MULTICOPY PROGRAMMABLE DISCRIMINATORS BETWEEN
TWO UNKNOWN QUBIT STATES WITH GROUP-THEORETIC APPROACH

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The discrimination between two unknown states can be performed by a universal programmable discriminator, where the copies of the two possible states are stored in two program systems respectively and the copies of data, which we want to confirm, are provided in the data system. In the present paper, we propose a group-thretic approach to the multi-copy programmable state discrimination problem. By equivalence of unknown pure states to known mixed states and with the representation theory of $U(n)$ group, we construct the Jordan basis to derive the analytical results for both the optimal unambiguous discrimination and minimum-error discrimination. The POVM operators for unambiguous discrimination and orthogonal measurement operators for minimum-error discrimination are obtained. We find that the optimal failure probability and minimum-error probability for the discrimination between the mean input mixd states are dependent on the dimension of the unknown qudit states. We applied the approach to generalize the results of He and Bergou (Phys. Rev. A **75**, 032316 (2007)) from qubit to qudit case, and we further solve the problem of programmable dicriminators with arbitrary copies of unknown states in both program and data systems.

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1 Introduction

As a recent development, the possibility of discrimination between quantum states can be potentially useful for many applications in quantum communication and quantum computation. In this problem, a quantum state is chosen from a set of known states but we do not know which and want to determine the actual states. This is a nontrivial problem since the states cannot be successfully identified with unit probability because of the non-cloning theorem [1]. Two basic strategies have been introduced to achieve the state discrimination, one of which is the minimum-error discrimination [2, 3, 4, 5, 6, 7] and the other is the unambiguous discrimination for linearly independent states [8, 9, 10, 11, 12, 13]. In the minimum-error discrimination, errors are permitted and the optimal measurement is required such that the probability of error is minimum, while in the unambiguous discrimination not errors but inconclusive results are permitted, and in the optimal strategy the probability of failure is a minimum. Recently, another approach for the linearly dependent states was proposed with the maximum confidence measurements [14].

A universal device that can unambiguously discriminate between two unknown qubit states has also been constructed by Bergou and Hillery [15]. In their work, the system consists of two program qubits A and C, and one data qubit B. It is assumed that the qubit A and B are prepared in the states $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively, and qubit A is prepared in either $|\psi_1\rangle$ or $|\psi_2\rangle$ with probabilities η_1 and η_2, where $\eta_1 + \eta_2 = 1$, guaranteeing that the state in system B is always one of the two states. Such a device can measure the total input states

$$
\begin{aligned}
|\Psi_1\rangle &= |\psi_1\rangle_A |\psi_1\rangle_B |\psi_2\rangle_C, \\
|\Psi_2\rangle &= |\psi_1\rangle_A |\psi_2\rangle_B |\psi_2\rangle_C,
\end{aligned}
\tag{1}
$$

where the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are both unknown,

$$
|\psi_1\rangle = a|0\rangle + b|1\rangle, \quad |\psi_2\rangle = c|0\rangle + d|1\rangle,
\tag{2}
$$

and the parameters a, b, c and d are all arbitrary unknown complex variables satisfying the normalization conditions $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. This universal discriminator is known as a sort of programmable quantum device, which has been studied in both theory and experiment recently [16, 17, 18, 19, 20, 21, 22, 23].

The generalization and the experimental realization aspects of the discriminator above have also been introduced and widely discussed [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. The optimal schemes, where the multiple copies of program and data are used in the input states, have been obtained for $n_A = n_C = n$, $n_B = 1$ [25, 26, 28], for $n_A = n_C = 1$, $n_B = n$ [27], for $n_A = n_C = n, n_B = m$ [31] and for arbitrary copies in both data and program systems [38, 40]. The unambiguous discrimination for qudit case has been considered with single program and data copies ($n_A = n_B = n_C = 1$) [39]. The minimum-error discrimination for qudit case arbitrary copies has also been discussed [40].

The most general problem is that there are n_A and n_C copies of states in the program system A and C respectively, and n_B copies of states in the data system B, and furthermore, the states are n-dimensional ($n \geqslant 2$) qudit states rather than qubit states only. Then, the task is to discriminate between two input states,

$$
\begin{aligned}
|\Phi_1\rangle &= |\phi_1\rangle_A^{\otimes n_A} |\phi_1\rangle_B^{\otimes n_B} |\phi_2\rangle_C^{\otimes n_C}, \\
|\Phi_2\rangle &= |\phi_1\rangle_A^{\otimes n_A} |\phi_2\rangle_B^{\otimes n_B} |\phi_2\rangle_C^{\otimes n_C},
\end{aligned}
\tag{3}
$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are two unknown states in n-dimensional Hilbert space.

In this paper, we study both the unambiguous discrimination and minimum-error discrimination between two unknown qudit states with the inputs prepared with arbitrary copies in program and data systems as in Eq. (3). Unlike the discrimination between two known states, we cannot only consider this problem in the subspace spanned by the two states $|\phi_1\rangle$ and $|\phi_2\rangle$, and we should consider it in the full n-dimensional space, as the two states are completely unknown to us. By the the equivalence of unknown pure states to known average mixed states as in Refs. [25, 26, 27, 31, 38, 39] and with the Jordan-basis method [41], we obtain the optimal detection operators and the results for the universal discrimination between the mean states.

The rest of the present paper is organized as follows. Sec. 2 is a preliminary section where we introduce some notations and discuss the average mixed states for the inputs. In Sec. 3, we will derive the Jordan-basis for the average input states by the representation theory of $U(n)$ group. The inner products and their multiplicities are given in Sec. 4 with the coupling theory of angular momenta. The main results of this paper are shown in Sec. 5 and Sec. 6 for optimal unambiguous discrimination and minimum-error discrimination, respectively, and some special examples are discussed in Sec. 7. Finally, we end this paper with a short summary in Sec. 8. Some basic concepts and methods about the group representation theory that are used in this paper are given in the appendix part.

