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COHERENCE PRESERVATION IN A Λ-TYPE THREE-LEVEL ATOM

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Coherence preservation of a multilevel system subject to Markovian decoherence is studied. A Λ-type three-level atom is selected as the system model. Coherence preservation between a ground state and the excited state of such a system is defined as the control object. A control field is designed by means of constraining the constant coherence condition. For the singularities of the control field, we qualitatively analyze the breakdown time, i.e. the time of control diverging. We obtain the region in which the state stays to maintain coherence forever in the case that the three-level system is reduced to a two-level one. For other cases, we investigate how different parameters affect the breakdown time qualitatively. Numerical experiments are implemented on a three-level quantum system and the experimental results are analyzed.

Keywords: quantum control, coherence preserving, Markovian decoherence, Λ-type
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1 Introduction

Decoherence is a serious obstacle against the preservation of quantum superposition and entanglement over long periods of time [1]. Decoherence entails non-unitary evolutions [2, 3, 4], and results in the loss of information and/or probability leakage toward the environment. This issue is recently attracting much attention and a number of interesting schemes have been proposed to preserve coherence. Among these schemes, there are quantum error-correcting codes [5-10], error-avoiding codes [11, 12, 13], decoherence-free subspaces [14-20], Bang-Bang control [21-30], and combinations thereof [31]. Error-avoiding codes encode information into the degenerate subspace of the error operators so that the information will not be affected by the error operators, while error-correcting codes restore the loss of information due to decoherence or quantum dissipation by monitoring the system and conditionally carrying on suitable feedback control. Their applications are limited by the requirement of using redundant qubit resources. Bang-Bang control uses high-frequency pulses to average out the decoherence effect. However, it requires in an essential manner that the bath retains some memory of its interaction with the system. So it is useless to Markovian open system. Therefore, reference [32] proposed a new method: tracking-control, to solve the problem of stabilizing the coherence of a single qubit subject to Markovian decoherence. Recently, J Zhang *et al* [33] proposed a feedback control strategy based on quantum weak measurements to protect coherence of a

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qubit system, where the stochastic noise was considered. However, their works were restricted to the single qubit system and they only considered the unital-decoherence channel. In fact, on the one hand, as three-level and higher dimension systems have attracted much interest for their applications in quantum control and quantum computation [34, 35], preserving the coherence of such a multilevel system has become a hot topic. On the other hand, non-unital decoherence channels are also frequently used, e.g. spontaneous emission, and they exhibit completely distinct properties from the unital one does. So we take the two points mentioned above into account and study coherence preservation of higher dimension system subject to Markovian decoherence characterized by non-unital decoherence channels. As reference [32] pointed out in their open questions, the controllability could be improved if one expands the Hilbert space of the quantum system by including additional levels, which is profitable to maintain the coherence. So the expected answer of our previous question is definite positive. We consider a simple but typical problem: preserving the coherence between two levels of a Λ-type three-level atom subject to Markovian decoherence. This problem involves two issues: the feasibility of coherence preservation of multilevel system and the singularities issues as encountered in [32].

First we wish to demonstrate the feasibility of preserving coherence of Markovian open quantum systems. For this goal, we choose the coherence between a ground state and the excited state as the control object, and apply a classical field to drive the states between the two levels. By imposing the constant coherence condition, the control parameters involving initial phase and the envelope of the field are designed. It is proven that such a control field does exist. Second, the critical factor in our work is the singularity issue that the control field diverges after the breakdown time. Namely, the coherence can be maintained only within the breakdown time. Although it is obtained analytically in the case of pure dephasing for a single qubit system in [32], we find in our case that the analytical solution of breakdown time cannot be derived. Nonetheless, we qualitatively analyze the breakdown time, and obtain the region in which the initial state resides to maintain coherence for a long time under the special condition. At the same time it is proved by the numerical experiment that the coherence can be maintained longer in the case that the initial state resides in this region than that in other cases.

The paper is arranged as follows. In Sec. 2, the models of a Λ-type three-level atom subject to Markovian decoherence is described and the control objectives in terms of the coherence are defined. The parameters of the control field are designed in Sec. 3 so as to realize coherence preservation of the system. In Sec. 4, we discuss the nature of the singularity of the control field. In Sec. 5, simulation examples are made to verify the effectiveness of the method proposed. We conclude in Sec. 6 with a brief summary.

