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QUANTUM FISHER INFORMATION FOR SUPERPOSITIONS OF SPIN STATES

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In terms of quantum Fisher information, a quantity χ^2 was introduced by Pezzé and Smerzi [Phys. Rev. Lett. 102, 100401 (2009)], which is a multiparticle entanglement measure, and provides a necessary and sufficient condition for sub-shot-noise phase estimation sensitivity. We derive a general expression of χ^2 for arbitrary symmetric multiqubit states with nonzero mean spins. It is shown that the entangled symmetric states are useful for phase sensitivity beyond the shot-noise limit. Using the expression, we explicitly examine a series of superpositions of spin states. We find that the superpositions of Dicke states perform better than Dicke states themselves in phase esitmation. Although the spin coherent states themselves only have a shot-noise limit phase sensitivity, their superpositions may reach the Heisenberg limit.

Keywords: Quantum Fisher Information, Dicke states, spin coherent states
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1 Introduction

Quantum precision measurement [1, 2] is a cross discipline among many fields, such as statistical and quantum mechanics. In particular, phase estimation is an important issue in quantum interferometry [3]. On one hand, in the theory of quantum statistical estimation, quantum Fisher information (QFI) [4, 5, 6] plays an important role, since it provides an ultimate precision via the quantum Cramer-Rao (QCR) theory [1, 2] [see Eq. (4)]. On the other hand, the estimation of a phase ϕ in an N-particle system is bounded by the shot-noise limit [7]

$$\Delta\phi_{SN} \equiv 1/\sqrt{N}, \tag{1}$$

which is attainable by separable states and can be beaten using quantum tricks such as entanglement. In addition, there is another fundamental limit—the Heisenberg limit [8, 9]

$$\Delta\phi_{HL} \equiv 1/N, \tag{2}$$

which is the ultimate limit in precision phase measurement. It has been proved by a large amount of works that quantum entangled states are able to achieve the precision beyond the shot-noise limit or even at the Heisenberg limit (see [10] and references therein). Therefore, to identify the entangled states that are useful for high sensitivity of phase estimation, recently, Pezzé and Smerzi [11] introduced a quantity

$$\chi^2 \equiv \frac{N}{F_Q\left[\rho_{in}, J_{\vec{n}}\right]} < 1, \tag{3}$$

where $F_Q\left[\rho_{in}, J_{\vec{n}}\right]$ is the QFI for the input state ρ_{in} and $J_{\vec{n}}$ is the collective spin operator along $\vec{n}$ direction.

The inequality $\chi^2 < 1$ has a twofold capability. One is a sufficient condition for ρ_{in} being multiparticle entangled as proved in [11]. The other is a criterion for states which are useful for phase sensitivity beyond the shot-noise limit. For an interferometer, the relation between the output state and the input state is generally described as $\rho_{out} = e^{i\phi J_{\vec{n}}}\rho_{in}e^{-i\phi J_{\vec{n}}}$ with ϕ the relative phase shift between the two arms. According to QCR theory, the phase sensitivity $\Delta\phi$ has a lower bound limit

$$\Delta\phi_{QCR} = \frac{1}{\sqrt{mF_Q\left[\rho_{in}, J_{\vec{n}}\right]}} = \frac{\chi}{\sqrt{mN}}, \tag{4}$$

where m is the number of independent measurements. In this paper, we only consider the case of a single measurement, i.e., $m = 1$. If $\chi < 1$, $\Delta\phi_{CR} < \Delta\phi_{\mathrm{SN}}$, which provides a sub-shot-noise sensitivity. This capability is not possessed by other entanglement measures [12, 13, 14, 15, 16]. Anyhow, the quantity χ^2 provides a criterion for a state that is not only entangled but also useful for phase estimation.

