical correlations in $\rho_{\mathcal{AB}}$ and $\rho_{\mathcal{AB}}$ is a classically correlated state when $D^Q_{\mathcal{A}}(\rho_{\mathcal{AB}}) = 0$ [5]. Similar interpretations hold for the case $D^Q_{\mathcal{B}}(\rho_{\mathcal{AB}}) = 0$.

Once quantum discord is recognized as a measure of non-classical correlations, it is natural to ask if it measures the same correlations as entanglement. Given that some separable mixed states have non-zero quantum discord and separable states by definition do not have entanglement, it is concluded that entanglement and quantum discord are in general different quantities. Nevertheless, they do coincide when $\rho_{\mathcal{AB}}$ is a pure state or a mixture of Bell states [6]. Furthermore, it has been found that in general the quantum discord of $\rho_{\mathcal{AB}}$ is not simply the sum of some measure of the entanglement in $\rho_{\mathcal{AB}}$ and some other non-classical correlation [6, 7].

It is the purpose of this article to study the entanglement, quantum discord, classical correlations, and degree of mixed-ness in the steady-state of the following open quantum system: two two-level atoms (qubits in the jargon of quantum information) at fixed positions driven by a resonant monochromatic laser field and interacting collectively with all the modes of the quantum electromagnetic field. In the notation used above, $\mathcal{A}$ will be one of the atoms (say, the atom at position $\mathbf{r}_1$) and $\mathcal{B}$ will be the other atom (say, the atom at position $\mathbf{r}_2$). Here we also have a third party $\mathcal{C}$ playing a decisive role in the dynamics: the quantum electromagnetic field which we will consider as a reservoir. The system is open because we are interested in studying only the subsystem $\mathcal{A}+\mathcal{B}$ of the complete system $\mathcal{A}+\mathcal{B}+\mathcal{C}$. Since the two atoms are interacting collectively with the reservoir, $\mathcal{C}$ acts as a medium that can allow quantum correlations to be formed between the two atoms for some time. In order that these correlations have the possibility of being long-lived and not being ultimately destroyed by the reservoir, the two atoms will be driven by a laser field.

Using the same system as ours, but without the driving field, the dynamical generation of entanglement between two atoms due to the collective interaction with the reservoir has been recently studied [10, 11]. It was found that, as a result of the interaction through the reservoir, the system develops non-negligible entanglement (as measured by the concurrence) for a period of time if the atoms are close enough and if initially one is in the excited state and the other is in the ground state. Also, it was reported that entanglement initially present in the system of two atoms is more robust when the two atoms are close when compared to the case where the atoms are far apart. The entanglement present in a system of two driven non-identical two-level atoms in a special configuration (namely one atom is located at a node while the other atom is placed at an antinode of a driving laser field with a standing wave cosine structure) has also been studied [12]. The steady-state density operator of the system was obtained numerically and the concurrence was evaluated. It was found that the two-atom system decays to a stationary entangled state only when the Rabi frequency equals the difference between the two atoms' transition frequencies. The entanglement in systems similar to ours has also been studied [13]-[15]. Using the same system as ours (but with a plane-wave laser field) the time-evolution of the entanglement of the two atoms has been studied when they are subject to the same Rabi frequency [13]. Analytical results were given for the steady-state density operator of the system and a mechanism was proposed to prepare the two atoms in a Bell state that involves two or zero excitations. Also, [14] studied a system composed of two driven qubits coupled through a dipole-dipole interaction and interacting with independent reservoirs. Analytical results characterizing the stationary entanglement were given and it was shown how to propagate this entanglement in a quantum network. Finally, [15] studied the pairwise entanglement of two qubits extracted from a driven multiparticle ensemble that interacts with a vacuum reservoir and that occupies a region much smaller than the wavelength associated with the qubits transition frequency. The dynamics of both the quantum discord and the entanglement have been studied also in other open quantum systems [16]-[22]. In particular, it has been found that the quantum discord can be more robust than entanglement in open systems interacting with dissipative/non-dissipative environments. Moreover, it appears that it can only decay exponentially and not abruptly (no sudden-death) under Markovian reservoirs.

The present article is organized as follows. In Sec. II we summarize some results on measures of correlations for two qubit systems. In Sec. III the system of interest is described and the master equation governing the dynamics of the two atoms is established. In Sec. IV, the steady-state density operator is calculated analytically for a special configuration of the two atoms. Analytic expressions for the populations of the eigenstates of the free Hamiltonian of the two atoms, the degree of entanglement, the quantum discord, and the degree of mixedness of the two atoms are calculated as functions of the laser field intensity and of the distance between the atoms. The conclusions are given in Sec. V.

2 Quantifying Correlations in Two Qubit Systems

In this article we will be considering two two-level atoms (qubits) which we will number by 1 and 2. In the following sections these labels will correspond to the atom at position $\mathbf{r}_1$ and to the atom at position $\mathbf{r}_2$, respectively. In terms of the notation of the Introduction, 1 will replace $\mathcal{A}$, while 2 will replace $\mathcal{B}$. We will now show how to calculate all correlations by simple formulas. We will only give the algorithms and refer the interested reader to the original articles for the proofs.

The kets $|j:+\rangle$ and $|j:-\rangle$ will denote the excited and ground states of the jth atom ($j=1,2$), respectively. In the following we will be making constant use of the triplet-singlet basis

$$\mathsf{B} \; = \; \{\; |1,1\rangle,\; |1,0\rangle,\; |1,-1\rangle,\; |0,0\rangle \;\} \;, \tag{8}$$

for the state space of the two atoms:

$$\begin{aligned} |1,1\rangle &= |1:+\rangle \otimes |2:+\rangle \;, \qquad |1,0\rangle = \frac{1}{\sqrt{2}}\,(\; |1:+\rangle \otimes |2:-\rangle + |1:-\rangle \otimes |2:+\rangle \;) \;, \\ |1,-1\rangle &= |1:-\rangle \otimes |2:-\rangle \;, \qquad |0,0\rangle = \frac{1}{\sqrt{2}}\,(\; |1:+\rangle \otimes |2:-\rangle - |1:-\rangle \otimes |2:+\rangle \;) \;. \end{aligned} \tag{9}$$

We shall also use the usual tensor product basis

$$\mathsf{B}' \; = \; \{\; |+,+\rangle,\; |+,-\rangle,\; |-,+\rangle,\; |-,-\rangle \;\} \;, \tag{10}$$

where

$$\begin{aligned} |+,+\rangle &= |1:+\rangle \otimes |2:+\rangle \;, \qquad |+,-\rangle = |1:+\rangle \otimes |2:-\rangle \;, \\ |-,+\rangle &= |1:-\rangle \otimes |2:+\rangle \;, \qquad |-,-\rangle = |1:-\rangle \otimes |2:-\rangle \;. \end{aligned} \tag{11}$$

Furthermore, we will denote the density operator of the two atoms by ρ_{12}.

We are interested in quantifying the degree of entanglement of the system of two atoms. We will use the concurrence C, which can be calculated as [23, 24]:

$$C = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \;, \tag{12}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of the matrix

$$M \;=\; [\rho_{12}]_{\mathsf{B}'}(\sigma_y \otimes \sigma_y)[\rho_{12}]^*_{\mathsf{B}'}(\sigma_y \otimes \sigma_y) \;. \tag{13}$$

Here σ_y is the well-known Pauli matrix, $[\rho_{12}]_{\mathsf{B}'}$ is the matrix representation of ρ_{12} in the basis B', and $[\rho_{12}]^*_{\mathsf{B}'}$ is the element-wise complex conjugate of the density matrix $[\rho_{12}]_{\mathsf{B}'}$. The concurrence takes values between 0 and 1. It is 1 when the atoms are in a maximally entangled state, while it is zero when the atoms are in a separable state.

Evaluation of the quantum discord [5]-[8] given by (6) in general requires considerable numerical minimization. Nevertheless, a method to calculate easily the classical correlations and quantum discord for a general two-qubit X-state has been developed [6].