2 preliminary

In this section, we will discuss the equivalence of unknown pure states to known mixed states. Since the two states $|\phi_1\rangle$ and $|\phi_2\rangle$ are two unknown states in a n-dimensional Hilbert space $\mathcal{H}$, they can change from preparation to preparation. It is only the permutation symmetry properties of $|\Phi_1\rangle$ and $|\Phi_2\rangle$ that is preserved and can be regarded as available information to distinguish $|\phi_1\rangle$ and $|\phi_2\rangle$. Therefore, we introduce two averaged states

$$
\begin{aligned}
\rho_1 &= \int d\mu(\phi_1)d\mu(\phi_2)[\phi_1^{\otimes n_A}]_A[\phi_1^{\otimes n_B}]_B[\phi_2^{\otimes n_C}]_C, \\
\rho_2 &= \int d\mu(\phi_1)d\mu(\phi_2)[\phi_1^{\otimes n_A}]_A[\phi_2^{\otimes n_B}]_B[\phi_2^{\otimes n_C}]_C,
\end{aligned}
\tag{4}
$$

where $d\mu(\phi)$ is the 'natural' measure for the pure state induced by the Haar measure on the unitary group $U(n)$ [42] with normalization condition $\int d\mu(\phi) = 1$. We use $[\phi]$ to denote $|\phi\rangle\langle\phi|$ as in the Refs. [43, 38] and similarly $[\phi\psi\cdots] = [\phi] \otimes [\psi] \otimes \cdots = |\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi| \otimes \cdots$. Without loss of generality, we assume that $n_A \geqslant n_C$, and $n_1 = n_A + n_B, n_2 = n_B + n_C, N = n_A + n_B + n_C$.

It should be noticed that there are schemes to discriminate between the original states in Eq. (3) but with less efficiency than the averaged states in Eq. (4), and generally, the discrimination between the original input states is not completely equivalent to that between their averaged states. For instance, the discrimination between the averaged input states provides an upper bound for the success of the two original states [28]. However, the discrimination of unknown pure states is, on average, equivalent to the discrimination between the known averaged mixed states.

lemma: For a pure state $|\psi\rangle$ in n-dimensional Hilbert space $\mathcal{H}$,

$$
\int d\mu(\psi)[\psi^{\otimes m}] = \frac{1}{d^{[m]}}I^{[m]},
$$

where $d^{[m]} = \begin{pmatrix} n+m-1 \\ n-1 \end{pmatrix}$ is the dimension of the fully symmetric space $\mathcal{H}^{[m]}$ and $I^{[m]}$ is the

projector onto this space.

Proof: Since for any vector $|\Psi^{[m]}\rangle \in \mathcal{H}^{[m]}$, we have $\int d\mu(\psi)[\psi^{\otimes m}]|\Psi^{[m]}\rangle \in \mathcal{H}^{[m]}$, and it is easy to see that $\int d\mu(\psi)[\psi^{\otimes m}]$ satisfies the additivity and scalar multiplication, then it is a linear operator on $\mathcal{H}^{[m]}$. Suppose $U^{[m]}$ is an irreducible representation for $U(n)$ on space $\mathcal{H}^{[m]}$, and therefore

$$U^{[m]}\left(\int d\mu(\psi)[\psi^{\otimes m}]\right) = \int d\mu(\psi)\left(U|\psi\rangle\langle\psi|\right)^{\otimes m}$$

$$= \int d\mu(U\psi)[(U|\psi\rangle)^{\otimes m}]U^{\otimes m} = \int d\mu(\psi)[\psi^{\otimes m}]U^{\otimes m}$$

$$= \left(\int d\mu(\psi)[\psi^{\otimes m}]\right)U^{[m]},$$

where we have used the property $d\mu(U\psi) = d\mu(\psi)$ for Haar measure. According to the Schur's lemma [44, 45], we have $\int d\mu(\psi)[\psi^{\otimes m}] = \lambda I^{[m]}$, where λ is a constant. Moreover, $\mathrm{Tr}\left(\int d\mu(\psi)[\psi^{\otimes m}]\right) = \int d\mu(\psi)\mathrm{Tr}([\psi^{\otimes m}]) = \int d\mu(\psi) = 1$ and $\mathrm{Tr}(I^{[m]}) = d^{[m]}$, so $\lambda = 1/d^{[m]}$, which accomplishes the demonstration of the lemma. $\qquad\square$

From the lemma above, one can obtain

$$\rho_1 = \frac{1}{d_1}I^{[n_1]} \otimes I^{[n_C]},$$

$$\rho_2 = \frac{1}{d_2}I^{[n_A]} \otimes I^{[n_2]}, \tag{5}$$

where $d_1 = d^{[n_1]}d^{[n_C]}$ and $d_2 = d^{[n_A]}d^{[n_2]}$ are the ranks for ρ_1 and ρ_2, respectively. These two mixed states are the ensembles of the two unknown pure states, and the problem becomes the discrimination between ρ_1 and ρ_2. In the following sections, we will show how to apply the Jordan-basis method to solve this problem.

3   Jordan basis for the average input states

Let us further discuss the structures of mixed states ρ_1 and ρ_2 in Eq. (5). Consider ρ_1 first, and it is obvious that $\mathbb{1}^{[n_1]} \otimes \mathbb{1}^{[n_C]}$ is the identity operator on the tensor space $\mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_c]}$, whose bases are $\left|\begin{array}{c}[n_1]\\\omega_1\end{array}\right\rangle \otimes \left|\begin{array}{c}[n_C]\\\omega_C\end{array}\right\rangle$. Here, the ω_1 and ω_C correspond to the Weyl tableaux of $[n_1]$ and $[n_C]$, respectively, and we have omitted the label m_1 and m_C because they can take only one value. The space $\mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_c]}$ is usually reducible under $[U(n)]^{\otimes N}$ [44, 45], and the two irreducible bases $\left|\begin{array}{c}[n_1]\\\omega_1\end{array}\right\rangle$ and $\left|\begin{array}{c}[n_C]\\\omega_C\end{array}\right\rangle$ can be coupled together to give the irreducible basis [45]

$$\left|\begin{array}{c}[\nu]\\\tau[n_1]m_1[n_C]m_2,\omega\end{array}\right\rangle = \sum_{\omega_1\omega_C} C^{[\nu]\tau,\omega}_{[n_1]\omega_1,[n_C]\omega_C}\left|\begin{array}{c}[n_1]\\\omega_1\end{array}\right\rangle\left|\begin{array}{c}[n_C]\\\omega_C\end{array}\right\rangle,$$

$$\tag{6}$$

where $C^{[\nu]\tau,\omega}_{[n_1]\omega_1,[n_C]\omega_C}$ are the CG coefficients of the $U(n)$ group, $\omega = 1, 2, \cdots, d^{[\nu]}$, and $\tau = 1, 2, \cdots, \{[n_1][n_C][\nu]\}$ is the multiplicity label. According to the Littlewood rule, one has

$$[n_1] \otimes [n_C] = \bigoplus_{[\nu]}\{[n_1][n_C][\nu]\}[\nu], \tag{7}$$

and it is easy to see $\{[n_1][n_C][\nu]\} = 1$ for all possible $[\nu]$ since $[n_1]$ and $[n_C]$ are both totally symmetric. With Young diagrams, Eq. (7) can be graphically expressed as