2 Models and objectives

As is well known, the dynamics of a Markovian open quantum system can be described by the following master equation [36]:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}(\rho) \tag{1}$$

where the quantum state is represented by the density matrix ρ. The system Hamiltonian H is comprised of field-free Hamiltonian H_0 and the control Hamiltonian H_C. The Lie-bracket on the operator space is defined as $[A,B]=AB-BA$. The Lindbladian $\mathcal{L}(\rho)$ is induced by interactions between the system and the environment. It can be written as

$$\mathcal{L}(\rho)=\frac{1}{2}\sum_k \gamma_k \left\{ \left[L_k,\rho L_k^\dagger\right]+\left[L_k\rho,L_k^\dagger\right]\right\} \tag{2}$$

with the system operator L_j with damping rates γ_j representing decoherence channels. Different decoherence channels exhibit distinct dynamics [37]. For instance, the unital Lindblad generator, i.e. $\mathcal{L}(I)=0$, can make the purity function be decreasing regardless of Hamiltonian control existing. The situation is different for non-unital decoherence channels. These channels can increase the purity even without the action of control. For example, under spontaneous emission an arbitrary qubit mixed state is gradually purified to the ground state. In this work, we consider only non-unital Lindbladian over finite dimensional Hilbert space.

We consider the atom in the Λ-type as shown in Fig. 1, in which two ground states namely $|1\rangle$, and $|2\rangle$ are coupled to an excited state $|e\rangle$ with resonance frequencies ω_1, and ω_2, respectively. The corresponding eigenvalues are labeled as E_1, E_2, and E_e, where $\omega_1=(E_e-E_1)/\hbar$, and $\omega_2=(E_e-E_2)/\hbar$. Similarly, we define $\omega_3=(E_1-E_2)/\hbar$ to be the resonant frequency between $|1\rangle$ and $|2\rangle$ for the atom and the laser field though it does not correspond to an available transition here.

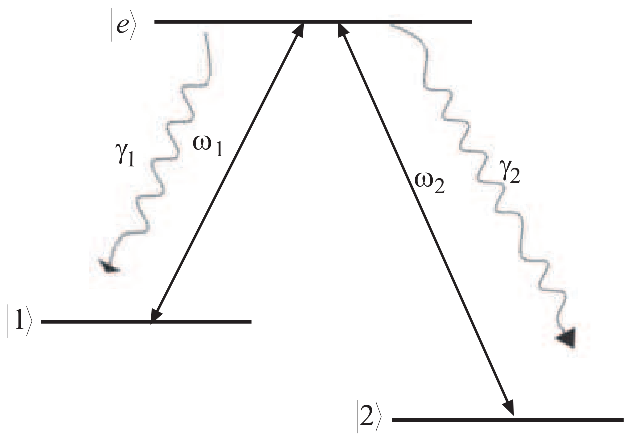

Fig. 1. Atomic configurations: two ground states coupled to an excited state $|e\rangle$ with coupling constant ω_1 and ω_2.

The following notations are defined as

$$\delta_z^{(j)}=|\,e\rangle\langle e|-|\,j\rangle\langle j|\,,\delta_x^{(j)}=|e\rangle\langle j|+|j\rangle\langle e|\,,\delta_y^{(j)}=-i\,|e\rangle\langle j|+i\,|j\rangle\langle e|$$

$$\delta_-^{(j)}=|j\rangle\langle e|\,,\delta_+^{(j)}=|e\rangle\langle j|\,,j=1,2 \tag{3}$$

Then, the field-free Hamiltonian of such a system can be written as

$$H_0=\frac{\omega_1}{3}\delta_z^{(1)}+\frac{\omega_2}{3}\delta_z^{(2)}+\frac{\omega_3}{3}\delta_z^{(3)} \tag{4}$$

where $\delta_z^{(3)}=|1\rangle\langle 1|-|2\rangle\langle 2|$, and the constant energy term $(E_1+E_2+E_e)/3$ has been ignored.

The excited state is not stable, it decays to the two ground states at rates γ_1 and γ_2, respectively, as shown in Fig. 1. Assume the decay process is Markovian, then the decoherence channel can be characterized by the Lindblad operator $\delta_-^{(j)} = |j\rangle\langle e|$, $j = 1, 2$, and the Lindbladian can be expressed as

$$\mathcal{L}(\rho) = \frac{1}{2}\sum_{k=1,2}\gamma_k\left\{\left[\delta_-^{(k)}, \rho\delta_+^{(k)}\right] + \left[\delta_-^{(k)}\rho, \delta_+^{(k)}\right]\right\} \tag{5}$$

One can verify $\mathcal{L}(I) \neq 0$, indicating the Lindbladian is non-unital.

Without loss of generality, we aim to preserve the coherence between the ground state $|1\rangle$ and the excited state $|e\rangle$, then the coherence function can be characterized by

$$C_1(\rho) = \sqrt{\left\langle \delta_x^{(1)} \right\rangle_\rho^2 + \left\langle \delta_y^{(1)} \right\rangle_\rho^2} \tag{6}$$

where $\langle A \rangle_\rho = \mathrm{Tr}(A\rho)$. The loss of coherence between two levels $|1\rangle$ and $|e\rangle$ is due to decay process from the higher level $|e\rangle$ to the lower level $|1\rangle$. So one can apply a classical field to drive the states between $|1\rangle$ and $|e\rangle$, thus the decay process may be inhibited.