In this work, we apply the quantity (3) to a class of pure states with exchange symmetry, and derive a general expression for χ^2 with arbitrary mean spin direction. It is shown that the entangled symmetric states can be used to enhance the phase sensitivity. To quantitatively examine the efficiency of these states for phase estimation, we explicitly investigate a series of superpositions of spin states. The states under consideration are all based on Dicke states [17], which are defined as

$$|n\rangle_N \equiv |\frac{N}{2}, -\frac{N}{2} + n\rangle, n = 0, \ldots, N, \tag{5}$$

where $|0\rangle_N \equiv |\frac{N}{2}, -\frac{N}{2}\rangle$ indicates a state for which all qubits are in the ground states, and n is the excitation number of qubits. Dicke states are elementary in atomic physics, and may be conditionally prepared in experiments by Quantum Non-Demolition technique [18, 19]. Moreover, quite recent experiments on the creation of photonic Dicke states are also demonstrated with parametric down conversion aimed at ultrasensitive interfermetry (see Ref. [20] and references therein). Therefore it is necessary and timely to quantitatively test the efficiency of the superpositions of Dicke states for the quantum precision measurements. We will show below that some choices of superposition states are possible reaching the Heisenberg limit.

This paper is organized as follows. In Sec. II, we show the general expression of χ^2 for symmetric multiqubit states with nonzero mean spin. In Sec. III, we quantitatively calculate the χ^2 for a class of superpositions of spin states, and find the special states making χ^2 minimum. Finally, a conclusion is given in Sec. IV.

2 QFI for symmetric multiqubit states

The expression of QFI in Eq. (3) is explicitly derived as [11]

$$F_Q\left[\rho_{in}, J_{\vec{n}}\right] = 2\sum_{i,j} \frac{\left(p_i - p_j\right)^2}{p_i + p_j} \left|\langle\psi_j|J_{\vec{n}}|\psi_i\rangle\right|^2, \quad (6)$$

in the basis of $\rho_{in} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ with p_i independent of the parameter ϕ. For a pure state $\rho_{in} = |\psi\rangle\langle\psi|$, the QFI is simplified as

$$F_Q\left[\rho_{in}, J_{\vec{n}}\right] = 4\left(\Delta J_{\vec{n}}\right)^2, \quad (7)$$

with $\left(\Delta J_{\vec{n}}\right)^2 \equiv \langle J_{\vec{n}}^2\rangle - \langle J_{\vec{n}}\rangle^2$ being the variance of the collective operator $J_{\vec{n}}$. Here $J_{\vec{n}} \equiv \vec{J}\cdot\vec{n} = \sum_{i=1}^{N} \sigma^{(i)}$ with $\sigma^{(i)}$ a Pauli operator on the ith particle. Its components along three axes obey the commutation relations as angular momentum

$$[J_z, J_\pm] = \pm J_\pm, [J_+, J_-] = 2J_z, \quad (8)$$

where $J_\pm = J_x \pm iJ_y$. It is noted that $\left(\Delta J_{\vec{n}}\right)^2 \le N^2/4$. Thus combining Eqs. (7) and (4), we find the Heisenberg limit that $\Delta\theta_{HL} = 1/N$, which is the ultimate limit of phase sensitivity. This is consistent with an recent proof using the fisher operator approach [21, 22], where it is pointed out that the upper limit to precision is provided only by the Semi-Norm of the Hamiltonian. In our work, the Hamiltonian reads $H = J_{\vec{n}}$, which results in a fundamental upper limit $\Delta\phi \ge \frac{1}{N}$.

In this work, we focus on the pure input states with exchange symmetry and choose $\vec{n}$ perpendicular to the mean spin direction. To make a minimum χ^2, we find the maximum variance $(\Delta J_{\vec{n}})^2$ in the plane perpendicular to the mean spin direction. In this case, $\chi^2 < 1$ means the input state ρ_{in} is not only entangled, but also the preferable state for sub-shot-noise sensitivity. In the following, we will prove that the entangled symmetric multiqubit states with nonzero mean spin are useful for sub-shot-noise phase sensitivity.