$$\tag{8}$$

The Eq. (8) above shows that besides the fully symmetric case, the Young diagram $[\nu]$ can take some special cases with only two rows, such as $[N-1,1], [N-2,2], \cdots, [n_1, n_C]$. The new irreducible basis in Eq. (6) have two labels $[\nu]$ and ω, and hence can be denoted as $\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle$ for simplify. Now, we can see that the basis $\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle$ also form the complete orthogonal basis for the space $\mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_C]}$. Therefore, ρ_1 can also be expressed as

$$\rho_1 = \frac{1}{d_1} \sum_{[\nu],\omega} \left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle \left\langle \begin{array}{c} [\nu] \\ \omega \end{array} \right|, \tag{9}$$

and if $H^{[\nu]}$ is defined as the space spanned by the basis vectors $\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle$ ($\omega = 1, 2, \cdots, d^{[\nu]}$), one has

$$\mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_C]} = \bigoplus_{[\nu]} H^{[\nu]}, \tag{10}$$

where we use H instead of $\mathcal{H}$ because the basis $\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle$ are not the standard basis of S_N. However, $\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle$ can be transformed to standard basis $\left| \begin{array}{c} [\nu] \\ m, \omega \end{array} \right\rangle$ by

$$\left| \begin{array}{c} [\nu] \\ \omega \end{array} \right\rangle = \sum_m \left\langle \begin{array}{c} [\nu] \\ m \end{array} \middle| [\nu], \begin{array}{c} [n_1][n_C] \\ m_1 m_C \end{array} \right\rangle \left| \begin{array}{c} [\nu] \\ m, \omega \end{array} \right\rangle, \tag{11}$$

where $\left\langle \begin{array}{c} [\nu] \\ m \end{array} \middle| [\nu], \begin{array}{c} [n_1][n_C] \\ m_1 m_C \end{array} \right\rangle$ are the $[\nu] \downarrow [n_1] \otimes [n_C]$ subduction coefficients (SDCs) of S_N [45].

Similar discussions can be carried on for ρ_2, and

$$\rho_2 = \frac{1}{d_2} \sum_{[\nu'],\omega'} \left| \begin{array}{c} [\nu'] \\ \omega' \end{array} \right\rangle \left\langle \begin{array}{c} [\nu'] \\ \omega' \end{array} \right|, \tag{12}$$

where

$$\left| \begin{array}{c} [\nu'] \\ \omega' \end{array} \right\rangle = \sum_{m'} \left\langle \begin{array}{c} [\nu'] \\ m' \end{array} \middle| [\nu'], \begin{array}{c} [n_A][n_2] \\ m_A m_2 \end{array} \right\rangle \left| \begin{array}{c} [\nu'] \\ m', \omega' \end{array} \right\rangle, \tag{13}$$

and

$$\mathcal{H}^{[n_A]} \otimes \mathcal{H}^{[n_2]} = \bigoplus_{[\nu']} H'^{[\nu']}. \tag{14}$$

It should be noticed here that the possible cases for the Young diagram $[\nu']$ are not all the same as those for $[\nu]$ if $n_A \neq n_C$. Since $n_A \geqslant n_C$, $[\nu']$ can take no fewer cases than $[\nu]$, which can be displayed as follows

$$\tag{15}$$

For the last Young diagram on the right hand of the equation above, the number of the cells for the first row is $\max(n_A, n_2)$. It is easy to see that $\max(n_A, n_2) \leqslant n_1$, and therefore, we can conclude that the Young diagrams $[\nu']$ and $[\nu]$ can take the same possible cases for $n_A = n_C$, while for $n_A > n_C$, $[\nu']$ can take more cases besides those of $[\nu]$. Based on this point, we can also know that $d_1 \leqslant d_2$.

Now, we can introduce some notations that will be useful in the following discussions. Let $\mathbb{H}_T = \left(\mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_C]}\right) \bigcup \left(\mathcal{H}^{[n_A]} \otimes \mathcal{H}^{[n_2]}\right)$, and this is the total space. For a Young diagram $[\lambda]$ that can be taken for both $[\nu]$ and $[\nu']$, we define $\mathbb{H} = \bigoplus_{[\lambda]} H^{[\lambda]}$, $\mathbb{H}' = \bigoplus_{[\lambda]} H'^{[\lambda]}$, $\mathbb{H}_0 = \mathbb{H} \bigcup \mathbb{H}'$, and $\mathbb{H}^{[\lambda]} = H^{[\lambda]} \bigcup H'^{[\lambda]}$. Meanwhile, for a Young diagram $[\mu]$ that $[\nu]$ cannot take but $[\nu']$ can, let $\mathbb{H}'^{\perp} = \bigoplus_{[\mu]} H'^{[\mu]}$. It is evident that these spaces have the following relationships

$$\mathbb{H} = \mathcal{H}^{[n_1]} \otimes \mathcal{H}^{[n_C]}, \qquad \mathbb{H}' \oplus \mathbb{H}'^{\perp} = \mathcal{H}^{[n_A]} \otimes \mathcal{H}^{[n_2]},$$
$$\mathbb{H}_0 = \bigoplus_{[\lambda]} \mathbb{H}^{[\lambda]}, \qquad \mathbb{H}_T = \mathbb{H}_0 \oplus \mathbb{H}'^{\perp}. \tag{16}$$

It has been well known that there exist the Jordan basis for two nonorthogonal Hilbert spaces [46], and this can be used to the discrimination between two mixed states if we can find their Jordan bases [41]. The Jordan bases are defined as follows. The sets of orthogonal and normalized basis $\{|f_1\rangle, |f_2\rangle, \cdots, |f_k\rangle\}$ in space V_1 and $\{|g_1\rangle, |g_2\rangle, \cdots, |g_k\rangle\}$ in space V_2 form the Jordan bases when

$$\langle f_i | g_j \rangle = \delta_{ij} \cos\theta_i, \tag{17}$$

where θ_i are the so-called Jordan angles ($\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_k$). Since the support of ρ_1 has no overlaps with the space $\mathbb{H}'^{\perp}$, there is no doubt that we should consider the Jordan bases of the two nonorthogonal spaces $\mathbb{H}$ and $\mathbb{H}'$ in subspace $\mathbb{H}_0$. With the Eqs. (11) and (13), we obtain the overlap

$$\left\langle \begin{matrix} [\nu] \\ \omega \end{matrix} \middle| \begin{matrix} [\nu'] \\ \omega' \end{matrix} \right\rangle = \sum_{m,m'} \left\langle \begin{matrix} [\nu] \\ m \end{matrix} \middle| [\nu], \begin{matrix} [n_1][n_C] \\ m_1 m_C \end{matrix} \right\rangle \left\langle \begin{matrix} [\nu'] \\ m' \end{matrix} \middle| [\nu'], \begin{matrix} [n_A][n_2] \\ m_A m_2 \end{matrix} \right\rangle \left\langle \begin{matrix} [\nu] \\ m, \omega \end{matrix} \middle| \begin{matrix} [\nu'] \\ m', \omega' \end{matrix} \right\rangle$$