We assume here that the transition dipole moments for the linearly polarized field are real. Then the control field, in the dipole approximation, can be expressed as [38]

$$E(t) = \varepsilon(t)\cos(\omega_d t + \phi_d)$$

The resulting expression for control Hamiltonian is

$$H_C = \varepsilon(t)\left(e^{i\phi_d}|1\rangle\langle e| + e^{-i\phi_d}|e\rangle\langle 1|\right)\cos\omega_d t \tag{7}$$

where the parameters ω_d and ϕ_d are the frequency and initial phase of the driving field; the parameter $\varepsilon(t)$ is the envelope of the field. They are the control parameters of the field to be designed. So according to (2), (4), (5) and (7), the decoherence equation under the action of control can be written as

$$\frac{\partial\rho}{\partial t} = -i\left[H_0 + H_C, \rho\right] + \frac{1}{2}\sum_{k=1,2}\gamma_k\left\{\left[\delta_-^{(k)}, \rho\delta_+^{(k)}\right] + \left[\delta_-^{(k)}\rho, \delta_+^{(k)}\right]\right\} \tag{8}$$

where the Planck constant has been assigned to be 1. Equation (8) is just the mathematic model of the control system. To keep the coherence constant during the whole evolution, we impose the following constraint:

$$C_1(\rho(t)) = C_1(\rho_0) \tag{9}$$

The following thing is to design the control parameters to satisfy constraint (9).

3 Design of the control field

In this section, we investigate the first issue as mentioned in the introduction, i.e. to design a control field to preserve the coherence of high-dimension system. One can see from (8) evidently that the drift Hamiltonian H_0 complicates the derivation of the control. So, for the sake of simplicity, we are going to analyze the system dynamics in the interaction picture, where an operator X in Hilbert space is transformed into

$$X^{\mathrm{I}} = e^{iH_0 t}Xe^{-iH_0 t} \tag{10}$$

Then the dynamics equation of $\rho^{\mathrm{I}}(t)$ in the interaction picture can be obtained:

$$\frac{\partial \rho^{\mathrm{I}}}{\partial t} = -i\left[H_C^{\mathrm{I}}, \rho^{\mathrm{I}}\right] + \frac{1}{2}\sum_k \gamma_k \left\{ \left[{\delta_-^{(k)}}^{\mathrm{I}}, \rho^{\mathrm{I}} {\delta_+^{(k)}}^{\mathrm{I}} \right] + \left[{\delta_-^{(k)}}^{\mathrm{I}} \rho^{\mathrm{I}}, {\delta_+^{(k)}}^{\mathrm{I}} \right] \right\} \tag{11}$$

And the coherence constraint (9) turns to be

$$C_1(\rho^{\mathrm{I}}(t)) = C_1(\rho_0) \tag{12}$$

The transformation maps the system states into a rotational coordinates, i.e. the local phases of the states are rotary. It can be verified that such transformation does not change the population distribution of the system state and the expectation value of operator [39]. In the interaction picture, the system Hamiltonian H turns to be

$$H_C^{\mathrm{I}} = e^{iH_0 t} H_c e^{-iH_0 t} = \varepsilon(t) e^{i\phi_d} \left|1\right\rangle \left\langle e\right| + \varepsilon(t) e^{-i\phi_d} \left|e\right\rangle \left\langle 1\right| = \varepsilon(t)\left(\cos\phi_d \delta_x^{(1)} + \sin\phi_d \delta_y^{(1)}\right) \tag{13}$$

where we assume the resonance condition, i.e. $\omega_d = \omega_1$. Further, one can derive that

$$\delta_-^{(1)\mathrm{I}} = e^{-i\omega_1 t}\delta_-^{(1)}, \delta_-^{(2)\mathrm{I}} = e^{-i\omega_2 t}\delta_-^{(2)} \tag{14}$$

Thus, substituting (14) into (11), equation (11) can be re-expressed as

$$\frac{\partial \rho^{\mathrm{I}}}{\partial t} = -i\left[H_C^{\mathrm{I}}, \rho^{\mathrm{I}}\right] + \frac{1}{2}\sum_{k=1,2} \gamma_k \left\{ \left[\delta_-^{(k)}, \rho^{\mathrm{I}} \delta_+^{(k)} \right] + \left[\delta_-^{(k)} \rho^{\mathrm{I}}, \delta_+^{(k)} \right] \right\} \tag{15}$$

which is just the models of the controlled system dynamics under the Markovian decoherence in the interaction picture. The transformation of the interaction picture is used to facilitate the mathematical treatment of the problem. The system after the transformation becomes more concise, in which the drift item H_0 disappears, and consequently greatly reduces the difficulty and complexity of the control design.