If ρ_{12} has a matrix representation $[\rho_{12}]_{\mathsf{B}'}$ with respect to the tensor-product basis B' of the form

$$[\rho_{12}]_{\mathsf{B}'} \;=\; \begin{pmatrix} R_{11} & 0 & 0 & R_{14} \\ 0 & R_{22} & R_{23} & 0 \\ 0 & R_{32} & R_{33} & 0 \\ R_{41} & 0 & 0 & R_{44} \end{pmatrix} \;, \tag{14}$$

then $[\rho_{12}]_{\mathsf{B}'}$ is called an X-state due to the visual appearance resembling the letter X. The eigenvalues of (14) are

$$\lambda_{0\pm} \;=\; \frac{1}{2}\left[\; R_{11} + R_{44} \pm \sqrt{(R_{11}-R_{44})^2 + 4|R_{14}|^2} \;\right] \;,$$

$$\lambda_{1\pm} = \frac{1}{2}\left[R_{22} + R_{33} \pm \sqrt{(R_{22} - R_{33})^2 + 4|R_{23}|^2} \right] . \quad (15)$$

The quantum mutual information (1) of the state ρ_{12} can then be calculated

$$I(\rho_{12}) = S(\rho_1) + S(\rho_2) + \sum_{j=0,1}\sum_{i=\pm}\lambda_{ji}\log_2\lambda_{ji} , \quad (16)$$

where ρ_j is the density operator of atom j obtained by tracing ρ_{12} over the degrees of freedom of the other atom, and

$$\begin{aligned} S(\rho_1) &= -[(R_{11}+R_{22})\log_2(R_{11}+R_{22}) + (R_{33}+R_{44})\log_2(R_{33}+R_{44})] , \\ S(\rho_2) &= -[(R_{11}+R_{33})\log_2(R_{11}+R_{33}) + (R_{22}+R_{44})\log_2(R_{22}+R_{44})] . \end{aligned} \quad (17)$$

We will consider one-dimensional projective measurements only for atom 2. Hence, we will calculate the right quantum discord $D_2^Q(\rho_{12})$ and right classical correlations $C_2^{cl}(\rho_{12})$. Later on we will also show how to calculate both the left quantum discord $D_1^Q(\rho_{12})$ and the left classical correlations $C_1^{cl}(\rho_{12})$, obtained when one-dimensional projective measurements are carried out on atom 1. It is very important to distinguish on which atom the measurements are made, since in general $D_1^Q(\rho_{12}) \neq D_2^Q(\rho_{12})$ and $C_1^{cl}(\rho_{12}) \neq C_2^{cl}(\rho_{12})$. This will be seen explicitly in the results of the following sections.

Define the quantities

$$\begin{aligned} p_0 &= (R_{11}+R_{33})k + (R_{22}+R_{44})l , \\ p_1 &= (R_{11}+R_{33})l + (R_{22}+R_{44})k , \\ \theta &= \sqrt{\frac{\Theta + [(R_{11}-R_{33})k + (R_{22}-R_{44})l]^2}{[(R_{11}+R_{33})k + (R_{22}+R_{44})l]^2}} , \\ \theta' &= \sqrt{\frac{\Theta + [(R_{11}-R_{33})l + (R_{22}-R_{44})k]^2}{[(R_{11}+R_{33})l + (R_{22}+R_{44})k]^2}} , \\ \Theta &= 4kl \left[|R_{14}|^2 + |R_{23}|^2 + 2\cdot \mathrm{Re}(R_{14}R_{23}) \right] - 16m\cdot \mathrm{Re}(R_{14}R_{23}) + 16n \cdot \mathrm{Im}(R_{14}R_{23}) , \\ S_0 &= -\frac{1-\theta}{2}\log_2\left(\frac{1-\theta}{2}\right) - \frac{1+\theta}{2}\log_2\left(\frac{1+\theta}{2}\right) , \\ S_1 &= -\frac{1-\theta'}{2}\log_2\left(\frac{1-\theta'}{2}\right) - \frac{1+\theta'}{2}\log_2\left(\frac{1+\theta'}{2}\right) , \end{aligned} \quad (18)$$

where $k + l = 1$ and Re and Im denote the real and imaginary parts of a complex number, respectively.

It can be shown that the classical correlations $C_2^{cl}(\rho_{12})$ in (2) are given by

$$C_2^{cl}(\rho_{12}) = S(\rho_1) - \min_{\{B_i\}} (p_0S_0 + p_1S_1) . \quad (19)$$

Furthermore, the minimum with respect to the variables k, l, m, and n in (19) is achieved at one of the following points:

$$(k,l,m,n) = (0,1,0,0),\ (1,0,0,0),\ \left(\frac{1}{2},\frac{1}{2},0,0\right) ,\ \left(\frac{1}{2},\ \frac{1}{2},0,\pm\frac{1}{8}\right),$$

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, 0\right) \quad \text{or} \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \pm\frac{1}{8}\right) \; . \tag{20}$$

Once the minimum has been identified, the classical correlations are easily calculated from (17) and (19). Finally, the quantum discord for the X-state is easily calculated from (16) and (19)

$$D_2^Q(\rho_{12}) \;=\; I(\rho_{12}) - C_2^{cl}(\rho_{12}) \; . \tag{21}$$

Here we must give a word of caution. This algorithm only works when the product $R_{23}R_{14} \neq 0$. It fails when such a condition is not met [6, 25].

Now suppose that we make measurements only on the atom at position $\mathbf{r}_1$. Instead of going through the whole process of calculating conditions analogous to those in (15)-(21), we only need to find the matrix representation of the density operator ρ_{12} with respect to the basis B'' in which the order of atoms 1 and 2 has been interchanged

$$\mathsf{B}'' \;=\; \{\; |2:+\rangle \otimes |1:+\rangle,\; |2:+\rangle \otimes |1:-\rangle,\; |2:-\rangle \otimes |1:+\rangle,\; |2:-\rangle \otimes |1:-\rangle \;\} \;, \tag{22}$$

and apply the procedure described above in (15)-(21) to obtain the quantum discord $D_1^Q(\rho_{12})$ and the classical correlations $C_1^{cl}(\rho_{12})$. It is easy to find such matrix representation. Suppose that the density operator ρ_{12} has the following matrix representation with respect to B'

$$[\rho_{12}]_{\mathsf{B}'} \;=\; \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix} \; . \tag{23}$$

Then the matrix representation of ρ_{12} with respect to B'' has the following form:

$$[\rho_{12}]_{\mathsf{B}''} \;=\; \begin{pmatrix} R_{11} & R_{13} & R_{12} & R_{14} \\ R_{31} & R_{33} & R_{32} & R_{34} \\ R_{21} & R_{23} & R_{22} & R_{24} \\ R_{41} & R_{43} & R_{42} & R_{44} \end{pmatrix} \; . \tag{24}$$

Note that $[\rho_{12}]_{\mathsf{B}''}$ is obtained by interchanging columns 2 and 3 and rows 2 and 3 of $[\rho_{12}]_{\mathsf{B}'}$.

For a general two-qubit density matrix the quantum discord is not easily calculated from its direct operational definition [8]. Therefore, several alternative measures have been proposed. One in particular calculates the distance of ρ_{12} to the set of zero discord states Ω_0 given by (7) if measurements are made on atom 1 [26]. It is called the geometric measure of left discord and it is denoted by $D_1^{(2)}(\rho_{12})$. It was found that

$$D_1^{(2)}(\rho_{12}) = \min_{\chi \in \Omega_0} ||\rho_{12} - \chi||_F^2 \;=\; \frac{1}{4}\left(\, \mathbf{x}^T\mathbf{x} + ||T||_F^2 - k_{\max} \,\right) \; . \tag{25}$$

Here $\mathbf{x}$ is a real column vector whose three components are given by $x_i = \mathrm{Tr}(\rho_{12}\sigma_i \otimes I)$, T is a 3×3 real matrix whose components are given by $T_{ij} = \mathrm{Tr}(\rho_{12}\sigma_i \otimes \sigma_j)$, and $k_{\max}$ is the largest eigenvalue of the matrix $K = \mathbf{x}\mathbf{x}^T + TT^T$. Note that σ_i is the i-th Pauli matrix. Also, the distance is measured using the usual Hilbert-Schmidt-Frobenius norm:

$$||A||_F^2 \;=\; \mathrm{Tr}(A^\dagger A) \;, \tag{26}$$

with A a linear operator in the state space of the system (it could also be a matrix). Using (23) and (24) one can determine the geometric measure of right discord $D_2^{(2)}(\rho_{12})$.