$$= \sum_{m} \left\langle \begin{matrix} [\nu] \\ m \end{matrix} \middle| [\nu], \begin{matrix} [n_1][n_C] \\ m_1 m_C \end{matrix} \right\rangle \left\langle \begin{matrix} [\nu'] \\ m \end{matrix} \middle| [\nu'], \begin{matrix} [n_A][n_2] \\ m_A m_2 \end{matrix} \right\rangle \delta_{[\nu][\nu']} \delta_{\omega\omega'}, \tag{18}$$

which shows that $\left| \begin{matrix} [\nu] \\ \omega \end{matrix} \right\rangle$ and $\left| \begin{matrix} [\nu'] \\ \omega' \end{matrix} \right\rangle$ have already been the Jordan bases of subspaces $\mathbb{H}$ and $\mathbb{H}'$. Next, we will consider how to calculate the inner products of Jordan bases and give their multiplicities.

4   Inner products of Joran basis and the multiplicities

Eq. (18) has shown that the inner products $\left\langle \begin{array}{c} [\nu] \\ \omega \end{array} \middle| \begin{array}{c} [\nu'] \\ \omega' \end{array} \right\rangle$ are dependent on the SDCs only. If we can calculate the SDCs, the inner products can easily be obtained. As we know, it is not a simple work to calculate the SDCs in this paper. However, in another way, the SDCs are connected with the permutation group only, and they are independent on the dimension of $\mathcal{H}$. This suggests that the inner products for any n can be solved as soon as one can calculate those for a special n. Fortunately, for $n = 2$, the qubit cases, the irreducible basis are just the angular momentum basis of the total system cosisting of A, B and C, and therefore, the inner products can be calculated by the coupling theory of angular momentum, without calculating the exact values of SDCs.

For $n = 2$, each copy is a qubit state, and then can been viewed as a spin-$1/2$ system with $j = 1/2$. The program systems A and C, and data system B can be regarded as angular systems with quantum numbers $j_A = n_A/2$, $j_C = n_C/2$ and $j_B = n_B/2$. The angular momentum basis $|(j_A j_B)j_{AB}, j_C; JM\rangle$ and $|j_A, (j_B j_C)j_{BC}; J'M'\rangle$ are the irreducible basis for ρ_1 and ρ_2, respectively. There is a a one-to-one relationship between the quantum numbers J or J' and the possible Young diagrams in the right hand of Eq. (8) or (15), and then the quantum numbers J for ρ_1 can take no more values than J' for ρ_2. In ρ_1 the first n_1 spins are couple in a symmetric way, therefore $j_{AB} = n_1/2$, and similarly $j_{BC} = n_2/2$ for the same reason. So we come to that $J = (n_1 - n_C)/2, \cdots, N/2$ and $J' = |n_A - n_2|/2, \cdots, N/2$. Now,

$$
\begin{aligned}
&\langle (j_A j_B)j_{AB}, j_C; JM | j_A, (j_B j_C)j_{BC}; J'M' \rangle \\
=\ & (-1)^{j_A + j_B + j_C + J} \sqrt{(2j_{AB} + 1)(2j_{BC} + 1)} \\
&\times \left\{ \begin{array}{ccc} j_A & j_B & j_{AB} \\ j_C & J & j_{BC} \end{array} \right\} \delta_{JJ'} \delta_{MM'} \\
=\ & \delta_{JJ'} \delta_{MM'} \sqrt{ \frac{ \left(\begin{array}{c} n_1 - k \\ n_B \end{array} \right) \left(\begin{array}{c} n_2 - k \\ n_B \end{array} \right) }{ \left(\begin{array}{c} n_1 \\ n_B \end{array} \right) \left(\begin{array}{c} n_2 \\ n_B \end{array} \right) } }
\end{aligned}
\tag{19}
$$

where we have set $J = N/2 - k$ ($k = 0, 1, \cdots, n_C$) and $\left\{ \begin{array}{ccc} j_A & j_B & j_{AB} \\ j_C & J & j_{BC} \end{array} \right\}$ are the Wigner's $6j$ symbols [48, 47]. The overlaps are independent of the quantum number M, and therefore, the inner products of Jordan basis $\left\langle \begin{array}{c} [\nu] \\ \omega \end{array} \middle| \begin{array}{c} [\nu'] \\ \omega' \end{array} \right\rangle$ are determined only by the Young diagram, which means that the inner products are the invariants of $U(n)$ group. For an inner product corresponding to a Young diagram $[\lambda]$ (this diagram can be taken by both $[\nu]$ for ρ_1 and $[\nu']$ for ρ_2), the multiplicity is the dimension $d^{[\lambda]}$, the number of values that ω can takes. The inner products of Jordan basis and their multiplicities are all listed in the following table:

$J\ (n=2)$	Young diagrams $[\lambda]$	Inner products $O_k = O^{[\lambda]}$	Multiplicity $d^k = d^{[\lambda]}$
$\frac{N}{2}$	$[N]$	1	$\binom{N+n-1}{n-1}$
$\frac{N}{2}-1$	$[N-1,1]$	$\sqrt{\frac{n_A n_C}{n_1 n_2}}$	$\frac{(N-1)(n-1)}{N}\binom{N+n-2}{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{N}{2}-k$	$[N-k,k]$	$\sqrt{\frac{\binom{n_1-k}{n_B}\binom{n_2-k}{n_B}}{\binom{n_1}{n_B}\binom{n_2}{n_B}}}$	$\frac{N-2k+1}{N-k+1}\binom{N+n-k-1}{n-1}\binom{n+k-2}{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{N}{2}-n_C$	$[n_1,n_C]$	$\sqrt{\frac{n_A!n_B!n_C!(n_1-n_C)!}{n_1!n_2!(n_A-n_C)!}}$	$\frac{n_1-n_C+1}{n_1+1}\binom{n_1+n-1}{n-1}\binom{n+n_C-2}{n-2}$

For the qubit case ($n=2$), the inner products and multiplicities will reduce to those in Ref. [31], if we further assume $n_A = n_C$. In that paper, the authors found an inherent symmetry to study the structures of the mean input states, which works for $n_A = n_C$.

