

CONSTRUCTING ARBITRARY STEANE CODE SINGLE LOGICAL QUBIT FAULT-TOLERANT GATES

AUSTIN G. FOWLER

*Centre for Quantum Computation and Communication Technology, School of Physics, University of Melbourne
Melbourne, Victoria 3010, Australia*

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We present a simple method for constructing optimal fault-tolerant approximations of arbitrary unitary gates using an arbitrary discrete universal gate set. The method presented is numerical and scales exponentially with the number of gates used in the approximation. However, for the specific case of arbitrary single-qubit gates and the fault-tolerant gates permitted by the concatenated 7-qubit Steane code, we find gate sequences sufficiently long and accurate to permit the fault-tolerant factoring of numbers thousands of bits long. A general scaling law of how rapidly these fault-tolerant approximations converge to arbitrary single-qubit gates is also determined.

Keywords: Quantum computing, fault-tolerant gates, Solovay-Kitaev theorem

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1 Introduction

In large-scale quantum computation, every qubit of data is encoded across multiple physical qubits to form a logical qubit permitting quantum error correction and fault-tolerant computation. Unfortunately, only very small sets of fault-tolerant gates G can be applied simply to logical qubits, where G depends on the number of logical qubits considered, the code used, and the level of complexity one is prepared to tolerate when implementing fault-tolerant gates. Gates outside G must be approximated with sequences of gates in G . The existence of efficient approximating sequences has been established by the Solovay-Kitaev theorem and subsequent work [1, 2, 3, 4]. In this paper, we describe a simple numerical procedure taking a universal gate set G , gate U , and integer l and outputting an optimal approximation of U using at most l gates from G . This procedure is used to explore the properties of approximations of the single-qubit phase rotation gates $R_{2^d} = \text{diag}(1, e^{i\pi/2^d})$ built out of fault-tolerant gates that can be applied to a single Steane code logical qubit. The average rate of convergence of Steane code fault-tolerant approximations to arbitrary single-qubit gates is also obtained.

Section 2 describes the numerical procedure used to find optimal gate sequences approximating a given gate. A convenient finite universal set of fault-tolerant gates that can be applied to a single Steane code logical qubit is given in Section 3. Section 4 contains a discussion of phase rotations R_{2^d} and their fault-tolerant approximations, followed by approximations of arbitrary gates in Section 5. Section 6 summarizes the results of this paper and their implications, and points to further work.

2 Finding optimal approximations

In this section, we outline a numerical procedure that takes a finite gate set $G \subset U(m)$ that generates $U(m)$, a gate $U \in U(m)$, and an integer l and outputs an optimal sequence U_l of at most l gates from G minimizing the metric

$$\text{dist}(U, U_l) = \sqrt{\frac{m - |\text{tr}(U^\dagger U_l)|}{m}}. \tag{1}$$

The rationale of Eq. (1) is that if U and U_l are similar, $U^\dagger U_l$ will be close to the identity matrix (possibly up to some global phase) and the absolute value of the trace will be close to m . By subtracting this absolute value from m and dividing by m a number between 0 and 1 is obtained. The overall square root is required to ensure that the triangle inequality

$$\text{dist}(U, W) \leq \text{dist}(U, V) + \text{dist}(V, W) \tag{2}$$

is satisfied. This metric has been used in preference to the trace distance used in the Solovay-Kitaev theorem [2, 3], as the trace distance does not ignore global phase, and hence leads to unnecessarily long global phase correct approximating sequences.

Finding optimal gate sequences is a difficult task, and the run-time of the numerical procedure presented here scales exponentially with l . Nevertheless, as we shall see in Section 4, gate sequences of sufficient length for practical purposes can be obtained.

For a set G of size $g = |G|$ and a maximum sequence length of l , the size of the set of all possible gate sequences of length up to l is approximately g^l . For even moderate g and l , this set cannot be searched exhaustively. To describe the basics of the actual method used, a few more definitions are required. Let G denote a gate in G . Order G , and denote the i th gate by G_i . Let S denote a sequence of gates in G . Order the possible gate sequences in the obvious manner and let S_n denote the n th sequence in this ordering. Let $\{S\}_l$ denote all sequences with length less than or equal to l . Let $\{Q\}_{l'}$, $l' < l$ denote the set of unique sequences of length at most l' . Naively, $\{Q\}_{l'}$ can be constructed by starting with the set containing the identity matrix, sequentially testing whether $S_n \in \{S\}_{l'}$ satisfies $\text{dist}(S_n, Q) > 0$ for all $Q \in \{Q\}_{l'}$, and adding S_n to $\{Q\}_{l'}$ if it does. A search for an optimal approximation of U using gates in G begins with the construction of a very large set of unique sequences $\{Q\}_{l'}$.