To quantify the degree of mixed-ness we will use the linear entropy S_L [27] defined as

$$S_L(\rho_{12}) \;=\; 1 - \mathrm{Tr}\left[\rho_{12}^2\right] \;. \tag{27}$$

Recall that $S_L(\rho_{12}) = 0$ if ρ_{12} is a pure state, while $S_L(\rho_{12}) = 3/4$ if ρ_{12} is a maximum mixed two-qubit state.

3 Two Driven Two-Level Atoms Collectively Interacting with a Reservoir

We consider two identical two-level atoms with transition frequency ω_A at fixed positions $\mathbf{r}_1$ and $\mathbf{r}_2$, driven by a classical monochromatic laser field, and interacting with the modes of the quantum electromagnetic field. In the following we will refer to the latter as the reservoir and, in some occasions, to the atom at position $\mathbf{r}_j$ as atom j.

The Hamiltonian of the system in the long-wavelength approximation and in the electric dipole representation is

$$H(t) \;=\; H_0 + V + V_{AL}(t) \tag{28}$$

where H_0 is the free hamiltonian of the two atoms and of the reservoir

$$H_0 \;=\; \frac{\hbar\omega_A}{2}\sigma_3 \;+\; \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2}\right) \;, \tag{29}$$

V is the electric dipole interaction between the two atoms and the electric field $\mathbf{E}(\mathbf{r})$ of the reservoir

$$V \;=\; -\mathbf{d}_1 \cdot \mathbf{E}(\mathbf{r}_1) \;-\mathbf{d}_2 \cdot \mathbf{E}(\mathbf{r}_2) \;, \tag{30}$$

and $V_{AL}(t)$ is the electric dipole interaction in the rotating-wave-approximation between the two atoms and the classical monochromatic electric field $\mathbf{E}_L(\mathbf{r}, t)$ of frequency ω_A

$$V_{AL}(t) \;=\; -\sum_{j=1}^{2} \hbar G(\mathbf{r}_j) \left(\sigma_{+j} e^{-i\omega_A t} + \sigma_{-j} e^{i\omega_A t}\right) \;. \tag{31}$$

Notice that we have assumed that the laser field is resonant with the atomic transition.

Here $|j:+\rangle$ and $|j:-\rangle$ are the excited and ground states of the jth atom ($j = 1, 2$), respectively, and $\sigma_3 = (\sigma_{31} + \sigma_{32})$ with $\sigma_{3j} = |j:+\rangle\langle j:+| - |j:-\rangle\langle j:-|$ is the inversion operator. Furthermore, $\mathbf{d}_j$ is the electric dipole moment operator of the jth atom and, since the atoms are identical and have two levels, can be expressed as

$$\mathbf{d}_j \;=\; \mathbf{d}_{01}\sigma_{+j} \;+\; \mathbf{d}_{01}^*\sigma_{-j} \;, \tag{32}$$

where $\sigma_{\pm j} = |j:\pm\rangle\langle j:\mp|$ are the transition operators for the j-th atom and $\mathbf{d}_{01} = \langle 1:+|\mathbf{d}_1|1:-\rangle$. The quantum electric field $\mathbf{E}(\mathbf{r})$ at position $\mathbf{r}$ is given by its expansion in terms of plane waves

$$\mathbf{E}(\mathbf{r}) \;=\; i \sum_j \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 \mathcal{V}}} a_j e^{i\mathbf{k}_j\cdot\mathbf{r}} \mathbf{e}_j + h.c. \;, \tag{33}$$

where $\mathcal{V}$ is the quantization volume, $a_j(a_j^\dagger)$ is the annihilation (creation) operator of a photon in mode j, and $\sum_j$ is a sum over the modes of the quantum electromagnetic field. The wave and polarization vectors of mode j are $\mathbf{k}_j$ and $\mathbf{e}_j$, respectively, while $\omega_j = ck_j$ is the corresponding angular frequency.

We have taken the driving electric field $\mathbf{E}_L(\mathbf{r}, t)$ of the form

$$\mathbf{E}_L(\mathbf{r}, t) \;=\; g(\mathbf{r}) \left(\boldsymbol{\mathcal{E}}_L e^{-i\omega_A t} + \boldsymbol{\mathcal{E}}_L^* e^{i\omega_A t} \right) \tag{34}$$

where $g(\mathbf{r})$ is a real-valued function describing the spatial structure of the field and $\boldsymbol{\mathcal{E}}_L$ a constant complex vector which contains the polarization of the electric field. Also, the Rabi frequency $G(\mathbf{r}) = g(\mathbf{r})(\mathbf{d}_{01} \cdot \boldsymbol{\mathcal{E}}_L)/\hbar$ is assumed to be a real quantity, and in the following we shall denote $G(\mathbf{r}_j)$ by G_j.

In the model presented above it is assumed that the two atoms interact with all the modes of the quantum electromagnetic field except for those in a very small solid angle with vertex at the atoms. In this solid angle there is a single mode that corresponds to the driving laser field. The solid angle has to be very small so one can approximate the interaction of the two atoms with the reservoir to be an interaction with all the modes of the quantum electromagnetic field. This model is useful in describing experiments with trapped cold atoms in free space or in a cavity and driven by a resonant monochromatic laser field in the optical regime. Here the transition frequency of the atoms must be different from that associated with the trap. In the case of atoms in a cavity, the aforementioned solid angle corresponds to that subtended by the cavity, the laser field would coincide with the mode of the cavity, and $g(\mathbf{r})$ is usually of the following form [28]

$$g(\mathbf{r}) \;=\; \cos\left(\frac{\omega_A}{c} z\right) \exp\left(-\frac{x^2 + y^2}{w_0^2}\right) \;, \tag{35}$$

where w_0 is the waist of the mode. Cavities that subtend a very small solid angle are common in cavity quantum electrodynamics experiments at optical frequencies [29].

We will denote the density operator of the complete system (two atoms plus reservoir) by $\rho(t)$, while $\rho_{12}(t)$ will denote the density operator of the two atoms. Recall that $\rho_{12}(t)$ is the reduced density operator obtained by tracing $\rho(t)$ over the reservoir degrees of freedom. Also, we assume that the initial state of the system is a separable state of the form

$$\rho(0) \;=\; \rho_{12}(0) \otimes \rho_B(0) \;, \tag{36}$$

where $\rho_B(0)$ is either the vacuum state of the quantum electromagnetic field or a thermal state at a temperature T such that the expected value of the number of photons in any mode of frequency ω_A is approximately zero, that is, $N(\omega_A) = \mathrm{Tr}_B[\rho_B(0) a_j^\dagger a_j] \simeq 0$ for all modes j such that $\omega_j = \omega_A$. For example, at optical frequencies $\omega_A \sim 10^{15}$ 1/s and temperatures T between 0.1 and 300 Kelvin one has $N(\omega_A) \leq 10^{-11}$.

When the interaction with the reservoir is neglected (V is zero in (28)), $\rho_{12}(t)$ is determined by von Neumann's equation:

$$i\hbar \frac{d}{dt} \rho_{12}(t) \;=\; \left[\frac{\hbar\omega_A}{2} \sigma_3 + V_{AL}(t),\; \rho_{12}(t) \right] \;. \tag{37}$$

On the other hand, when there is no driving field ($V_{AL}(t)$ is zero in (28)), the dynamics of the density operator $\rho_{12}(t)$ of the two atoms can be described by a Born-Markov-secular master equation [30]:

$$\frac{d}{dt}\rho_{12}(t) \;=\; -\frac{i}{\hbar}\left[\frac{\hbar(\omega_A+\Delta_A)}{2}\sigma_3 + H_{dd},\;\rho_{12}(t)\right] \;+\; \mathcal{D}[\rho_{12}(t)]\;. \tag{38}$$

The dissipator $\mathcal{D}$ is given by

$$\mathcal{D}(\rho) \;=\; \mathcal{D}_1(\rho) \;+\; \mathcal{D}_2(\rho) \;+\; \mathcal{D}_3(\rho)\;, \tag{39}$$

where $\mathcal{D}_j$ $(j=1,2)$ is the same as the dissipator of a single atom interacting with the reservoir [27]

$$\mathcal{D}_j(\rho) \;=\; \gamma_1\left(\sigma_{-j}\rho\sigma_{+j} - \frac{1}{2}\{\,\sigma_{+j}\sigma_{-j},\rho\,\}\right) \qquad (j=1,2), \tag{40}$$

and $\mathcal{D}_3$ describes an effective dissipative interaction between the two atoms induced by the collective interaction with the reservoir

$$\mathcal{D}_3(\rho) \;=\; \frac{3}{2}\gamma_1 F_{12}\left(\sigma_{-1}\rho\sigma_{+2} - \frac{1}{2}\{\sigma_{+2}\sigma_{-1},\rho\} + \sigma_{-2}\rho\sigma_{+1} - \frac{1}{2}\{\sigma_{+1}\sigma_{-2},\rho\}\right)\;. \tag{41}$$