Then what flows is to design the parameters ϕ_d, $\varepsilon(t)$ of the control field so as to satisfy the coherence constraint (12). We start from deriving the equations of motion of $\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$, and $\left\langle \delta_y^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$. According to (15), one has

$$\begin{aligned}
\frac{d\left\langle \delta_y^{(1)} \right\rangle_{\rho^{\mathrm{I}}}}{dt} &= \mathrm{Tr}(\delta_y^{(1)} \dot{\rho}^{\mathrm{I}}) \\
&= \mathrm{Tr}\left(\delta_y^{(1)}\left(-i\left[H_C^{\mathrm{I}}, \rho^{\mathrm{I}}\right] + \mathcal{L}(\rho)\right)\right) \\
&= -i\mathrm{Tr}\left(\left[\delta_y^{(1)}, H_C^{\mathrm{I}}\right]\rho^{\mathrm{I}}\right) + \mathrm{Tr}\left(\delta_y^{(1)}\mathcal{L}(\rho)\right) \\
&= -2\varepsilon(t)\cos\phi_d \left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \frac{\gamma_1+\gamma_2}{2}\left\langle \delta_y^{(1)} \right\rangle_{\rho^{\mathrm{I}}}
\end{aligned} \tag{16}$$

Likewise, the motion of equation for $\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ can be obtained:

$$\frac{d\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}}}{dt} = \mathrm{Tr}(\delta_x^{(1)} \dot{\rho}^{\mathrm{I}}) = 2\varepsilon(t)\sin\phi_d \left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \frac{\gamma_1+\gamma_2}{2}\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \tag{17}$$

Obviously, by setting the phase ϕ_d and the amplitude $\varepsilon(t)$ as the following form:

$$\varepsilon(t) = \frac{(\gamma_1 + \gamma_2)\, C_1(\rho_0)}{4\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}} \tag{18}$$

$$\phi_d = \begin{cases} \arctan\left(-\left\langle \delta_x^{(1)} \right\rangle_{\rho_0} \Big/ \left\langle \delta_y^{(1)} \right\rangle_{\rho_0} \right), & \left\langle \delta_y^{(1)} \right\rangle_{\rho_0} < 0 \\ \pi \big/ 2, & \left\langle \delta_y^{(1)} \right\rangle_{\rho_0} = 0 \\ \pi + \arctan\left(-\left\langle \delta_x^{(1)} \right\rangle_{\rho_0} \Big/ \left\langle \delta_y^{(1)} \right\rangle_{\rho_0} \right), & \left\langle \delta_y^{(1)} \right\rangle_{\rho_0} > 0 \end{cases} \tag{19}$$

we have $d\left\langle \delta_y^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \Big/ dt \equiv 0$ and $d\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \Big/ dt \equiv 0$, this means that $\left\langle \delta_x^{(1)} \right\rangle_{\rho^{\mathrm{I}}} = \left\langle \delta_x^{(1)} \right\rangle_{\rho_0}$ and $\left\langle \delta_y^{(1)} \right\rangle_{\rho^{\mathrm{I}}} = \left\langle \delta_y^{(1)} \right\rangle_{\rho_0}$, which naturally lead to $C_1(\rho^{\mathrm{I}}(t)) = C_1(\rho_0)$, i.e. the coherence between $|1\rangle$ and $|e\rangle$ is preserved under the control field with parameters ϕ_d, $\varepsilon(t)$ specified by (18) and (19).

The purity and the coherence are both important quality of quantum dynamics. The former reflects the entire unitary dynamics, and the latter reflects the partial quantum dynamics [33]. So the purity in general comprises of coherence and other variables related to diagonal elements of density matrix. In the uncontrolled dynamics, the coherence keeps decreasing under decoherence effect until it vanishes. In the controlled scenario, the control fields trade changing variation trend of other variables in purity for fixing the coherence. In general, the purity for N-dimension quantum system can be defined as $p = \frac{N\mathrm{tr}(\rho^2)-1}{N-1}$. By the definition, the purity of the pure state and the maximum mixed state I_N/N are one and zero, respectively. In our case, the purity can be computed as

$$p = \frac{3}{2}tr({\rho^{\mathrm{I}}}^2) - \frac{1}{2} = \frac{3}{2}\sum_{i,k=1,2,e} \left|\rho_{ik}^{\mathrm{I}}\right|^2 - \frac{1}{2} = \frac{3}{4}\sum_{j=1,2,3}\left(\left\langle \delta_x^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 + \left\langle \delta_y^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 \right) + \frac{1}{2}\sum_{j=1,2,3}\left\langle \delta_z^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 \tag{20}$$

where $\delta_x^{(3)} = |1\rangle\langle 2| + |2\rangle\langle 1|$ and $\delta_y^{(3)} = -i\,|\,1\rangle\langle 2\,| + i\,|\,2\rangle\langle 1\,|$. Considering (6), equation (20) can be expressed as

$$p = \frac{3}{4}C_1^2 + \frac{3}{4}\sum_{j=2,3}\left(\left\langle \delta_x^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 + \left\langle \delta_y^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 \right) + \frac{1}{2}\sum_{j=1,2,3}\left\langle \delta_z^{(j)} \right\rangle_{\rho^{\mathrm{I}}}^2 \tag{21}$$