The utility of $\{Q\}_{l'}$ lies in its ability to predict which sequences in $\{S\}_l$, $l > l'$ do not need to be compared with U to determine whether they are good approximations, and what the next sequence worth comparing is. To be more precise, assume every sequence up to S_{n-1} has been compared with U . Let $\{S_{n-1}\}$ denote this set of compared sequences. Consider subsequences of S_n of length l' . If any subsequence is not in $\{Q\}_{l'}$, there exists a sequence in $\{S_{n-1}\}$ equivalent to S_n . In other words, a sequence equivalent to S_n has already been compared with U , and S_n can be skipped. Furthermore, let

$$S_n = G_{i_N} \dots G_{i_{k+l'+1}} G_{i_{k+l'}} \dots G_{i_{k+1}} G_{i_k} \dots G_{i_1}, \tag{3}$$

where $G_{i_{k+l'}} \dots G_{i_{k+1}}$ is the subsequence not in $\{Q\}_{l'}$. Let $Q(G_{i_{k+l'}} \dots G_{i_{k+1}})$ denote the next sequence in $\{Q\}_{l'}$ after $G_{i_{k+l'}} \dots G_{i_{k+1}}$. The next sequence with the potential to not be equivalent to a sequence in $\{S_{n-1}\}$ is

$$G_{i_N} \dots G_{i_{k+l'+1}} Q(G_{i_{k+l'}} \dots G_{i_{k+1}}) G_1 \dots G_1. \tag{4}$$

The process of checking subsequences is then repeated on this new sequence. Skipping sequences in this manner is vastly better than an exhaustive search, and enables optimal sequences of interesting length to be obtained. It should be stressed, however, that the runtime is still exponentially in l .

Highly non-optimal but polynomial runtime sequence finding techniques do exist [2, 3, 5, 6]. The fast runtimes come at the cost of long sequences, ranging from $O(\ln^2(1/\epsilon) \ln(\ln(1/\epsilon)))$ [5] to $O(\ln^{3.97}(1/\epsilon))$ [6]. The method presented here obtains sequences of optimal length $O(\ln(1/\epsilon))$. Given classical computing is much cheaper than quantum computing, it is far better to have an exponential runtime classical algorithm that obtains optimal length quantum gate sequences, provided sufficiently long sequences can still be obtained, which we will show is the case.

3 Simple Steane code single-qubit gates

For the remainder of the paper we will restrict our attention to fault-tolerant single-qubit gates that can be applied to the 7-qubit Steane code. The Steane code representation of states $|0\rangle$ and $|1\rangle$ is [7]

$$\begin{aligned}
 |0_L\rangle = & \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle \\
 & + |1100110\rangle + |0001111\rangle + |1011010\rangle \\
 & + |0111100\rangle + |1101001\rangle), \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 |1_L\rangle = & \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle \\
 & + |0011001\rangle + |1110000\rangle + |0100101\rangle \\
 & + |1000011\rangle + |0010110\rangle). \tag{6}
 \end{aligned}$$

An equivalent description of this code can be given in terms of stabilizers [8] which are operators that map the logical states $|0_L\rangle$ and $|1_L\rangle$ to themselves.

$$\text{IIIXXXX} \tag{7}$$

$$\text{IXXIIXX} \tag{8}$$

$$\text{XIXIXIX} \tag{9}$$

$$\text{IIIZZZZ} \tag{10}$$

$$\text{IZZIIZZ} \tag{11}$$

$$\text{ZIZIZIZ} \tag{12}$$

States $|0_L\rangle$ and $|1_L\rangle$ are the only two that are simultaneously stabilized by Eqs (7–12).

The minimal universal set of single-qubit fault-tolerant gates that can be applied to a Steane code logical qubit consists of just the Hadamard gate and the T -gate [3]

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}. \tag{13}$$

For practical purposes, the gates X , Z , S , S^\dagger should be added to this set, where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{14}$$

along with all gates generated by H, X, Z, S, S^\dagger . The complete list of gates that we shall consider is shown in Eq. (15). This is our set G . Note that gates $\{I, G_1, \dots, G_{23}\}$ form a group under multiplication.

$$\begin{array}{ll}
 G_1 = H & G_{13} = HS \\
 G_2 = X & G_{14} = HS^\dagger \\
 G_3 = Z & G_{15} = ZXH \\
 G_4 = S & G_{16} = SXH \\
 G_5 = S^\dagger & G_{17} = S^\dagger XH \\
 G_6 = XH & G_{18} = HSH \\
 G_7 = ZH & G_{19} = HS^\dagger H \\
 G_8 = SH & G_{20} = HSX \\
 G_9 = S^\dagger H & G_{21} = HS^\dagger X \\
 G_{10} = ZX & G_{22} = S^\dagger HS \\
 G_{11} = SX & G_{23} = SHS^\dagger \\
 G_{12} = S^\dagger X & G_{24} = T
 \end{array} \tag{15}$$