For a special case that the mean spin direction of the state is along z axis, i.e., $\langle J_x\rangle = \langle J_y\rangle = 0$, and $\langle J_z\rangle \ne 0$, we set its perpendicular direction is $\vec{n} = (\cos\omega, \sin\omega, 0)$. The corresponding collective operator reads

$$J_\omega = \vec{J}\cdot\vec{n} = \cos\omega J_x + \sin\omega J_y. \quad (9)$$

Since $\langle J_\omega\rangle = 0$, the maximum variance of J_ω in the range $\omega \in [0, 2\pi)$ is easily obtained as [23]

$$\max_\omega \left(\Delta J_\omega\right)^2 = \frac{1}{2}\left[\frac{N}{2}\left(\frac{N}{2}+1\right) - \langle J_z^2\rangle + \left|\langle J_+^2\rangle\right|\right]. \quad (10)$$

Combining Eqs. (10), (7) and (3), we get

$$\chi^2 = \frac{N}{4\max_\omega\left(\Delta J_\omega\right)^2} = \frac{1}{1+\frac{N}{2}-\frac{2}{N}\left(\langle J_z^2\rangle - \left|\langle J_+^2\rangle\right|\right)}, \quad (11)$$

which only depends on the expectation values of the collective operators J_z^2 and J_+^2. Interestingly, from the inequality $\langle J_z^2\rangle \le N^2/4$, we find

$$\chi^2 \le \frac{1}{1+\frac{2}{N}\left|\langle J_+^2\rangle\right|} \le 1, \quad (12)$$

where $\chi^2 = 1$ is obtained when $\langle J_z^2 \rangle = N^2/4$ and $\langle J_+^2 \rangle = 0$, which corresponds to the Dicke states $|0\rangle_N$ and $|N\rangle_N$ and their superpositions, i.e.,

$$|\psi\rangle = \cos\alpha|0\rangle_N + e^{i\beta}\sin\alpha|N\rangle_N, \tag{13}$$

where $\alpha \neq \pi/4 + k(\pi/2)$ with k arbitrary integer. Overall, an arbitrary symmetric state with mean spin along z direction, except for the Dicke states $|0\rangle_N$ and $|N\rangle_N$ and their superpositions, is not only entangled, but also useful for sub-shot-noise phase estimation.

It should be pointed out that, when $\alpha = \pi/4 + k(\pi/2)$, it corresponds a kind of states called GHZ states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_N + e^{i\beta}|N\rangle_N\right), \tag{14}$$

with zero mean spin, i.e., $\langle J_x \rangle = \langle J_y \rangle = \langle J_z \rangle = 0$. In this case, the expression shown in Eq. (11) is no longer valid. From Eq. (7), we can get the $\chi^2 = 1/N$ [24]. That is, the GHZ states produce a phase sensitivity at the Heisenberg limit.

For a general case that the mean spin of the state is along an arbitrary direction, we can redefine the mean spin direction as a direction $Z = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$,with $\Theta = \arccos\left[\langle J_z\rangle/\sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2}\right]$, and $\Phi = \arctan\left[\langle J_y\rangle/\langle J_x\rangle\right]$. Accordingly, the new X and Y direction are set to be $X = (-\sin\Phi, \cos\Phi, 0)$, $Y = (-\cos\Theta\cos\Phi, -\cos\Theta\sin\Phi, \sin\Theta)$. If we denote the new collective spin operator as $\mathcal{J}$, the corresponding operators are

$$\begin{aligned} \mathcal{J}_Z &= \sin\Theta\cos\Phi J_x + \sin\Theta\sin\Phi J_y + \cos\Theta J_z, \\ \mathcal{J}_X &= -\sin\Phi J_x + \cos\Phi J_y, \\ \mathcal{J}_Y &= -\cos\Theta\cos\Phi J_x - \cos\Theta\sin\Phi J_y + \sin\Theta J_z. \end{aligned} \tag{15}$$