Here $\{\cdot,\cdot\}$ is the anti-commutator and γ_1 is equal to the spontaneous emission rate of a two-level atom interacting with all the modes of the quantum electromagnetic field

$$\gamma_1 = \frac{1}{4\pi\epsilon_0}\cdot\frac{4|\mathbf{d}_{01}|^2\omega_A^3}{3\hbar c^3}\;.$$

The function F_{12} is defined by[a]

$$F_{12} \equiv \frac{d_\perp^2}{|\mathbf{d}_{01}|^2}\cdot\frac{\sin(x)}{x} + \left(3\frac{d_\perp^2}{|\mathbf{d}_{01}|^2} - 2\right)\frac{\cos(x) - \frac{\sin(x)}{x}}{x^2}\;, \tag{42}$$

with

$$x = \frac{\omega_A}{c}|\mathbf{r}_1 - \mathbf{r}_2|\;, \tag{43}$$

and $d_\perp^2$ the square of the norm of the projection of $\mathbf{d}_{01}$ onto the plane perpendicular to $\mathbf{r}_1 - \mathbf{r}_2$.

The coherent dipole-dipole interaction H_{dd} induced by the collective interaction with the reservoir is given by

$$H_{dd} \;=\; \hbar\Omega_{12}(\sigma_{+1}\sigma_{-2} + \sigma_{-1}\sigma_{+2})\;, \tag{44}$$

where[a]

$$\Omega_{12} \;=\; \frac{3}{4}\gamma_1\left[\,-\frac{d_\perp^2}{|\mathbf{d}_{01}|^2}\frac{\cos(x)}{x} + \left(3\frac{d_\perp^2}{|\mathbf{d}_{01}|^2} - 2\right)\left(\frac{\sin(x)}{x^2} + \frac{\cos(x)}{x^3}\right)\,\right]\;. \tag{45}$$

Meanwhile, Δ_A is a frequency shift produced by the interaction with the reservoir. We note that, in writing the master equation (38), we have already used $N(\omega_A) \simeq 0$.

[a]Notice that we have chosen a slightly different notation than the one in [30]. In particular, that article uses the component of $\mathbf{d}_{01}$ along $\mathbf{r}_1 - \mathbf{r}_2$ and $(3/2)F_{12}$ instead of $d_\perp$ and F_{12}, respectively.

Applying the approximation of independent rates of variation [31], the master equation for $\rho_{12}(t)$ which takes into account the interaction of the two atoms with the reservoir and with the laser field is given by

$$\frac{d}{dt}\rho_{12}(t) = -\frac{i}{\hbar}\left[\frac{\hbar(\omega_A+\Delta_A)}{2}\sigma_3 + H_{dd} + V_{AL}(t),\ \rho_{12}(t)\right] + \mathcal{D}\left[\rho_{12}(t)\right] \qquad (t\geq 0). \tag{46}$$

In the following we will neglect the frequency shift Δ_A since $|\Delta_A| \ll \omega_A$.

Passing to the interaction picture (IP) defined by the unitary transformation $U_0(t,0) = \exp(-i\omega_A\sigma_3 t/2)$ we obtain the master equation

$$\frac{d}{dt}\rho_{12}(t) = -\frac{i}{\hbar}\left[V_{AL}^I + H_{dd},\ \rho_{12}(t)\right] + \mathcal{D}\left[\rho_{12}(t)\right] \qquad (t\geq 0)\ , \tag{47}$$

where

$$V_{AL}^I = -\sum_{j=1}^{2}\hbar G(\mathbf{r}_j)\left(\sigma_{+j}+\sigma_{-j}\right)\ , \tag{48}$$

and $\rho_{12}(t)$ is now the IP density operator of the two atoms.

The dipole-dipole coupling Ω_{12} is zero for certain values of $d_\perp$ whenever $x \leq 2.8$, see (43) and (45) and Fig. (1). In the following we will restrict to the region $0 < x \leq 2.8$ and we will assume that $d_\perp$ is chosen for each x such that $\Omega_{12} \simeq 0$. In this case the master equation (47) reduces to

$$\frac{d}{dt}\rho_{12}(t) = -\frac{i}{\hbar}\left[V_{AL}^I,\ \rho_{12}(t)\right] + \mathcal{D}\left[\rho_{12}(t)\right] \qquad (t\geq 0)\ . \tag{49}$$

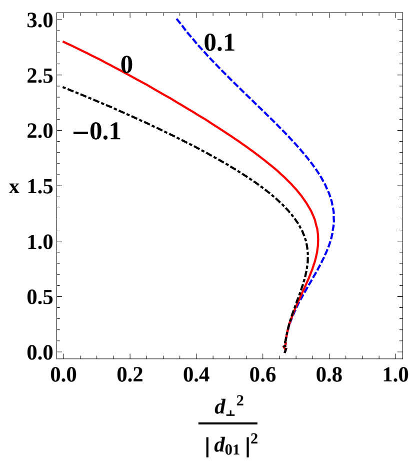


Fig. 1. (Color online) Contour plot of Ω_{12}/γ_1 as a function of $d_\perp^2/|\mathbf{d}_{01}|^2$ and $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$. The contours $\Omega_{12}/\gamma_1 = 0.1$ (blue-dashed line), 0 (red-solid line), and -0.1 (black-dot-dashed line) are shown. They indicate the region in which $\Omega_{12}/\gamma_1 \simeq 0$.

4 $G(\mathbf{r}_2) = 0$

We assume that one of the atoms is fixed at a position where the classical electric field (34) is zero, while the other atom is fixed at a position where it is not zero. Therefore, we take $G_2 = G(\mathbf{r}_2) = 0$. Notice that increasing (decreasing) the intensity of the laser field increases (decreases) $|G_1| = |G(\mathbf{r}_1)|$ (see the definition of G_1 following (34)). Therefore, the Rabi frequency G_1 can be made to vary by increasing or decreasing the intensity of the electric field (34). Furthermore, the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ between the two atoms can also be varied independently of G_1. For example, one of the atoms could be placed at a node of a stationary wave to have $G_2 = 0$, the other atom could be placed anywhere else (except at a position where $G_1 = 0$), and the intensity of the laser field could be varied to have $|G_1| = |G(\mathbf{r}_1)|$ take on any positive value.

We will calculate the steady-state solution ρ_{12}^{ST} of (49) for this special configuration. Recall that ρ_{12}^{ST} is a steady-state solution of (49) if ρ_{12}^{ST} is not explicitly time dependent and

$$-\frac{i}{\hbar}\left[V_{AL}^{I},\ \rho_{12}^{ST}\right] + \mathcal{D}\left[\rho_{12}^{ST}\right] = 0 \ . \tag{50}$$

It is found that the populations of the solution ρ_{12}^{ST} of (50) in the triplet-singlet basis (9) are given by

$$\begin{aligned}
\langle 1,1|\rho_{12}^{ST}|1,1\rangle &= \frac{18}{\kappa}\left[\ 16F_{12}^2\bar{G}_1^6 + 9(4-F_{12}^2)F_{12}^2\bar{G}_1^4\ \right] \ , \\
\langle 1,0|\rho_{12}^{ST}|1,0\rangle &= \frac{1}{\kappa}\left[\ \mu_-\bar{G}_1^6 + \nu_-\bar{G}_1^4 + \eta_-\bar{G}_1^2\ \right] \ , \\
\langle 1,-1|\rho_{12}^{ST}|1,-1\rangle &= 1 - \langle 1,1|\rho_{12}^{ST}|1,1\rangle - \langle 1,0|\rho_{12}^{ST}|1,0\rangle - \langle 0,0|\rho_{12}^{ST}|0,0\rangle \ , \\
\langle 0,0|\rho_{12}^{ST}|0,0\rangle &= \frac{1}{\kappa}\left[\ \mu_+\bar{G}_1^6 + \nu_+\bar{G}_1^4 + \eta_+\bar{G}_1^2\ \right] \ ,
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\bar{G}_1 &= G_1/\gamma_1 \ , \\
\kappa &= \bar{G}_1^6(2048 + 1152F_{12}^2) + \bar{G}_1^4(2560 + 5184F_{12}^2 - 1296F_{12}^4) \\
&\quad + \bar{G}_1^2(864 + 144F_{12}^2 + 486F_{12}^4) + \frac{9}{8}(2+3F_{12})^2(2-3F_{12})^2(4-F_{12}^2) \ , \\
\mu_\pm &= 32(16 \pm 12F_{12} + 9F_{12}^2) \ , \\
\nu_\pm &= 18(32 \pm 48F_{12} + 68F_{12}^2 - 9F_{12}^4) \ , \\
\eta_\pm &= 9(2 \pm 3F_{12})^2(4 - F_{12}^2) \ .
\end{aligned} \tag{52}$$