With the Jordan basis above, the subspace $\mathbb{H}^{[\lambda]}$ can be further decomposed into $\mathbb{H}^{[\lambda]} = \bigoplus_\omega \mathbb{H}_\omega^{[\lambda]}$, where $\mathbb{H}_\omega^{[\lambda]}$ is the subspace spanned by $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_1$ and $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_2$, $\omega = 1, 2, \cdots, d^{[\lambda]}$, and the indexes "1" and "2" are used to label the bases for ρ_1 and ρ_2, respectively. In the overall ensemble, since ρ_1 occurs with probability η_1 and ρ_2 with η_2, the probability of occurrence for $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_1$ is η_1/d_1 and that of $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_2$ is η_2/d_2. Therefore, the probability for the occurrence of a vector in $\mathbb{H}_\omega^{[\lambda]}$ is

$$p_\omega^{[\lambda]} = \frac{\eta_1}{d_1} + \frac{\eta_2}{d_2}. \tag{20}$$

The probability that $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_1$ occurs conditioned on that $\mathbb{H}_\omega^{[\lambda]}$ has occurred is $\eta_{\omega,1}^{[\lambda]} = \eta_1/(d_1 p_\omega^{[\lambda]})$ and similarly $\eta_{\omega,2}^{[\lambda]} = \eta_2/(d_2 p_\omega^{[\lambda]})$. Finally, the problem to discriminate between ρ_1 and ρ_2 is reduced to deriving the optimal schemes for the unambiguous and the minimum-error discrimination between two pure states occurring with probabilities $\eta_{\omega,1}^{[\lambda]}$ and $\eta_{\omega,2}^{[\lambda]}$ in each subspace $\mathbb{H}_\omega^{[\lambda]}$.

5 Optimal unambiguous discrimination

To discriminate between $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_1$ and $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_2$ with the *a priori* probabilities $\eta_{\omega,1}^{[\lambda]}$ and $\eta_{\omega,2}^{[\lambda]}$ in space $\mathbb{H}_\omega^{[H]}$ unambiguously, we can introduce the POVM operators in the following form [41, 31]

$$\begin{aligned}
\Pi_{\omega,1}^k(q_{k,1}, q_{k,2}) &= \frac{1-q_{k,1}}{1-O_k^2}|\psi_{k,\omega}^\perp\rangle_2\langle\psi_{k,\omega}^\perp|, \\
\Pi_{\omega,2}^k(q_{k,1}, q_{k,2}) &= \frac{1-q_{k,2}}{1-O_k^2}|\psi_{k,\omega}^\perp\rangle_1\langle\psi_{k,\omega}^\perp|, \\
\Pi_{\omega,0}^k(q_{k,1}, q_{k,2}) &= \mathbb{1}_\omega^k - \Pi_{\omega,1}^k - \Pi_{\omega,2}^k
\end{aligned} \tag{21}$$

where we have used k to denote the Young diagrams listed in the table in Sec. 4. $q_{k,1}$ or $q_{k,2}$ is the failure probability for $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_1$ or $\left|\begin{smallmatrix}[\lambda]\\\omega\end{smallmatrix}\right\rangle_2$ in the unambiguous discrimination, the normalized vector

$|\psi_{k,\omega}^{\perp}\rangle_{1(2)}$ is orthogonal to $\left| \begin{smallmatrix} [\lambda] \\ \omega \end{smallmatrix} \right\rangle_{1(2)}$ in the subspace $\mathbb{H}_{\omega}^{[\lambda]}$, and $\mathbb{1}_{\omega}^{k}$ is the unit operator in $\mathbb{H}_{\omega}^{[\lambda]}$. The parameters $q_{k,1}$ and $q_{k,2}$ are independent of ω because the inner products are independent of ω. The total failure probability for the unambiguous discrimination between ρ_1 and ρ_2 is

$$Q = \sum_{k=0}^{n_C} d^k Q_k = \sum_{k=0}^{n_C} d^k \left(\frac{\eta_1 q_{k,1}}{d_1} + \frac{\eta_2 q_{k,2}}{d_2} \right), \tag{22}$$

with $d^k = d^{[N-k,k]}$ the multiplicities for the inner products of Jordan basis. We can find the the optimal settings

$$q_{k,1}^{\text{opt}} = \begin{cases} 1 & \text{if } \eta_1 < c_k \\ \sqrt{\frac{\eta_2 d_1}{\eta_1 d_2}} O_k & \text{if } c_k \leqslant \eta_1 \leqslant d_k \\ O_k & \text{if } \eta_1 \geqslant d_k \end{cases}, \tag{23}$$

and $q_{k,2}^{\text{opt}} = O_k^2 / q_{k,1}^{\text{opt}}$, where Q_k attains its minimum,

$$Q_k^{\text{opt}} = \begin{cases} \frac{\eta_1}{d_1} + \frac{\eta_2}{d_2} O_k^2 & \text{if } \eta_1 < c_k \\ 2\sqrt{\frac{\eta_1 \eta_2}{d_1 d_2}} O_k & \text{if } c_k \leqslant \eta_1 \leqslant d_k \\ \frac{\eta_1}{d_1} O_k^2 + \frac{\eta_2}{d_2} & \text{if } \eta_1 \geqslant d_k \end{cases}. \tag{24}$$

The boundaries c_k and d_k are as follows

$$c_k = \frac{d_1 O_k^2}{d_2 + d_1 O_k^2}, \ d_k = \frac{d_1}{d_1 + d_2 O_k^2}. \tag{25}$$

Finally, the optimal failure probability for the unambiguous discrimination between ρ_1 and ρ_2 is

$$Q^{\text{opt}} = \sum_{k=0}^{n_C} d^k Q_k^{\text{opt}} \tag{26}$$

and the corresponding optimal POVM are

$$\begin{aligned} \Pi_1 &= \sum_{k,\omega} \Pi_{\omega,1}^k (q_{k,1}^{\text{opt}}, q_{k,2}^{\text{opt}}), \\ \Pi_2 &= \sum_{k,\omega} \Pi_{\omega,2}^k (q_{k,1}^{\text{opt}}, q_{k,2}^{\text{opt}}) + I^{\perp}, \\ \Pi_0 &= I_{\text{T}} - \Pi_1 - \Pi_2, \end{aligned} \tag{27}$$

where I_{T} is the identity operator on the space $\mathbb{H}_{\text{T}}$, and $I^{\perp}$ is the projector onto the subspace $\mathbb{H}^{\perp}$. The projector $I^{\perp}$ appears in Π_2 because the occurrence in $\mathbb{H}^{\perp}$ always means the input state is ρ_2 (or $|\Phi_2\rangle$). We see from the equations above that both the POVM operators and the optimal failure probability of unambiguous discrimination between ρ_1 and ρ_2 are dependent on the dimension n and the numbers of copies in data system and program systems, since the parameters such as O_k and d^k are dependent on them.