It is easy to check that the three components also obey the commutation relationship as

$$[\mathcal{J}_Z, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, [\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_Z, \tag{16}$$

where $\mathcal{J}_\pm = \mathcal{J}_X \pm i\mathcal{J}_Y$. Moreover, the operator $\mathcal{J}^2$ is Casimir invariant, i.e.,

$$\mathcal{J}^2 = \mathcal{J}_X^2 + \mathcal{J}_Y^2 + \mathcal{J}_Z^2 = J_x^2 + J_y^2 + J_z^2 = J^2. \tag{17}$$

Since $\langle \mathcal{J}_X \rangle = \langle \mathcal{J}_Y \rangle = 0$, and $\langle \mathcal{J}_Z \rangle \neq 0$, we can redefine a new operator in the $X - Y$ plane

$$\mathcal{J}_\Omega = \cos\Omega\mathcal{J}_X + \sin\Omega\mathcal{J}_Y. \tag{18}$$

Then the maximum variance of $\mathcal{J}_\Omega$ is obtained similarly as

$$\max_\Omega (\Delta\mathcal{J}_\Omega)^2 = \frac{1}{2}\left[\frac{N}{2}\left(\frac{N}{2}+1\right) - \langle\mathcal{J}_Z^2\rangle + \left|\langle\mathcal{J}_+^2\rangle\right|\right]. \tag{19}$$

Finally, we get a generalized expression of Eq. (11) as

$$\chi^2 = \frac{N}{4\max_\Omega (\Delta\mathcal{J}_\Omega)^2} = \frac{1}{1 + \frac{N}{2} - \frac{2}{N}\left(\langle\mathcal{J}_Z^2\rangle - \left|\langle\mathcal{J}_+^2\rangle\right|\right)}. \tag{20}$$

From the inequality $\langle\mathcal{J}_Z^2\rangle \leq N^2/4$, we can still find

$$\chi^2 \leq \frac{1}{1 + \frac{2}{N}\left|\langle\mathcal{J}_+^2\rangle\right|} \leq 1, \tag{21}$$

where the equal sign is reached when $\langle \mathcal{J}_Z^2 \rangle = N^2/4$ and $\left|\langle \mathcal{J}_+^2 \rangle\right| = 0$, which also corresponds to the Dicke states $|0\rangle_N$ and $|N\rangle_N$ and their superpositions in the new coordinate frame, as shown in Eq. (13).

In conclusion, the quantity χ^2 implies that the entangled multiqubit symmetric states with nonzero mean spins are helpful for sub-shot-noise phase sensitivity. The expressions (11) and (20) are general, and will be used in the following to quantitatively study a series of superpositions of spin states. It is noted that, since the Dicke states (5) is symmetric, all the states below has exchange symmetry.

3 Quantum Fisher Information for superpositions of spin states

3.1 Superposition of Dicke states

First, we discuss the properties of χ^2 for the superposition of Dicke states such as

$$|\psi_D\rangle = \cos\theta|n\rangle_N + e^{i\varphi}\sin\theta|n+2\rangle_N, n = 0, \ldots, N-2, \tag{22}$$

with the angle $\theta \in [0, \pi)$ and the relative phase $\varphi \in [0, 2\pi)$. The definition of Dicke states is already given in Eq. (5). According to the operations of collective operators J_α $(\alpha = x, y, z)$ on the Dicke states

$$\begin{aligned} J_z|n\rangle_N &= \left(-\frac{N}{2}+n\right)|n\rangle_N, \\ J_+|n\rangle_N &= \sqrt{(n+1)(N-n)}|n+1\rangle_N, \\ J_-|n\rangle_N &= \sqrt{n(N-n+1)}|n-1\rangle_N, \end{aligned} \tag{23}$$

we can easily check that, for the superposition state (22), the mean spin direction is along z axis. Thus χ^2 of state $|\psi_D\rangle$ is given by Eq. (11) with