On the other hand, the coherences of ρ_{12}^{ST} are given by

$$\langle 0,0|\rho_{12}^{ST}|1,1\rangle = i\frac{9\sqrt{2}}{\kappa}\bar{G}_1^3F_{12}\left[\ 3(2+3F_{12})(4-F_{12}^2) + 8(4+2F_{12}-3F_{12}^2)\bar{G}_1^2\ \right] \ ,$$

$$\begin{aligned}
\langle 1,0|\rho_{12}^{ST}|1,1\rangle &= i\frac{9\sqrt{2}}{\kappa}\bar{G}_1^3 F_{12}\left[\, 3(2-3F_{12})(4-F_{12}^2)+8(4-2F_{12}-3F_{12}^2)\bar{G}_1^2 \,\right] \;, \\
\langle 1,-1|\rho_{12}^{ST}|1,0\rangle &= -i\frac{\sqrt{2}}{4\kappa}\bar{G}_1\Big[\, 32(32-12F_{12}-18F_{12}^2+27F_{12}^3)\bar{G}_1^4 \\
&\quad + 36(32-24F_{12}-4F_{12}^2+54F_{12}^3-9F_{12}^4)\bar{G}_1^2 \\
&\quad + 9(2-3F_{12})^2(2+3F_{12})(4-F_{12}^2)\,\Big] \;, \\
\langle 1,-1|\rho_{12}^{ST}|0,0\rangle &= -i\frac{\sqrt{2}}{4\kappa}\bar{G}_1\Big[\, 32(32+12F_{12}-18F_{12}^2-27F_{12}^3)\bar{G}_1^4 \\
&\quad + 36(32+24F_{12}-4F_{12}^2-54F_{12}^3-9F_{12}^4)\bar{G}_1^2 \\
&\quad + 9(2+3F_{12})^2(2-3F_{12})(4-F_{12}^2)\,\Big] \;, \\
\langle 1,-1|\rho_{12}^{ST}|1,1\rangle &= \frac{3}{2\kappa}F_{12}\bar{G}_1^2\left[\, -256\bar{G}_1^4-9(64)F_{12}^2\bar{G}_1^2+9(16-40F_{12}^2+9F_{12}^4)\,\right] \;, \\
\langle 0,0|\rho_{12}^{ST}|1,0\rangle &= \frac{\bar{G}_1^2}{\kappa}\left[\, 512\bar{G}_1^4+36(16+16F_{12}^2-9F_{12}^4)\bar{G}_1^2+9(16-40F_{12}^2+9F_{12}^4)\,\right] \;.
\end{aligned} \tag{53}$$

The rest of the matrix elements can be obtained using the hermiticity of ρ_{12}^{ST}.

Using the following argument it can be shown that the density operator $\rho_{12}(t)$ of the two atoms tends to the state ρ_{12}^{ST} above for any initial state $\rho_{12}(0)$ if $G_2 = 0$. Equation (49) with the initial condition $\rho_{12}(0)$ can be expressed as an initial value problem (IVP) of the form

$$\dot{\mathbf{x}}(t) \;=\; A\mathbf{x}(t) \;, \qquad \mathbf{x}(0)=\mathbf{x}_0 \;, \tag{54}$$

where $\mathbf{x}(t)$ is a 16 component column complex vector associated with $\rho_{12}(t)$, A is a 16×16 constant complex matrix, and $\mathbf{x}_0$ is a 16 component column complex vector defined by $\rho_{12}(0)$. The IVP in (54) has a unique solution $\mathbf{x}(t)$ [32]. It can be shown that the matrix A has 15 eigenvalues with negative real part and one eigenvalue is zero. Hence it follows that $\mathbf{x}(t)\to\mathbf{x}^{ST}$ as $t\to+\infty$ where $\mathbf{x}^{ST}$ is associated with ρ_{12}^{ST} above [32]. Therefore, $\rho_{12}(t)\to\rho_{12}^{ST}$ as $t\to+\infty$ for any initial condition $\rho_{12}(0)$.

There are several limiting cases of interest. First, if $\bar{G}_1\to 0$, then it is seen from (51) and (53) that $\rho_{12}^{ST}\to|1,-1\rangle\langle 1,-1|$. This result is expected since the atoms are located at different positions and, without the driving field, the reservoir ultimately leaves the two atoms in their respective ground states.

Another limiting case of more interest occurs when the laser field is very intense ($|\bar{G}_1|\to+\infty$). From (51)-(53) we see that the matrix representation of ρ_{12}^{ST} in the tensor product basis B' (11) takes the form of an X-state

$$[\rho_{12}^{ST}]_{\mathsf{B}'} \;=\; \begin{pmatrix}
\frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2} & 0 & 0 & -\frac{3F_{12}}{16+9F_{12}^2} \\
0 & \frac{1}{2}-\frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2} & -\frac{3F_{12}}{16+9F_{12}^2} & 0 \\
0 & -\frac{3F_{12}}{16+9F_{12}^2} & \frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2} & 0 \\
-\frac{3F_{12}}{16+9F_{12}^2} & 0 & 0 & \frac{1}{2}-\frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2}
\end{pmatrix} \;. \tag{55}$$

Therefore, one can prepare the system of two atoms in an X-state by simply adjusting the laser field intensity regardless of the initial state of the two atoms.

Continuing with the case of high laser field intensities, the corresponding populations of the triplet-singlet basis B (9) take the form

$$\begin{aligned}
\langle 1,1|\rho_{12}^{ST}|1,1\rangle &= \frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2}\ , \\
\langle 1,0|\rho_{12}^{ST}|1,0\rangle &= \frac{1}{4}-\frac{3F_{12}}{16+9F_{12}^2}\ , \\
\langle 1,-1|\rho_{12}^{ST}|1,-1\rangle &= \frac{1}{2}-\frac{\frac{9}{4}F_{12}^2}{16+9F_{12}^2}\ , \\
\langle 0,0|\rho_{12}^{ST}|0,0\rangle &= \frac{1}{4}+\frac{3F_{12}}{16+9F_{12}^2}\ .
\end{aligned} \tag{56}$$

Noting that $-0.224 < F_{12} \leq 2/3$, we can establish bounds on the populations in (56). When the laser field is very intense, the populations of the states $|1,1\rangle$ and $|0,0\rangle$ cannot grow more than $1/20$ and $1/4+1/10$, respectively, and these maximum values are achieved when the atoms are close. At the same time the populations of the states $|1,-1\rangle$ and $|1,0\rangle$ take on their respective minimum values of $1/2-1/20$ and $1/4-1/10$. We note that the dissipative interaction of the two atoms embodied by $\mathcal{D}_3(\rho)$ in (41), is responsible for the non-zero population of the state $|1,1\rangle$ (recall that the dipole-dipole interaction is zero). Moreover, this interaction is not very effective, since this population can only grow to a relatively small value of $1/20$ for high laser field intensities.