We use this large set G as H, X, Z, S, S^\dagger and their products can all be easily implemented with transversal single-qubit gates. In contrast, the T -gate is extremely complicated to implement [9]. Since we are interested in minimal complexity as well as minimum length sequences of gates in G , it would be unreasonable to count G_{23} as three gates when in reality it can be implemented as easily as any other gate $\{G_1, \dots, G_{22}\}$. Since $\{I, G_1, \dots, G_{23}\}$ is a group under multiplication, minimum length sequences of gates approximating some U outside G will alternate between an element of $\{G_1, \dots, G_{23}\}$ and a T -gate. Note that the T^\dagger -gate is not required in G for universality or efficiency as, in gate sequences of length $l \geq 2$, it is equally efficient to use $S^\dagger T$ or TS^\dagger . The extra S^\dagger -gate is absorbed into neighboring G_i -gates, $i < 24$.

4 Approximations of phase gates

We now use the simple algorithm described in this paper to construct optimal fault-tolerant approximations of phase rotation gates

$$R_{2^d} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2^d} \end{pmatrix}. \tag{16}$$

Gates R_{2^d} are examples of gates used in the single-qubit quantum Fourier transform that forms part of the Shor circuits described in [10, 11]. Note that phase rotations of angle $2\pi x/2^d$, where x is a d -digit binary number, are also required, but the properties of fault-tolerant approximations of such gates can be inferred from R_{2^d} .

For a given R_{2^d} , and maximum number of gates l in G , Fig. 1 shows $\text{dist}(R_{2^d}, U_l)$ where U_l is an optimal sequence of at most l gates in G minimising $\text{dist}(R_{2^d}, U_l)$. For $d \geq 3$, U_1 is equivalent to the identity. Note that as d increases, R_{2^d} becomes closer and closer to the identity, lowering the value of $\text{dist}(R_{2^d}, U_l)$, and increasing the value of l required to obtain an approximation U_l that is closer to R_{2^d} than the identity. In fact, for R_{128} the shortest sequence of gates that provides a better approximation of R_{128} than the identity has length $l = 31$. There are a very large number of optimal sequences of this length. An example of one with a minimal number of T -gates is

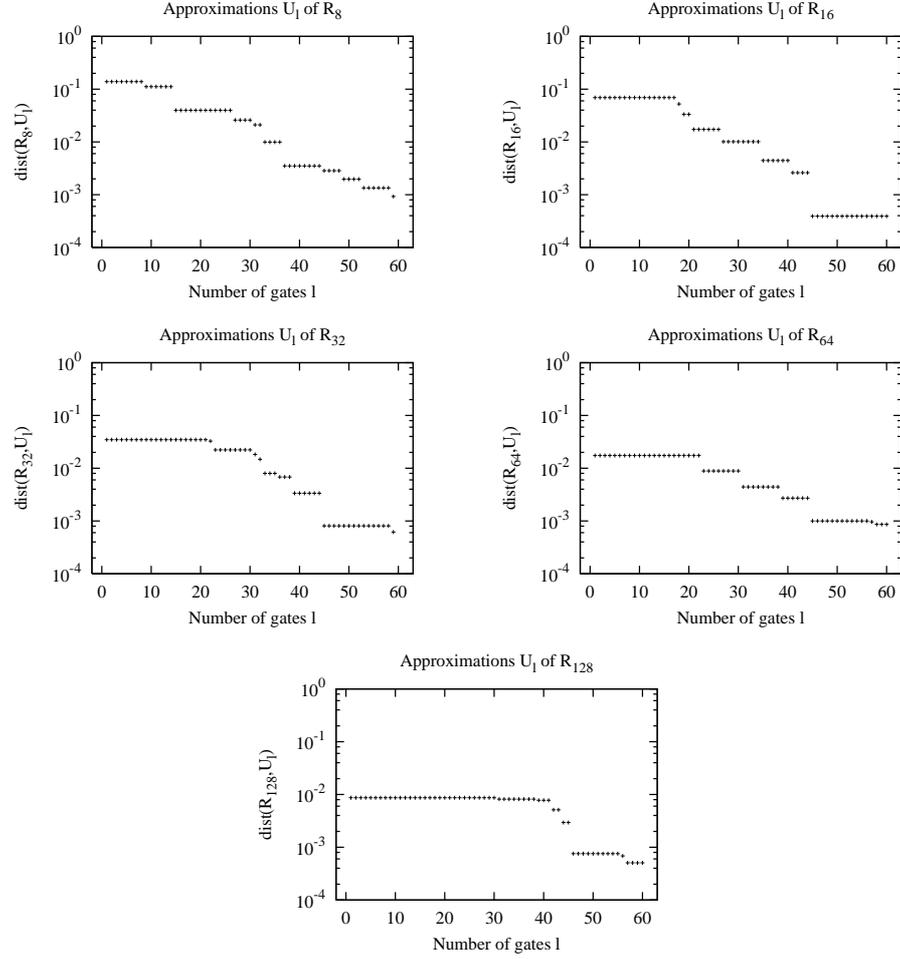


Fig. 1. Optimal fault-tolerant approximations U_l of phase rotation gates R_{2^d} .