$$\begin{aligned} \langle J_z^2\rangle &= \left(-\frac{N}{2}+n\right)^2\cos^2\theta + \left(-\frac{N}{2}+n+2\right)^2\sin^2\theta, \\ \langle J_+^2\rangle &= \frac{1}{2}e^{i\varphi}\sin 2\theta\sqrt{\mu_n}, \end{aligned} \tag{24}$$

where $\mu_n = (n+1)(n+2)(N-n)(N-n-1)$. We see that χ^2 does not depend on the relative phase φ, and it is a function of the angle θ and the excitation number n. Since $|\psi_D\rangle$ has exchange symmetry, except for some states, it is always entangled and useful for sub-shot-noise sensitivity. However, the problem we are interested in is finding the special states making the χ^2 minimum over the whole parameter space $\theta \in [0, \pi)$ and $n \in [0, N-2]$.

For comparison, we first check χ^2 for a Dicke state $|n\rangle_N$ (see also Ref. [25]). When $\theta = 0$, we obtain

$$\chi^2_{|n\rangle_N} = \frac{1}{1+\frac{N}{2}-\frac{2}{N}\left(-\frac{N}{2}+n\right)^2}, \tag{25}$$

with $n \in [0, N]$. Obviously, except for $n = 0$ or N, $\chi^2_{|n\rangle_N}$ is always smaller than 1, i.e., for $n \in [1, N-1]$, a Dicke state is always entangled and useful for phase estimation. Moreover, for even N, $\chi^2_{|n\rangle_N}$ reaches its minimum over n at $n = N/2$, where the Dicke state reads

$$|n\rangle_N = |\frac{N}{2}\rangle_N, \tag{26}$$

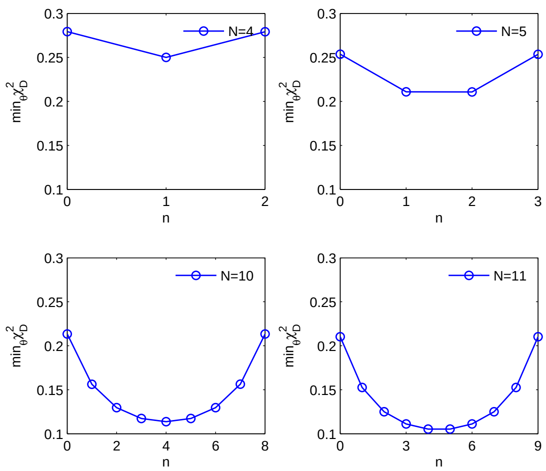

Fig. 1. The minimum value of χ^2 for the state $|D\rangle$ versus the excitation number n for different spin numbers N. The minimum is obtained at $\theta = \pm\frac{1}{2}\arctan\left[\mu_n/4\left(N/2-n-1\right)\right]$.

and χ^2 is

$$\min_n \chi^2_{|n\rangle_N} = \frac{1}{1+\frac{N}{2}}. \tag{27}$$

Whereas, for odd N, at $n = N/2 \pm 1/2$, i.e., for the Dicke state

$$|n\rangle_N = |\frac{N}{2} \pm \frac{1}{2}\rangle_N, \tag{28}$$

$\chi^2_{|n\rangle_N}$ attains its minimum

$$\min_n \chi^2_{|n\rangle_N} = \frac{1}{1+\frac{N}{2}-\frac{1}{2N}}. \tag{29}$$

Obviously, both Eqs. (27) and (29) result in a phase sensitivity limit $\Delta\phi_{QCR} \sim \sqrt{2}/N$ for sufficient large N, which reaches the Heisenberg limit $\Delta\phi_{HL} = 1/N$.