Thus, the purity related to not only the coherence function we concerned, but also the coherence between $|2\rangle$ and $|e\rangle$, as well as the coherence between $|2\rangle$ and $|1\rangle$, and the population distribution. In the two-level system subject to dephasing decoherence, the control fields trade decrease in purity in return for stabilization of coherence until the purity is equal to coherence ($c = C_1^2$) [32]. However, the trade-off for the three-level system becomes impossible as soon as $\left\langle \delta_z^{(1)} \right\rangle = 0$ at some time t_b. One can see from (21) that the purity could be larger than coherence at the time t_b. Namely, there is remaining purity not traded for the stabilization of coherence. Thus the time one can preserve the coherence in high-dimension system is shorter than the case of lower-dimension system under the same initial conditions if only the unital decoherence channels is considered. The situation is different for the non-unital decoherence. Because the purity increases probably, the time at which the trade-off becomes impossible is not determined. Therefore we must analyze the time of $\left\langle \delta_z^{(1)} \right\rangle = 0$, this is just the second issue as mentioned in the introduction. And, we will work out the singularities of the control field in the next section.

4 Analysis of Singularities issues

Although we have demonstrated the coherence of multilevel system can be preserved, and illustrated the process of the control field design in Section 3, we have to study singularities of the control field, i.e. the control field diverges as soon as $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ turns to zero. Therefore, the breakdown time, i.e. the time of the control field diverging, represents how long one can maintain the coherence. And we analyze the breakdown time in this section. First, we derive the equation of motion for $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$. According to (15), one has

$$\frac{d\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}}{dt} = \frac{-(\gamma_1+\gamma_2)\, C_1^2(\rho_0)}{2\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}} - (2\gamma_1+\gamma_2)\, \rho_{ee}^{\mathrm{I}} \tag{22}$$

where $\rho_{ee}^{\mathrm{I}} = \langle e \left| \rho^{\mathrm{I}} \right| e\rangle$. Due to the variation of $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ depending on ρ_{ee}^{I}, one needs to derive the equation of motion for ρ_{ee}^{I}. Similarly, one has

$$\frac{d\rho_{ee}^{\mathrm{I}}}{dt} = \langle e|\, \dot{\rho}^{\mathrm{I}}\, |e\rangle = \frac{-(\gamma_1+\gamma_2)\, C_1^2(\rho_0)}{4\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}} - (\gamma_1+\gamma_2)\, \rho_{ee}^{\mathrm{I}} \tag{23}$$

Obviously, equations (22) and (23) are not analytically solvable, so the analytical solution of the breakdown time can not be derived. Nonetheless, it is evident that the breakdown time is concerned with some parameters (e.g. $C_1(\rho_0)$, $\left\langle \delta_z^{(1)} \right\rangle_{\rho^0}$, and $\rho_{0,ee}$). Then it is important to ascertain how these parameters affect the breakdown time, this motivates us to make qualitative analysis for it.

In fact, there is no need to discuss the case of $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0} \geq 0$, because, in this case, one can directly see from (22) and (23) that the breakdown time is inversely proportional to $C_1^2(\rho_0)$ and $\rho_{0,ee}$, and proportional to $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0}$.

Thus, in what follows we assume $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0} < 0$ and analyze the motion of $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ based on the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane. From (22) and (23) it is evident that $d\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \Big/ d\rho_{ee}^{\mathrm{I}} \approx 2$ holds if the decay rates satisfy $\gamma_2/\gamma_1 \ll 1$. In such a case, there exist many pairs of real numbers (c_1, c_2) such that $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} = c_1$, $\rho_{ee}^{\mathrm{I}} = c_2$, and $4c_1c_2 = -C_1^2(\rho_0)$, leading to $d\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} = d\rho_{ee}^{\mathrm{I}} = 0$, thus $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ is kept to be c_1. In fact, such pairs of real numbers form the curve $4\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \rho_{ee}^{\mathrm{I}} = -C_1^2(\rho_0)$ in the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane. Therefore for some initial states, the control fields make $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ vary to and remain at some point on the curve, and these initial states form the region S_0. In other words, one can preserve the coherence almost forever if the initial states of the system reside in S_0 in the case of $\gamma_2/\gamma_1 \ll 1$. In the other case, for any initial state, there exist t_M such that $\left\langle \delta_z^{(1)}(t_M) \right\rangle_{\rho^{\mathrm{I}}} = 0$ and t_M is just the breakdown time. So the problem is divided into two parts: the first is to seek S_0, and the other is to analyze how different parameters influence the breakdown time.