$$U_{31} = \begin{matrix} HTHT(SH)T(SH)T(SH)THTHT(SH) \\ THTHT(SH)THTHTHT(SH)T(S^\dagger H) \end{matrix} \quad (17)$$

Note that $\text{dist}(R_{128}, I) = 8.7 \times 10^{-3}$ whereas $\text{dist}(R_{128}, U_{33}) = 8.1 \times 10^{-3}$. In other words Eq. (17) is only slightly better than the identity. This immediately raises the question of how many gates are required to construct a sufficiently good approximation.

In [11], it was shown that

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\pi/128+\pi/512)} \end{pmatrix} \quad (18)$$

was sufficiently close to R_{128} . This is, of course, only a property of Shor's algorithm, not a universal property of quantum circuits. Given $\text{dist}(R_{128}, U) = 2.2 \times 10^{-3}$, a fault-tolerant approximation U_l of R_{128} must therefore satisfy $\text{dist}(R_{128}, U_l) < 2.2 \times 10^{-3}$ to have a high

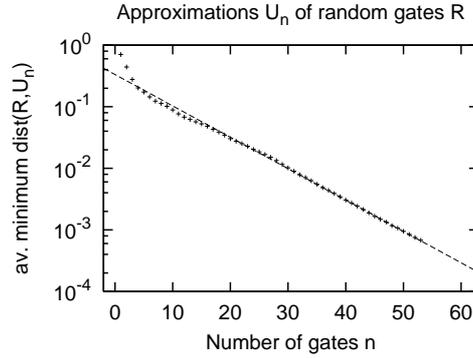


Fig. 2. Average accuracy of optimal fault-tolerant gate sequence approximations of length l .

chance of being sufficiently accurate. The smallest value of l for which this is true is 46, and one of the many optimal gate sequences satisfying $\text{dist}(R_{128}, U_{46}) = 7.5 \times 10^{-4}$ is

$$\begin{aligned}
 U_{31} = & HTHTHT(SH)THT(SH)T(SH)T(SH)THT \\
 & (SH)T(SH)THTHT(SH)T(SH)THT(SH)T \\
 & (SH)T(SH)THT(SH)THT(HS^\dagger)T
 \end{aligned} \tag{19}$$

Note that implementing this long sequence of fault-tolerant gates would necessitate the use of concatenation to ensure the inevitable multiple errors during execution are reliably corrected.

5 Approximations of arbitrary gates

In this section, we investigate the properties of fault-tolerant approximations of arbitrary single-qubit gates

$$U = \begin{pmatrix} \cos(\theta/2)e^{i(\alpha+\beta)/2} & \sin(\theta/2)e^{i(\alpha-\beta)/2} \\ -\sin(\theta/2)e^{i(-\alpha+\beta)/2} & \cos(\theta/2)e^{i(-\alpha-\beta)/2} \end{pmatrix}. \tag{20}$$

Consider Fig. 2. This was constructed using 1000 random matrices U of the form Eq. 20 with α, β, θ uniformly distributed in $[0, 2\pi)$. Optimal fault-tolerant approximations U_l were constructed of each, with the average $\text{dist}(U, U_l)$ plotted for each l . The indicated line best fit has the form

$$\delta = 0.292 \times 10^{-0.0511l}. \tag{21}$$

This equation characterizes the average number l of Steane code single-qubit fault-tolerant gates required to obtain a fault-tolerant approximation U_l of an arbitrary single-qubit gate U to within $\delta = \text{dist}(U, U_l)$.

6 Conclusion

We have described an algorithm enabling the optimal approximation of arbitrary unitary matrices given a discrete universal gate set. We have used this algorithm to investigate the properties of fault-tolerant approximations of arbitrary single-qubit gates using the gates that can be applied to a single Steane code logical qubit. We have found that on average an l gate approximation can be found within $\delta = 0.292 \times 10^{-0.0511l}$ of the ideal gate. The work

here suggests that practical quantum algorithms should avoid, where possible, logical gates that must be implemented using a sequence of fault-tolerant gates since even the rotation gates used in Shor's algorithm, which do not need to be implemented with great accuracy, still require lengthy sequences. Quantum simulation algorithms are expected to require far greater precision and thus far longer sequences, and will be studied in future work.

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