In fact, when $\theta = \pm\frac{1}{2}\arctan\left[\mu_n/4\left(N/2-n-1\right)\right]$, we can obtain the minimum value for χ^2 over θ as

$$\min_\theta \chi^2_{|\psi_D\rangle} = \frac{1}{1+\frac{N}{2}-\frac{2}{N}T}, \tag{30}$$

with

$$T = \frac{1}{2}\left[\left(-\frac{N}{2}+n\right)^2 + \left(-\frac{N}{2}+n+2\right)^2\right] - \sqrt{4\left(-\frac{N}{2}+n+1\right)^2 + \frac{1}{4}\mu_n}, \tag{31}$$

which is indeed the minimum value of $\langle J_z^2\rangle - \left|\langle J_+^2\rangle\right|$. The change of $\min_\theta \chi^2_{|\psi_D\rangle}$ along with the excitation number n for different particle numbers N is shown in Fig. 1. For even N, the minimum value of $\min_\theta \chi^2_{|\psi_D\rangle}$ over n is at $n = N/2-1$, where

$$|\psi_D\rangle = \frac{1}{\sqrt{2}}\left(|\frac{N}{2}-1\rangle_N \pm e^{i\varphi}|\frac{N}{2}+1\rangle_N\right), \tag{32}$$

and

$$\min_{\theta,n}\chi^2_{|\psi_D\rangle}=\frac{1}{-\frac{2}{N}+\frac{3N}{4}+\frac{3}{2}}. \tag{33}$$

One can check that, when $N=2$, $\min_{\theta,n}\chi^2_{|\psi_D\rangle}=\min_n\chi^2_{|n\rangle_N}$. Whereas, when $N\geq 4$, $\min_{\theta,n}\chi^2_{|\psi_D\rangle}<\min_n\chi^2_{|n\rangle_N}$. That is, the superposition state $|\psi_D\rangle$ is a little better than the Dicke state itself for the enhancement of phase estimation. And $\min_{\theta,n}\chi^2_{|\psi_D\rangle}$ becomes smaller and smaller as N increases. For odd N, the minimum value of $\min_\theta\chi^2_{|\psi_D\rangle}$ over n is at $n=N/2-3/2$ or $N/2-1/2$, where the superposition states are

$$\begin{aligned}|\psi_D^{(1)}\rangle &= \cos\theta'|\frac{N}{2}-\frac{3}{2}\rangle+e^{i\varphi}\sin\theta'|\frac{N}{2}+\frac{1}{2}\rangle,\\ |\psi_D^{(2)}\rangle &= \cos\theta'|\frac{N}{2}-\frac{1}{2}\rangle+e^{i\varphi}\sin\theta'|\frac{N}{2}+\frac{3}{2}\rangle,\end{aligned} \tag{34}$$

with $\theta'=\pm\frac{1}{2}\arctan\left[\mu_{\left(\frac{N}{2}-\frac{1}{2}\right)}/2\right]$, and the corresponding χ^2 is

$$\min_{\theta,n}\chi^2_{|\psi_D\rangle}=\frac{1}{1+\frac{N}{2}-\frac{5}{2N}+\sqrt{\frac{61}{16N^2}-\frac{1}{4N}+\frac{1}{8}+\frac{N}{4}+\frac{N^2}{16}}}. \tag{35}$$

When $N\geq 3$, the superposition state is better than the Dicke state itself for phase sensitivity. In the limit that $N\gg 1$, both Eqs. (33) and (35) approximately become $\min_{\theta,n}\chi^2_{|\psi_D\rangle}\sim 4/(3N)$. This leads to a phase sensitivity as $\Delta\phi_{QCR}\sim 2/\left(\sqrt{3}N\right)$, which achieves the Heisenberg limit and is better than the Dicke state case for phase estimation. This result is consistent with that of Ref. [26], where they have shown that Eqs. (33) and (35) almost saturate the Heisenberg limit with a single measurement and with a sensitivity higher than other states.