For the first case of $\gamma_2/\gamma_1 \ll 1$, it can be shown that the variable $\left\langle \delta_z^{(1)}(t) \right\rangle_{\rho^{\mathrm{I}}}$ is constant for $\forall t > t_b$ if $\left\langle \delta_z^{(1)}(t_b) \right\rangle_{\rho^{\mathrm{I}}} = -\frac{C_1^2(\rho_0)}{4(1-\tau)\rho_{ee}^I(t_b)}$, where $\tau = \gamma_2/(\gamma_1+\gamma_2) \ll 1$. Considering the natural condition $\rho_{11}^{\mathrm{I}}(t) + \rho_{ee}^{\mathrm{I}}(t) \leq 1$, i.e. $\left\langle \delta_z^{(1)}(t) \right\rangle_{\rho^{\mathrm{I}}} \geq 2\rho_{ee}^{\mathrm{I}}(t) - 1$, the region S_0 is given by

$$\begin{aligned} S_0 = \{\rho_0 : C_1^2(\rho_0) \leq \frac{1-\tau}{2}, -1 \leq \left\langle \delta_z^{(1)} \right\rangle_{\rho_0} - 2\rho_{0,ee} \leq \frac{-2C_1(\rho_0)}{\sqrt{1-\tau}}, \\ C_1^2(\rho_0) + 4(1-\tau)\left\langle \delta_z^{(1)} \right\rangle_{\rho_0} \rho_{0,ee} \leq 0, \rho_{0,ee} \geq 0\} \end{aligned} \tag{24}$$

Then, we can conclude that if the initial states of the system reside in S_0 and the decay rates satisfy $\gamma_2/\gamma_1 \ll 1$, the coherence can be maintained for a long time. In this case, due to the weak coupling between $|2\rangle$ and $|e\rangle$, the quantum system is nearly equivalent to a two-level system. In such a system, the coherence can be preserved almost forever if the initial state resides in S_0. By contrast, the coherence is preserved within the break down time in [32]. This obvious difference comes from the fact that the dynamics equation of our system has a non-unital Lindblad operator $\delta_- = |1\rangle\langle e|$ that describes the relaxation effect, while the decoherence-channel in the reference [32] is unital. This difference will be verified by the first numerical experiment in next section.

In addition, the reduced two-level system can be represented by Bloch sphere, and the trade-off between coherence and purity can be interpreted geometrically. The uncontrolled relaxation channel makes any point in the Bloch sphere flow toward the stable point at the North Pole [40]. This process can be represented by the transformation of Block vector, that is

$$(v_x, v_y, v_z) \to (v_x\sqrt{1-\Gamma}, v_y\sqrt{1-\Gamma}, v_z(1-\Gamma)+\Gamma) \tag{25}$$

where Γ is time-depend function that converges to one. Equation (25) indicates that the x-y plane is contracted, at the same time the z-component v_z move toward the North Pole. In the controlled scenario, the components of x-y plane (coherence) are invariant, and the z-component move to the South Pole. The control field is thus able to trade the contraction in the x-y plane for the motion of z-component to the South Pole. So for all the initial states in Northern hemisphere, v_z turns to zero certainly. Whereas for some initial state in the southern hemisphere, e.g. the states in S_0, v_z does not reach zero almost forever. For the two-level system subject to dephasing decoherence, the trade-off between coherence and purity has distinct geometric interpretation [32]. The uncontrolled phase-flip channel maps the Bloch sphere to an ellipsoid with the z-axis as major axis and minor axis in the x-y plane. The major axis is invariant under the uncontrolled dynamics, while the minor axis is contracted. The control field attempts to rotate the ellipsoid so that the minor axis becomes as aligned as possible with the z-axis, where it would experience no contraction. The fields diverge as long as the contraction is so strong that the control field is no longer capable of sustaining the required rotation. In this case, two initial states symmetric about x-y plane have the same breakdown time.

For the other case, the atom is a generic three-level system, and the analytical solution of t_M is unable to be obtained as well. Similarly we qualitatively analyze the motion of $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ in the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane. Now, the special region is defined as

$$S = \{\rho^{\mathrm{I}} : C_1^2(\rho^{\mathrm{I}}) \leq \frac{1-\tau}{2}, -1 \leq \left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - 2\rho_{ee}^{\mathrm{I}} \leq \frac{-2C_1(\rho^{\mathrm{I}})}{\sqrt{1-\tau}}, \\ C_1^2(\rho^{\mathrm{I}}) + 4(1-\tau)\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} \rho_{ee}^{\mathrm{I}} \leq 0, \rho_{ee}^{\mathrm{I}} \geq 0\} \tag{26}$$

Note that if the initial state resides in S_0, the state of the system first evolves in the region S, then leaves it, and ultimately reaches some point that makes $\left\langle \delta_z^{(1)}(t) \right\rangle_{\rho^{\mathrm{I}}}$ be zero.

It inspires us to make a conjecture that the breakdown time is longer for the case of the initial state residing in S_0 than that for other cases. In addition, the breakdown time is proportional to the distance between the initial state and S_0.