A final limiting case of interest occurs when the laser field intensity is low ($|\bar{G}_1| \ll 1$). From (51)-(53) we find to second order in $\bar{G}_1$ that the steady-state density matrix in the triplet-singlet basis B (9) takes the form

$$[\rho_{12}^{ST}]_{\mathsf{B}} = \begin{pmatrix} 0 & 0 & \frac{3F_{12}\bar{G}_1^2}{1-\left(\frac{3}{2}F_{12}\right)^2} & 0 \\ 0 & \frac{2\bar{G}_1^2}{\left(1+\frac{3}{2}F_{12}\right)^2} & i\frac{\sqrt{2}\bar{G}_1}{1+\frac{3}{2}F_{12}} & \frac{2\bar{G}_1^2}{1-\left(\frac{3}{2}F_{12}\right)^2} \\ \frac{3F_{12}\bar{G}_1^2}{1-\left(\frac{3}{2}F_{12}\right)^2} & -i\frac{\sqrt{2}\bar{G}_1}{1+\frac{3}{2}F_{12}} & 1-\frac{2\bar{G}_1^2}{\left(1+\frac{3}{2}F_{12}\right)^2}-\frac{2\bar{G}_1^2}{\left(1-\frac{3}{2}F_{12}\right)^2} & -i\frac{\sqrt{2}\bar{G}_1}{1-\frac{3}{2}F_{12}} \\ 0 & \frac{2\bar{G}_1^2}{1-\left(\frac{3}{2}F_{12}\right)^2} & i\frac{\sqrt{2}\bar{G}_1}{1-\frac{3}{2}F_{12}} & \frac{2\bar{G}_1^2}{\left(1-\frac{3}{2}F_{12}\right)^2} \end{pmatrix} . \tag{57}$$

Notice that $\bar{G}_1^2$ must be sufficiently small in order for (57) to make sense, since it can be seen that the population of the state $|0,0\rangle$ (component $4,4$ above) diverges as the distance between the atoms tends to zero. In fact, it must occur that

$$\bar{G}_1^2 \ \leq\ \frac{1}{4}\cdot\frac{\left(1-\frac{3}{2}F_{12}\right)^2\left(1+\frac{3}{2}F_{12}\right)^2}{1+\left(\frac{3}{2}F_{12}\right)^2}\ , \tag{58}$$

to have $\langle 1,0|\rho_{12}^{ST}|1,0\rangle + \langle 0,0|\rho_{12}^{ST}|0,0\rangle \leq 1$. If one substitutes in (57) the bound in (58) it follows that

$$\begin{aligned}
\langle 1,0|\rho_{12}^{ST}|1,0\rangle &= \frac{1}{2}-\frac{\frac{3}{2}F_{12}}{1+\left(\frac{3}{2}F_{12}\right)^2}\to 0 \qquad \text{as } |\mathbf{r}_1-\mathbf{r}_2|\to 0^+ \\
\langle 0,0|\rho_{12}^{ST}|0,0\rangle &= \frac{1}{2}+\frac{\frac{3}{2}F_{12}}{1+\left(\frac{3}{2}F_{12}\right)^2}\to 1 \qquad \text{as } |\mathbf{r}_1-\mathbf{r}_2|\to 0^+.
\end{aligned} \tag{59}$$

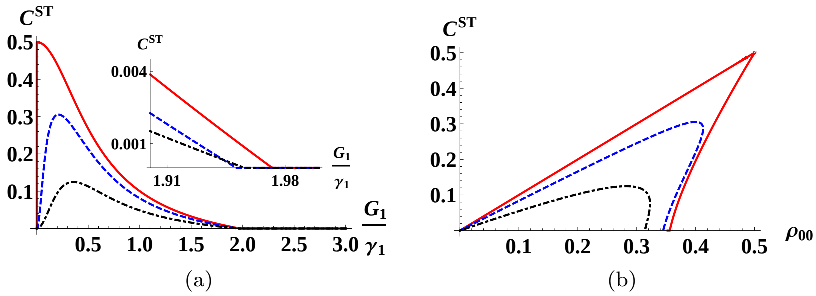


Fig. 2. (Color online) Figures (2a) and (2b) show the steady-state concurrence C^{ST} of the two atoms as a function of $\bar{G}_1 = G_1/\gamma_1$ and the population ρ_{00} of the state $|0,0\rangle$, respectively. The inside figure in (2a) shows a close-up of the entanglement sudden death. Results are shown when $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c = 0.01$ (red-solid line), 1 (blue-dashed line), and 2 (black-dot-dashed line).

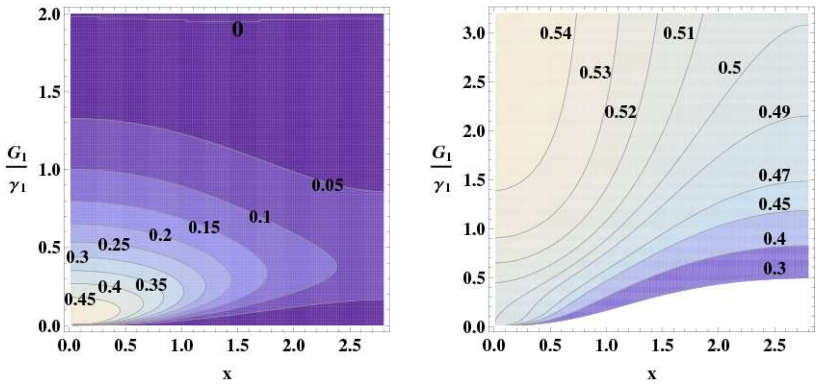
Hence, for a weak driving field and small distance between the atoms, the level $|0,0\rangle$ is much more populated than the level $|1,0\rangle$.

We now discuss the degree of entanglement of ρ_{12}^{ST} as measured by the concurrence C^{ST}. We observe that as the intensity of the laser field increases ($\bar{G}_1$ increases), C^{ST} increases, takes on a maximum value, then decreases, and finally dies abruptly, Figs. (2a) and (3a). Therefore, for a fixed distance between the two atoms, increasing/decreasing the laser field intensity allows one to control the amount of entanglement between the two atoms. Also, the *sudden death* of entanglement as a function of the Rabi frequency G_1 is found for all values of the distance between the atoms $0 < x \leq 2.8$, Fig. (3a). Moreover, C^{ST} takes on large values only when the intensity of the laser field is such that $\bar{G}_1 = G_1/\gamma_1 \leq 1$. Notice that the atoms become entangled by exchanging spontaneously emitted photons, a process embodied by the term $\mathcal{D}_3(\rho)$ in (41).

To understand the behavior of the steady-state concurrence we consider the case of a weak driving field. The concurrence of the density operator in (57) is easily calculated to be

$$\begin{aligned} C^{ST} &= \frac{3}{2}|F_{12}| \left[\frac{2\bar{G}_1^2}{\left(1 - \frac{3}{2}F_{12}\right)^2} + \frac{2\bar{G}_1^2}{\left(1 + \frac{3}{2}F_{12}\right)^2} \right] , \\ &= \frac{3}{2}|F_{12}| \cdot \left(\langle 0,0|\rho_{12}^{ST}|0,0\rangle + \langle 1,0|\rho_{12}^{ST}|1,0\rangle \right) , \end{aligned} \tag{60}$$

to second order in $\bar{G}_1$. Hence, we observe that for weak driving fields the steady-state concurrence is determined by the populations of the states $|0,0\rangle$ and $|1,0\rangle$. Since $x \leq 2.8$, the atoms are close and from (59) we conclude that the population of the level $|0,0\rangle$ dominates over that of the state $|1,0\rangle$. Hence, the population of $|0,0\rangle$ and the function F_{12} determine the behavior of the steady-state concurrence. This is illustrated in Fig. (2b) where it is seen that the population of $|0,0\rangle$ is responsible to a large extent of the behavior of the concurrence not only for weak laser field intensities. From that figure we observe that the concurrence grows with the population of the level $|0,0\rangle$ and is maximized when the latter is maximized. As the population of the state $|0,0\rangle$ decreases to its asymptotic value (that is, when $\bar{G}_1 \to +\infty$), the concurrence decreases to zero. Notice that when the atoms are more separated, F_{12} is smaller and, hence, the population of $|1,0\rangle$ has a greater contribution in the concurrence, see (59) and (60). This is observed in the case $x = 2$ in Fig. (2b), where the concurrence does not exhibit such a sharp dependence on the population ρ_{00} of $|0,0\rangle$ when compared to the cases $x = 0.01$ and $x = 1$. Why the concurrence disappears when the laser field becomes very intense can be explained by the asymptotic X-form of the density matrix in the product state basis (55). There we observe that the laser field has reached such a high intensity that many coherences have decreased to zero and the atomic transitions are saturated. The latter is taken to mean that the populations have reached their asymptotic value. Furthermore, the concurrence of this particular X-state can be easily calculated to be zero.