6 Minimum-error discrimination

For the minimum-error discrimination between the two mixed states ρ_1 and ρ_2, the inconclusive results do not occur, so $\Pi_0 = 0$, and we require that the probability of errors in the discrimination procedure is a minimum. The error probability can be expressed as [2]

$$P_E = \eta_1 \mathrm{Tr}(\rho_1 \Pi_2) + \eta_2 \mathrm{Tr}(\rho_2 \Pi_1) = \eta_1 + \mathrm{Tr}(\Lambda \Pi_1), \tag{28}$$

where $\Lambda = \eta_2 \rho_2 - \eta_1 \rho_1 = \sum_i \lambda_i |\omega_i\rangle\langle\omega_i|$, with λ_i the eigenvalue spectrum of the operator Λ. It is obvious that the minimum of the error probability is obtained when Π_1 is the projector onto the space spanned by those eigenstates $|\omega_i\rangle$ that belong to negative eigenvalues λ_i. The optimal detection operators therefore read

$$\Pi_1 = \sum_{i < i_0} |\omega_i\rangle\langle\omega_i|, \ \ \Pi_2 = \sum_{i \geqslant i_0} |\omega_i\rangle\langle\omega_i|, \tag{29}$$

where $\omega_i < 0$ for $1 \leqslant i \leqslant i_0$ and $\omega_i \geqslant 0$ for $i \geqslant i_0$. Clearly, the minimum-error measurement for discriminating between two quantum states is a von Neumann measurement. The resulting minimum-error probability is

$$P_{\mathrm{ME}} = \frac{1}{2}(1 - \mathrm{Tr}|\Lambda|), \tag{30}$$

where $|\Lambda| = \sqrt{\Lambda^\dagger \Lambda}$.

With Eq. (9) and Eq. (12), the operator Λ can be expressed as

$$\Lambda = \sum_{[\lambda],\omega} \Lambda_\omega^{[\lambda]} + \frac{\eta_2}{d_2} \sum_{[\mu],\omega} \left| \begin{array}{c} [\mu] \\ \omega \end{array} \right\rangle_2 \left\langle \begin{array}{c} [\mu] \\ \omega \end{array} \right|, \tag{31}$$

with

$$\Lambda_\omega^{[\lambda]} = \frac{\eta_2}{d_2} \left| \begin{array}{c} [\lambda] \\ \omega \end{array} \right\rangle_2 \left\langle \begin{array}{c} [\lambda] \\ \omega \end{array} \right| - \frac{\eta_1}{d_1} \left| \begin{array}{c} [\lambda] \\ \omega \end{array} \right\rangle_1 \left\langle \begin{array}{c} [\lambda] \\ \omega \end{array} \right|, \tag{32}$$

where the Young diagram $[\lambda]$ can be taken for both ρ_1 and ρ_2, while $[\mu]$ for ρ_2 only. The eigenvalues of $\Lambda^{[\lambda],\omega}$ can be easily obtained as

$$\begin{aligned} \lambda_{\omega,+}^k &= \frac{1}{2}\left(c_- + \sqrt{c_+^2 - (c_+^2 - c_-^2)O_k^2}\right), \\ \lambda_{\omega,-}^k &= \frac{1}{2}\left(c_- - \sqrt{c_+^2 - (c_+^2 - c_-^2)O_k^2}\right), \end{aligned} \tag{33}$$

with $c_\pm = \eta_2/d_2 \pm \eta_1/d_1$, and we have used k to denote the Young diagram $[N-k, k]$. The eigenvalue spectrum of $\Lambda^{[\lambda],\omega}$ is therefore as follows,

$$\Lambda^{[\lambda],\omega} = \lambda_{\omega,+}^k |\lambda_{\omega,+}^k\rangle\langle\lambda_{\omega,+}^k| + \lambda_{\omega,-}^k |\lambda_{\omega,-}^k\rangle\langle\lambda_{\omega,-}^k|, \tag{34}$$

where $|\lambda_{\omega,+}^k\rangle$ and $|\lambda_{\omega,-}^k\rangle$ are the eigenvectors corresponding to the eigenvalues $\lambda_{\omega,+}^k$ and $\lambda_{\omega,-}^k$, respectively. By some algebra, one can easily know $\lambda_{\omega,+}^k \geqslant 0$ and $\lambda_{\omega,-}^k \leqslant 0$, so we can get

$$P_{\mathrm{ME}} = \frac{1}{2}\left(\eta_1 + \frac{\eta_2 d_1}{d_2} - \sum_{k=0}^{n_C} d^k \sqrt{c_+^2 - (c_+^2 - c_-^2)O_k^2}\right), \tag{35}$$
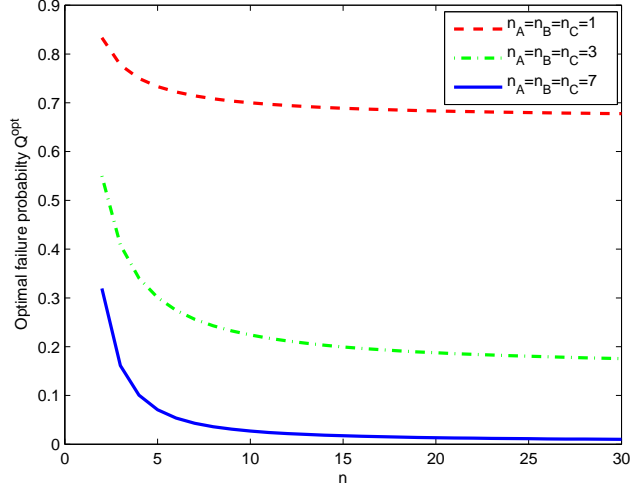
Fig. 1   (Color online) Graphs of the optimal failure probability Q^{opt} as a function of the dimension n for $n_A = n_B = n_C = 1$(dashed, red), 3 (dotted-dashed, green) and 7 (solid, blue).

and the corresponding measurement operators read

$$
\begin{aligned}
\Pi_1 &= \sum_{k=0}^{n_C} \sum_{\omega} |\lambda_{\omega,-}^k\rangle\langle\lambda_{\omega,-}^k|, \\
\Pi_2 &= \sum_{k=0}^{n_C} \sum_{\omega} |\lambda_{\omega,+}^k\rangle\langle\lambda_{\omega,+}^k| + \sum_{[\mu],\omega} \left| \begin{matrix} [\mu] \\ \omega \end{matrix} \right\rangle_2 \left\langle \begin{matrix} [\mu] \\ \omega \end{matrix} \right|.
\end{aligned}
\tag{36}
$$

Obviously, P_E is also dependent on the dimension n and the numbers of copies in systems A, B and C, and for the equal *a priori* probabilities $\eta_1 = \eta_2 = 0.5$, the express in Eq. (35) reproduces the results in the Ref. [40]. With Jordan basis approach, the geometric structure of the Hilbert space are shown, and it is more convenient to obtain these results. Next, we will give some special examples to show the influence of the dimension n on both unambiguous discrimination and minimum-error discrimination.