This conjecture cannot be demonstrated theoretically and rigorously, however, we can verify it by numerical experiments. Obviously, one can see from (24) that the shape of S_0 relies on some parameters such as $C_1(\rho_0)$, τ, $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0}$ and $\rho_{0,ee}$. Thus the breakdown time t_M is concerned with these parameters. We will take $C_1(\rho_0)$, and τ for examples to verify the conjecture by the second simulation example in next section.

5 Numerical experiments and discussions

To demonstrate the effectiveness of the strategy proposed in this paper, we will implement some numerical examples with different parameters in this section and give some analysis. The propagation of the dynamical equation in (13) is carried out by fourth order Runge-Kutta integration.

The first simulation example is to verify that the control field can preserve the coherence of the system for a long time, provided that the initial states reside in S_0 and $\tau \ll 1$. Therefore the decay rates can be chosen as $\gamma_1 = 0.1, \gamma_2 = 0.001$ such that $\tau \ll 1$ and the initial state is assumed to be

$$\rho_0 = \begin{pmatrix} 0.21 & 0.195 - 0.195i & 0 \\ 0.195 + 0.195i & 0.78 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

which resides in S_0. The parameters of control field are designed according to (18) and (19). The dynamical equations are solved over the total propagation time intervals of $T = 500$ with a time step of 0.01.

The evolution of coherence function $C_1(\rho^{\mathrm{I}})$ is displayed in Fig. 2, from which one can see that the coherence between the ground state $|1\rangle$ and the excited state $|e\rangle$, is completely and quickly lost in absence of control, while one can keep the coherence be constant almost forever under the action of control field with the designed control parameters (18) and (19). In fact, the initial state indicates that the initial populations distribute mainly on the level

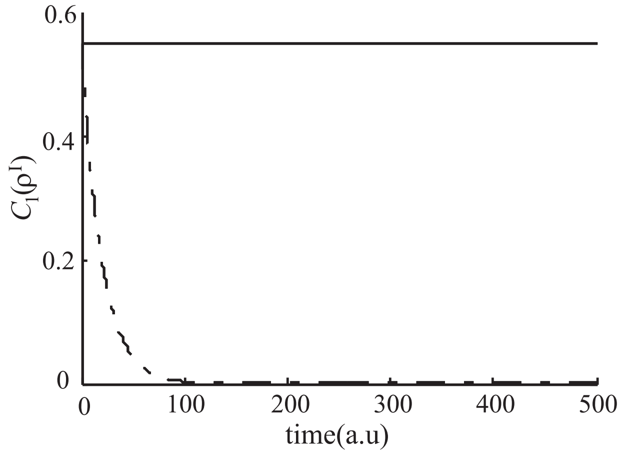

Fig. 2. Evolution of coherence function $C_1(\rho^{\mathrm{I}})$. The solid line corresponds to the trajectories under the action of control; and the dashed lines corresponds to the uncontrolled trajectories with decoherence.

$|e\rangle$ and $|1\rangle$, and the relaxation rate γ_2 is much smaller compared with γ_1, so the three-level system can be regarded as a two-level one. Furthermore, the initial state also indicates that the purity is $p = 0.8047$ and the initial coherence is $C_1(\rho_0) = 0.55$(according to the definition of the coherence c in [32], $c = C_1^2$ holds for the two-level system, so the coherence is $c = 0.3$). Then, we can deem that all the parameters in this simulation are the same as that in [32], except the decoherence-channels. The simulation results are different: the coherence in [32] can only be maintained within 8 a.u., which is very short compared to the case of non-unital channel in our case.

The second simulation example is to analyze how the different parameters influence the breakdown time, and verify our conjecture in previous section. Here the parameters $C_1(\rho_0)$ and τ are considered separately. Firstly, we study the influence of the parameter $C_1(\rho_0)$. The other parameters are fixed as $\tau = \frac{\gamma_2}{\gamma_1+\gamma_2} = \frac{0.1}{0.1+0.1} = 0.5$, $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0} = -0.5$, and let $C_1(\rho_0)$ be equal to 0.5, 0.6, and 0.7 respectively, the corresponding initial states are $\rho_{0,1}, \rho_{0,2}$, and $\rho_{0,3}$, respectively, where

$$\rho_{0,1} = \begin{pmatrix} 0.2 & 0.25 & 0 \\ 0.25 & 0.7 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \rho_{0,2} = \begin{pmatrix} 0.2 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \rho_{0,3} = \begin{pmatrix} 0.2 & 0.35 & 0 \\ 0.35 & 0.7 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Obviously, the three initial states do not reside in S_0, in addition, $\rho_{0,1}$ is nearest to S_0, and $\rho_{0,2}$ is nearer to S_0 than $\rho_{0,3}$ in the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane. The simulation results are shown in Fig.3, from which one can see that the breakdown time is inversely proportional to the desired constant initial coherence value. It indicates that the nearer the initial state is to S_0, the longer the breakdown time is. This result coincides with our conjecture.