(a) (b)

Fig. 3. (Color online) Figures (3a) and (3b) show contour plots of the steady-state concurrence C^{ST} and steady-state linear entropy S_L^{ST} as a function of $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$ and $\bar{G}_1 = G_1/\gamma_1$, respectively.

In the preceding paragraphs we have established that for any distance $0 < x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c \leq 2.8$ between the two atoms, the amount of entanglement present in the system can be controlled adjusting the laser field intensity. Moreover, the entanglement can be made to disappear if the laser field intensity is high enough (that is, $\bar{G}_1$ has achieved a sufficiently high value). Therefore, it is natural to ask whether all quantum correlations disappear if the laser field intensity is sufficiently high. This is specially important in quantum technologies, because other quantum correlations constitute a resource that might be exploited to bring advantage in information transmission and processing [33, 34].

Recall that ρ_{12}^{ST} has the form of an X-state in the limit $|\bar{G}_1| \to +\infty$, see (55). Therefore, we can use the method introduced in (21)-(24) to obtain analytic expressions of the left and right quantum discord of ρ_{12}^{ST} in the limit of high laser field intensities (that is, $|\bar{G}_1| \to +\infty$).

We will first consider the case of the right quantum discord. It is found that the steady-state quantum mutual information $I(\rho_{12}^{ST})$ is given by (16):

$$\begin{aligned} I(\rho_{12}^{ST}) \;=\; & \frac{-4+\sqrt{y}}{2\sqrt{y}}\log_2(-4+\sqrt{y}) \;-\; \frac{\frac{9}{2}F_{12}^2}{y}\log_2\left(\frac{9}{2}F_{12}^2\right) \;+\; \frac{1}{2}\log_2(y) \\ & +\frac{4+\sqrt{y}}{2\sqrt{y}}\log_2(4+\sqrt{y}) \;-\; \frac{16+\frac{9}{2}F_{12}^2}{y}\log_2\left(16+\frac{9}{2}F_{12}^2\right) \;-1\;, \end{aligned} \qquad (61)$$

where $y = 16 + 9F_{12}^2$. Meanwhile, the classical correlation $C_2^{cl}(\rho_{12}^{ST})$ is given by (19):

$$C_2^{cl}(\rho_{12}^{ST}) \;=\; 1 + \frac{1-4|b|}{2}\log_2\left(\frac{1-4|b|}{2}\right) + \frac{1+4|b|}{2}\log_2\left(\frac{1+4|b|}{2}\right) \;, \tag{62}$$

where $b = -3F_{12}/(16+9F_{12}^2) = -3F_{12}/y$. The right quantum discord $D_2^Q(\rho_{12}^{ST})$ of the state ρ_{12}^{ST} is given by (21):

$$D_2^Q(\rho_{12}^{ST}) \;=\; I(\rho_{12}^{ST}) - C_2^{cl}(\rho_{12}^{ST}) \;. \tag{63}$$

Notice that all these correlations are functions of F_{12}. The origin of this is that these correlations build up as the atoms exchange spontaneously emitted photons, a process that depends on F_{12} and that is embodied by $\mathcal{D}_3(\rho)$ in (41) (recall that the dipole-dipole interaction is zero).

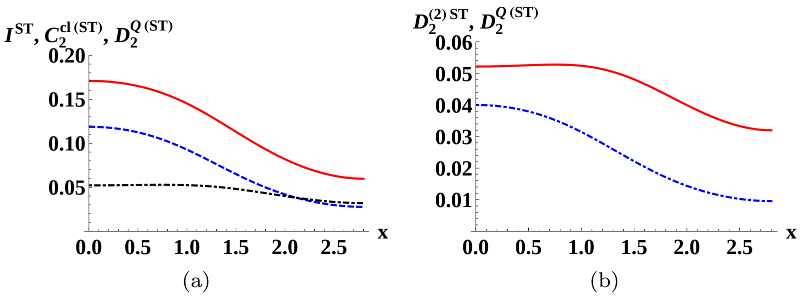


Fig. 4. (Color online) Figure (4a) shows the steady-state quantum mutual information I^{ST} (red-solid line), classical correlations $C_2^{cl(ST)}$ (blue-dashed lined), and right quantum discord $D_2^{Q(ST)}$ (black-dot-dashed line) as a function of $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$ in the limit $|\bar{G}_1| \to +\infty$. Here measurements are performed on the atom at position $\mathbf{r}_2$. Figure (4b) illustrates the right quantum discord $D_2^{Q(ST)}$ (red-solid line) and the right geometric discord $D_2^{(2)(ST)}$ (blue-dashed line) as a function of $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$ in the limit $|\bar{G}_1| \to +\infty$..

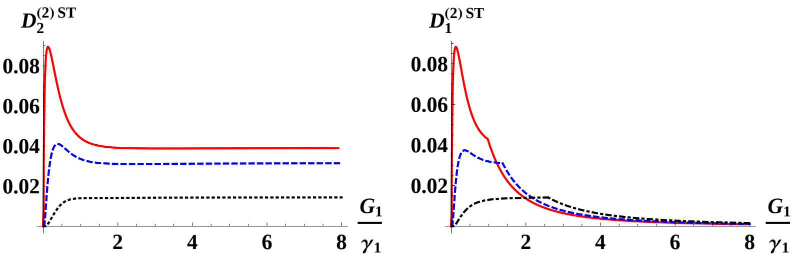
Figure (4a) shows these three correlations as a function of $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$. The first observation is that ρ_{12}^{ST} has both quantum and classical correlations even though the entanglement is long gone (from Fig. (3a) we see that it died abruptly around $\bar{G}_1 = 2$). Also, the quantum discord is approximately constant for $0 < x \leq 2.8$, while the quantum mutual information and the right classical correlations decrease as x increases. Therefore, increasing/decreasing the distance between atoms allows one to control the amount of right classical correlations maintaining (approximately) the same amount of right quantum discord. In particular, the latter is smaller than the right classical correlations except for $2 < x \leq 2.8$.

Given that the quantum discord can be calculated exactly for X-states, it is of great interest to compare it with the geometric measure of quantum discord given in (25) to see how well the latter estimates the exact result. Using that equation, the right geometric discord $D_2^{(2)}(\rho_{12}^{ST})$ is easily calculated to be

$$D_2^{(2)}(\rho_{12}^{ST}) \;=\; \frac{36F_{12}^2}{(16+9F_{12}^2)^2} \;. \tag{64}$$

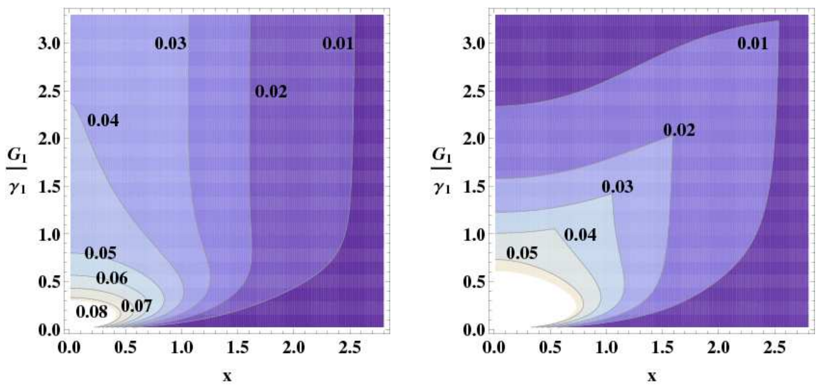
From Fig. (4b) we observe that there is good qualitative agreement between the estimate $D_2^{(2)}(\rho_{12}^{ST})$ and the exact $D_2^{Q}(\rho_{12}^{ST})$. Nevertheless, notice that the exact quantum discord is considerably larger than that given in (64), so, in general, the quantitative agreement between $D_2^{Q}(\rho_{12}^{ST})$ and $D_2^{(2)}(\rho_{12}^{ST})$ is not so good.

The geometric discord in (25) allows us to calculate easily an estimate of the quantum discord for the general case (51)-(53). The results are analytic, but the expressions are so large that they are not presented. Figures (5a) and (6a) show that for a fixed distance between the atoms the geometric discord quickly achieves the stationary value given in (64).