7   Some examples

In previous works, the authors have already given some examples to show the fact that more copies in program and data systems will give lower inconclusive probability and lower minimum-error probability for unambiguous discrimination and minimum-error discrimination between the mean input states ρ_1 and ρ_2. The results also hold for qudit cases, and therefore we do not focus on this question here. In this section, we mainly provide some examples to show the relation between the dimension n and the unambiguous discrimination and minimum-error discrimination between ρ_1 and ρ_2. For convenience sake, we set $\eta_1 = \eta_2 = 0.5$.

First, we consider the unambiguous discrimination between ρ_1 and ρ_2. If $n_A = n_C$, we have $d_1 = d_2$, and the inequality $c_k \leqslant \eta_1 \leqslant d_k$ always holds for $0 \leqslant k \leqslant n_C$. Therefore, the total POVM

is valid, and the the optimal inconclusive probability is reduced to

$$Q^{\text{opt}} = \frac{1}{d_1} \sum_{k=0}^{n_C} d^k O_k. \tag{37}$$

For the cases $n_A = n_B = n_C = 1, 3$ and 7, the numerical results of the failure probability Q^{opt} as a function of the dimension n are displayed in Fig. 1. One can see that the optimal failure probability decreases as the dimension n increases. For large n, there is a low bound for Q^{opt}, and the low bound can be obtained as

$$Q_0 = \frac{\Gamma(n_A + 1)\Gamma(n_B/2 + 1)}{\Gamma(n_A + n_B/2 + 1)} \tag{38}$$

for arbitrary $n_A = n_C$ and n_B when $n \to \infty$. The results also show that Q^{opt} decreased as the number of the copies is added.

Next, we consider the minimum-error discrimination between ρ_1 and ρ_2. When $n_A = n_C$, Eq. (35) becomes

$$P_{\text{ME}} = \frac{1}{2}(1 - \frac{1}{d_1} \sum_{k=0}^{n_C} d^k \sqrt{1 - O_k^2}). \tag{39}$$

We plot the minimum-error probability versus the dimension n for the cases $n_A = n_B = n_C = 3, 7$ and 10 in Fig. 2. We see that the minimum-error probability also decreases as the dimension increases, and also decreases as the number of the copies is added. Similarly, when $n \to \infty$, the low bound for P_{ME} is obtained as

$$P_0 = \frac{1}{2}\left(1 - \sum_{k=0}^{n_C} \frac{(N - 2k + 1)n_1!n_C!}{(N - k + 1)k!(N - k)!} \sqrt{1 - O_k^2}\right) \tag{40}$$

for arbitrary n_A, n_B and n_C.

For the case $n_A = n_B = n_C$, the low bounds for Q^{opt} and P_{ME} as a function of n_A are depicted in Fig. 3. The bounds decrease as the copies are added and they both approach 0 as $n_A \to \infty$.

8   Conclusions

In summary, we investigate the universal discrimination between two unknown qudit states with arbitrary numbers of copies in both data system and program systems. We demonstrate that the average input states are the maximally mixed states in the tensor spaces consisting of two totally symmetric spaces. The tensor spaces are reducible, and with the reducibility of $U(n)$ group, it can be decomposed into some irreducible subspaces denoted by the Young diagrams. The Jordan bases of the mean input states are just the irreducible basis of each irreducible subspace. We also find that the inner products of the Jordan bases are determined only by the corresponding Young diagrams and thus are independent on the dimension n. By the coupling theory of angular momentum, the explicit expressions of the inner products are derived. The multiplicities of the inner products are just the the dimensions of the irreducible subspaces, which can be given by the Robinson formula.

Then, we apply the Jordan-basis method, and the problem is reduced to the discrimination between two known pure states in each two-dimensional subspace $\mathbb{H}_\omega^{[\lambda]}$. We give the optimal measurement
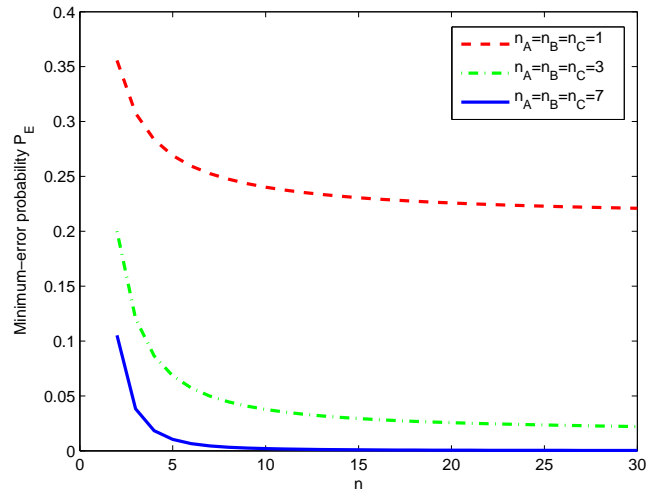
Fig. 2   (Color online) Graphs of the minimum-error probability P_E as a function of the dimension n for $n_A = n_B = n_C = 1$(dashed, red), 3 (dotted-dashed, green) and 7 (solid, blue).
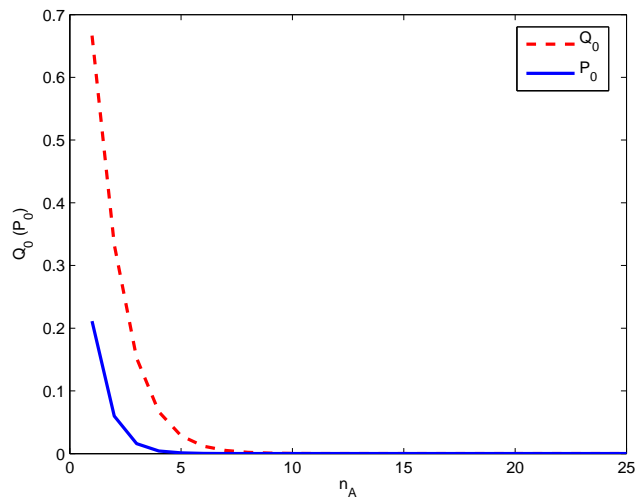

Fig. 3   (Color online) Graphs of the low bounds Q_0 (dashed, red) and P_0 (solid, blue) for $n \to \infty$ as a function of n_A for the case $n_A = n_B = n_C$.

operators for both unambiguous discrimination and minimum-error discrimination between the mixed states ρ_1 and ρ_2, where the optimal failure probability and the minimum-error probability are obtained in Eq. (26) and Eq. (35), respectively. For the qubit case ($n = 2$), the results in the previous works can be reproduced.