Secondly, to study the parameter τ, the other parameters are fixed as $C_1(\rho_0) = 0.4$,

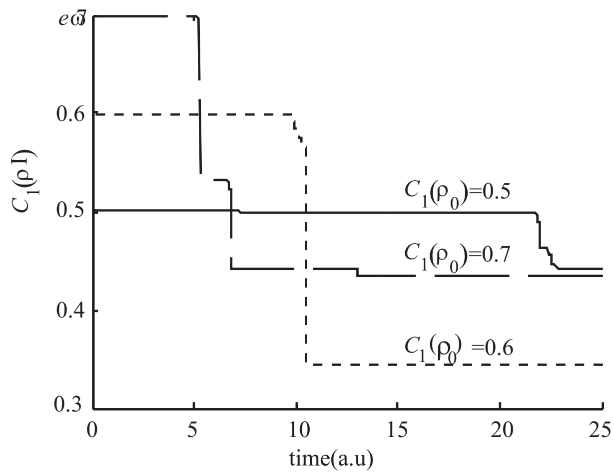

Fig. 3. Evolution of coherence function $C_1(\rho^{\mathrm{I}})$ for different $C_1(\rho_0)$. The solid line, the dotted line and the dashed line correspond to $C_1(\rho_0) = 0.5$, 0.6, and 0.7 respectively.

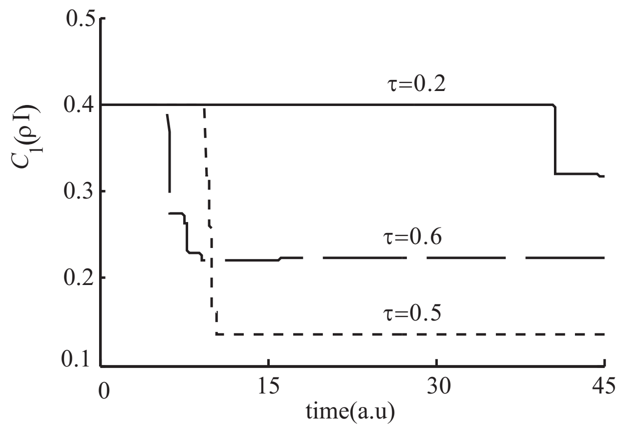

Fig. 4. coherence function $C_1(\rho^{\mathrm{I}})$ for different τ. The solid line, the dotted line and the dashed line correspond to $\tau = 0.2$, 0.5, and 0.6 respectively.

$\left\langle \delta_z^{(1)} \right\rangle_{\rho_0} = -0.5$. The initial state is assumed to be

$$\rho_0 = \begin{pmatrix} 0.2 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

One can see from (24) that different τ reflect whether the initial state resides in S_0. Let τ be equal to 0.2, 0.5, and 0.6, respectively. The simulation results are displayed in Fig. 4, from which one can see that the breakdown time is longer and longer as τ increases. In other words, the closer the initial state is to S_0, the longer the breakdown time is. This result also coincides with our conjecture.

In the simulation examples, we demonstrated that the designed control fields can preserve the coherence of the three-level system subject to non-unital decoherence channels. At the same time, the results also illustrated that the non-unital decoherence channels exhibited distinct property from unital ones. The second simulation example is to verify our conjecture that is derived from qualitative analysis. And we must stress here that only one parameter is allowed to vary in the simulations, because we cannot measure the distance between initial state and the region S_0 as soon as two or more parameters varying. In fact, the shape of the region S_0 is concerned with four parameters $C_1(\rho_0)$, τ, $\left\langle \delta_z^{(1)} \right\rangle_{\rho_0}$, and $\rho_{0,ee}$, but which parameter plays a dominant role is not clear, so the distance measure makes sense only if one parameter varies.

6 Conclusions

Decoherence is a natural part of the dynamics in open quantum systems, and the suppression or control of decoherence is a central issue for many applications. In this paper, taking the Λ-type three-level atom subject to Markovian decoherence for example we demonstrated the feasibility of preserving coherence of high-level Markovian open quantum systems. Concretely, the coherence between a ground state and the excited state is defined as the control object and a control field was designed by imposing the constant coherence condition. Moreover by means of analyzing the motion of $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}}$ in the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane, we investigated the singularities of the control field, where the breakdown time is studied. As a consequence, we obtained the region S_0 in which the initial state resides to maintain coherence for a long time when $\gamma_1/\gamma_2 \ll 1$. In fact, the three-level system, in such a case, is nearly equivalent to be a two-level system. For the other case, by qualitatively analyzing how the different parameters influence the breakdown time, the following conclusion was obtained: the breakdown time is longer for the case of the initial state residing in S_0 than that for other cases, in addition, the breakdown time is proportional to the distance between the initial state and S_0 in the $\left\langle \delta_z^{(1)} \right\rangle_{\rho^{\mathrm{I}}} - \rho_{ee}^{\mathrm{I}}$ phase plane. At last we presented some numerical examples to demonstrate the results.

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