(a) (b)

Fig. 5. (Color online) Figures (5a) and (5b) show the steady-state right $D_2^{(2)ST}$ and left $D_1^{(2)ST}$ geometric discords as a function of $\bar{G}_1 = G_1/\gamma_1$ for $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c = 0.355$ (red-solid line), 1 (blue-dashed line), and 2 (black-dot-dashed-line).



(a) (b)

Fig. 6. (Color online) Figures (6a) and (6b) show contour plots of the steady-state right $D_2^{(2)ST}$ and left $D_1^{(2)ST}$ geometric discords as a function of $\bar{G}_1 = G_1/\gamma_1$ and $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c$.

We now consider the case of the left quantum discord in the limiting case where $|\bar{G}_1| \to +\infty$. Using (16), (19), and (21) we find that the quantum mutual information $I(\rho_{12}^{ST})$ is exactly the same as before (61), but the left quantum discord $D_1^{Q}(\rho_{12}^{ST})$ is now zero and the left classical correlations $C_1^{cl}(\rho_{12}^{ST})$ are equal to $I(\rho_{12}^{ST})$. Hence, all information of the atom at position $\mathbf{r}_2$

contained in the correlations between the atoms in ρ_{12}^{ST} can be extracted by (one-dimensional projective) measurements on the atom at $\mathbf{r}_1$ [5]. Also, ρ_{12}^{ST} is not perturbed by (certain) non-selective measurements on the atom at $\mathbf{r}_1$ [5]. This contrasts greatly with the case considered above, that is, with the case were the measurements are performed on the atom at $\mathbf{r}_2$. There we found that ρ_{12}^{ST} presents non-negligible quantum discord. Thus, when measurements are performed only on the atom at $\mathbf{r}_2$, ρ_{12}^{ST} is perturbed and there is information about the atom at $\mathbf{r}_1$ that cannot be extracted [5]. Furthermore, we have here another example were both the quantum discord and the classical correlations are not symmetric: $D_1^Q(\rho_{12}^{ST}) \neq D_2^Q(\rho_{12}^{ST})$ and $C_1^{cl}(\rho_{12}^{ST}) \neq C_2^{cl}(\rho_{12}^{ST})$. Given that the atoms are very close it would be difficult (if not impossible) to carry out measurements on only one of the atoms. Nevertheless, the quantum discord can still be used as a measure of quantum correlations and its interpretations are still valid. Moreover, we find that by fixing the distance between the two atoms and adjusting the laser field intensity one is able to prepare the system of two atoms in a mixed separable state with non-zero right quantum discord and zero left quantum discord.

Again, the use of the geometric discord (25) allows us to give an estimate of the quantum discord in the general case (51)-(53). From Figs. (5b) and (6b) we note that the left geometric discord is smooth by parts as a function of $\bar{G}_1$, since it exhibits edges. Also notice that there is no sudden death of $D_1^{(2)}(\rho_{12}^{ST})$ and, hence, of the left quantum discord. This behavior was also observed in the time-evolution of the quantum discord of a similar system in [20]. Finally, observe that $D_1^{(2)}(\rho_{12}^{ST})$ is drastically different from $D_2^{(2)}(\rho_{12}^{ST})$ (compare Figs. (6a) and (6b)).

We now turn to discuss the degree of mixed-ness of ρ_{12}^{ST} as measured by the steady state linear entropy $S_L^{ST} \equiv S_L(\rho_{12}^{ST})$, see Fig. (3b). We observe that, as the intensity of the laser field is increased ($\bar{G}_1$ increases), S_L^{ST} increases until it acquires an asymptotic value which is easily calculated from (55):

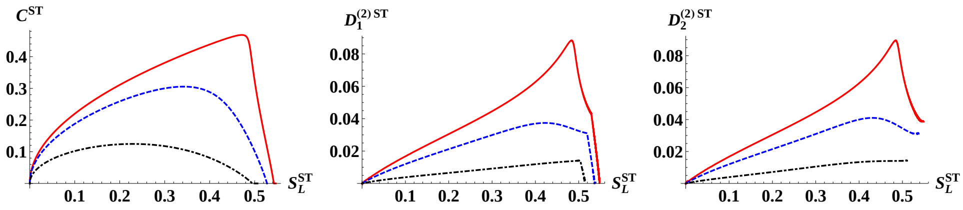
$$S_L^{ST} = \frac{3}{4} - \frac{4}{16 + 9F_{12}^2} \,. \tag{65}$$

Notice that the system is never in a maximum mixed state and that the maximum value $11/20$ of S_L^{ST} occurs when the atoms are very close. Moreover, one can control the degree of mixedness of the state of the two atoms by adjusting the laser field intensity.

To end this section we discuss the relationship between the steady-state quantum correlations considered above (concurrence and right and left geometric discords) and the degree of mixed-ness as measured by the linear entropy S_L^{ST}. From Fig. (7a) we observe that, as ρ_{12}^{ST} becomes more mixed, the entanglement (as measured by the concurrence) increases until S_L^{ST} reaches a certain value, after which the entanglement decays to zero. This behavior illustrates the known fact that a mixed state of two qubits cannot contain an arbitrary amount of entanglement, and that the more mixed the state becomes, the less entanglement it can have [35]. To the best of our knowledge, an analogous result has not been established for the quantum and geometric discords. Nevertheless, figures (7b) and (7c) show that the left and right geometric discords exhibit a similar behavior to that of the concurrence as a function of S_L^{ST}. Notice that only the left geometric discord decreases to zero, Fig. (7b).

5 Conclusions

In this article we considered two two-level atoms (qubits) fixed at different positions, driven by a resonant monochromatic laser field, and interacting collectively with the quantum elec-



(a) (b) (c)

Fig. 7. (Color online) Figures (7a), (7b), and (7c) show the steady-state concurrence C^{ST}, left $D_1^{(2)ST}$ and right $D_2^{(2)ST}$ geometric discords as a function of the steady-state linear entropy S_L^{ST} for $x = \omega_A|\mathbf{r}_1 - \mathbf{r}_2|/c = 0.355$ (red-solid line), 1 (blue-dashed line), and 2 (black-dot-dashed line).

tromagnetic field. A Born-Markov-secular master equation was used to describe the dynamics of the two atoms and their steady-state was studied for a configuration in which one atom was located in a region where the driving electric field is zero while the other was in a position where it is not zero. Furthermore, the direction of the matrix element of the electric dipole moment operator between the excited and ground states of each atom was chosen in such a way that the dipole-dipole coupling was zero.

The steady-state density operator ρ_{12}^{ST} was obtained analytically and it was shown that any initial state of the two atoms tends to ρ_{12}^{ST}. Therefore, it does not matter how the system of two atoms is initially prepared, the quantum state of the two atoms is ρ_{12}^{ST} once the transient terms are negligible. The steady-state density operator is entangled and has non-zero left and right quantum discords if the laser field intensity is not very high. High laser field intensities turn ρ_{12}^{ST} into a separable mixed X-state that has non-zero right quantum discord. Moreover, the amount of entanglement is highly dependent on the population of $|0,0\rangle$.

Steady-state quantum correlations (entanglement and quantum discord) are created as the atoms exchange spontaneously emitted photons and can be controlled by adjusting the laser field intensity. Increasing the laser field intensity eventually leads to steady-state entanglement sudden death, but the quantum discord still survives. For high laser field intensities, increasing/decreasing de distance between the atoms allows one to control the amount of classical correlations while maintaining the quantum discord (approximately) constant.

The model studied could be useful in experiments with trapped cold atoms in free space or in a cavity and driven by a resonant monochromatic laser field. In the case of atoms in a cavity, the laser field would coincide with the cavity mode and the solid angle subtended by the cavity would have to be very small. The results of the article are relevant for experimental studies, because it was found that, regardless of the initial state of the two atoms, the steady-state density operator of the two atoms is ρ_{12}^{ST} and the amount of (quantum and classical) correlations in it can be controlled by adjusting the laser field intensity. Moreover, by adjusting the laser field intensity one can prepare the system of two atoms in a separable mixed state with non-zero left and right quantum discords. Increasing the laser field intensity allows one to prepare the two atoms in a separable mixed X-state whose classical correlations can be manipulated by varying the distance between the two atoms while maintaining the non-zero right quantum discord approximately constant.

Acknowledgements

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