Finally, some special examples are given to show the relation between the dimension n and the discrimination between ρ_1 and ρ_2. We find that both the optimal failure probability and the minimum-error probability of unambiguous discrimination and minimum-error discrimination are decreased as the dimension n increases.

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Appendix A Partitions, Young diagrams and Young tableaux

A partition is a way of writing a positive integer n as a sum of k ($k \leqslant n$) integers λ_i satisfying

$$n = \sum_{i=1}^{k} \lambda_i, \ \ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0. \tag{A.1}$$

It can be pictured as a Young diagram $[\lambda] = [\lambda_1 \lambda_2 \cdots \lambda_k]$, which consists of n cells arranged in left-justified rows, with λ_i cells in the ith row. Since a partition of n corresponds to a inequivalent and irreducible representation of the permutation group S_n, a Young diagram $[\lambda]$ can usually be used to label an inequivalent and irreducible representation of S_n.

A Young tableau is an arrangement of the numbers $1, 2, \cdots, n$ in a Young diagram. If the numbers increase as one moves to the right and goes down, the Young tableau is a standard Young tableau. For a given Young diagram $[\lambda]$, the number of standard Young tableaux $f^{[\lambda]}$ is equal to the dimension of the irreducible representation $[\lambda]$ of the permutation group S_n, and can be calculated by the formula

$$f^{[\lambda]} = \frac{n!}{\prod_{ij} g_{ij}}, \tag{A.2}$$

where g_{ij} is the hook length for the cell in the ith row and jth column of the Young diagram $[\lambda]$. A hook of a cell consists of this given cell together with all those to the right in the same row and lower in the same column, and the number of cells in the hook is called the hook length. The m-th standard Young tableau can be denoted by $T_m^{[\lambda]}$ for a given Young diagram $[\lambda]$, where $m = 1, 2, \cdots, f^{[\lambda]}$.

Appendix B Representation theory of $U(n)$ group

The tensor product $U^{\otimes k}$ is a faithfull representation of the n-dimensional unitary group $U(n)$ on the tensor space $\mathcal{H}^{\otimes k}$, and it is reducible. Therefore, it can be decomposed into [44]

$$U^{\otimes k} = \bigoplus_{[\lambda], m} U_m^{[\lambda]}, \tag{B.1}$$

and correspondingly, $\mathcal{H}^{\otimes k}$ can be decomposed into

$$\mathcal{H}^{\otimes k} = \bigoplus_{[\lambda], m} \mathcal{H}_m^{[\lambda]}. \tag{B.2}$$

Here, $U_m^{[\lambda]}$ is an irreducible representation of $U(n)$ group on the subspace $\mathcal{H}_m^{[\lambda]}$, corresponding to a standard Young tableau $T_m^{[\lambda]}$, where the number of rows in $[\lambda]$ is no more than n. For $[\lambda] = [k]$ or $[\lambda] = [1^k]$, m can take only one value, and we can omit the label m in these cases.

The irreducible subspaces $\mathcal{H}_m^{[\lambda]}$ can be constructed via a standard way,

$$\mathcal{H}_m^{[\lambda]} = O_{mm}^{[\lambda]} \mathcal{H}^{\otimes k}, \tag{B.3}$$

where the projector operator $O_{mm}^{[\lambda]}$ are the orthogonal units of the permutation group S_k. For the details of $O_{mm}^{[\lambda]}$, see Ref. [44]. If $\{e_i\}$ ($i = 1, 2, \cdots, n$) form the complete orthogonal basis of $\mathcal{H}$, the complete orthogonal bases of the irreducible space $\mathcal{H}_m^{[\lambda]}$ can be constructed by $O_{mm}^{[\lambda]}$, and a basis vector of $\mathcal{H}_m^{[\lambda]}$ can be obtained as

$$\xi_{m, i_1 i_2 \cdots i_k}^{[\lambda]} = O_{mm}^{[\lambda]} e_{i_1 i_2 \cdots i_k}, \tag{B.4}$$

where $e_{i_1 i_2 \cdots i_k} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ $(i_1, i_2, \cdots, i_k = 1, 2, \cdots, n)$ are the complete orthogonal basis of $\mathcal{H}^{\otimes k}$. However, the bases $\{\xi^{[\lambda]}_{m, i_1 i_2 \cdots i_k}\}$ are linearly dependent, and to give the independent basis of $\mathcal{H}^{[\lambda]}_m$, we should first introduce the concept of Weyl tableaux.

A Weyl tableau is a Young diagram $[\lambda]$ whose cells are filled with some of the numbers $1, 2, \cdots, n$ under the restrictions that

(a) the numbers do not decrease in the row as one moves to the right;

(b) the numbers increase in the column as one goes down.

If the indices $i_1, i_2, \cdots, i_k$ take the values in the Wely tableaux, $\xi^{[\lambda]}_{m, i_1 i_2 \cdots i_k}$ are independent and form a complete orthogonal bases of $\mathcal{H}^{[\lambda]}_m$. Thus, the dimension of subspace $\mathcal{H}^{[\lambda]}_m$ is the number of the Weyl tableaux for $[\lambda]$, which has been given by the Robinson formula

$$d^{[\lambda]} = \prod_{ij} \frac{n - i + j}{g_{ij}}, \tag{B.5}$$

where g_{ij} is the hook length. For convenience sake, we use $\left| \begin{array}{c} [\lambda] \\ m, \omega \end{array} \right\rangle$ to denote the *normalized* basis vectors for $\mathcal{H}^{[\lambda]}_m$, where m and ω correspond to the standard Young tableaux and Weyl tableaux, respectively, $m = 1, 2, \cdots, f^{[\lambda]}, \omega = 1, 2, \cdots, d^{[\lambda]}$. Furthermore, $\left| \begin{array}{c} [\lambda] \\ m, \omega \end{array} \right\rangle$ are the standard basis of S_k and the irreducible basis of $U(n)$ [45].

Suppose $U^{[\lambda]}$ and $U^{[\mu]}$ are two irreducible representation for the unitary group $U(n)$, and the tensor product $U^{[\lambda]} \otimes U^{[\mu]}$ is also a representation of $U(n)$, but usually reducible. With Littlewood rule, $U^{[\lambda]} \otimes U^{[\mu]}$ can be decomposed into

$$U^{[\lambda]} \otimes U^{[\mu]} = \bigoplus_{[\sigma]} \{[\lambda][\mu][\sigma]\} U^{[\sigma]}, \tag{B.6}$$

with $\{[\lambda][\mu][\sigma]\}$ the multiplicity for $[\sigma]$. One should notice that he irreducible basses for $U^{[\sigma]}$ are usually not the standard basis of permutation group.