3.2 Even and odd spin coherent states

Following the superposition of Dicke states $|\psi_D\rangle$, we consider two more complex superpositions of Dicke states, i.e., the superpositions of the form $\sum_n d_n|n\rangle_N$ with the excitation numbers n even and odd respectively [27]. They can be written in terms of spin coherent states (SCSs) [28, 29] as

$$|\eta\rangle_\pm=\frac{1}{\sqrt{2\left(1\pm\gamma^N\right)}}\left(|\eta\rangle\pm|-\eta\rangle\right), \tag{36}$$

with $\gamma=\left(1-|\eta|^2\right)/\left(1+|\eta|^2\right)$. $|\eta\rangle_+$ and $|\eta\rangle_-$ are called even and odd SCSs respectively. A SCS is obtained only by a rotation of the Dicke state $|0\rangle_N$:

$$\begin{aligned}|\eta\rangle &= \left(1+|\eta|^2\right)^{-N/2}\exp\left(\eta J_+\right)|0\rangle_N\\ &= \left(1+|\eta|^2\right)^{-N/2}\sum_{n=0}^{N}\left(C_N^n\right)^{1/2}\eta^n|n\rangle_N,\end{aligned} \tag{37}$$

with η being complex and $C_N^n=\frac{N!}{n!(N-n)!}$. In the following, we restrict $|\eta|\in[0,1]$ due to the fact that the probability distribution $|{}_N\langle n|\eta\rangle|^2$ is a binomial distribution.

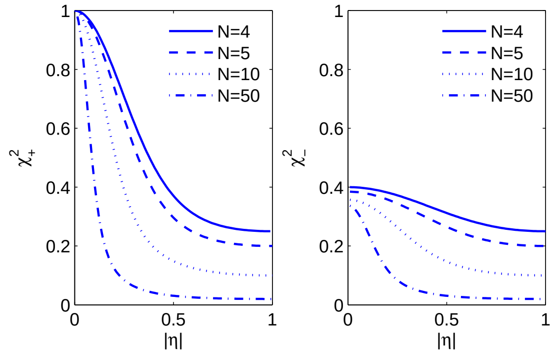

Fig. 2. χ^2 of even (left) and odd (right) SCSs versus the parameter $|\eta|$ for different spin numbers N.

Similar to the superposition of Dicke states $|\psi_D\rangle$, the mean spins of the even and odd SCSs are also along z direction, which can be seen from Eq. (45) as well. From the expectations for the even and odd SCSs

$$\begin{aligned} \langle J_z^2\rangle_\pm &= \frac{N^2}{4} \pm \frac{1}{(1 \pm \gamma^N)} N(N-1)|\eta|^2 \nu_\eta^\mp, \\ \langle J_+^2\rangle_\pm &= \pm \frac{1}{(1 \pm \gamma^N)} N(N-1)\eta^2 \nu_\eta^\pm, \end{aligned} \tag{38}$$

with $\nu_\eta^\pm = \gamma^N \left(1 - |\eta|^2\right)^{-2} \pm \left(1 + |\eta|^2\right)^{-2}$, we can easily take out the expressions of χ^2 as

$$\chi_\pm^2 = \frac{1}{1 + \frac{1}{1 \pm \gamma^N} 4(N-1)|\eta|^2 \left(1 + |\eta|^2\right)^{-2}}, \tag{39}$$

which only depends on the module of η. When $|\eta| \in (0, 1)$, we see that $\chi_\pm^2 < 1$ always holds. Fig. 2 displays $\chi_\pm^2$ versus $|\eta|$ for different system sizes N. It is seen that when $|\eta| \to 1$, $\chi_\pm^2$ approach their minimum values, which can be analytically derived from Eq. (39)

$$\min_\eta \chi_\pm^2 \to \frac{1}{N}. \tag{40}$$

That is, for even and odd SCSs, the phase sensitivity approaches the Heisenberg limit, i.e., $\Delta\phi_{QCR} \to 1/N$. In fact, in the limit $\eta = 1$, the two states $|\pm\eta\rangle$ are orthogonal with each other, and the even and odd SCSs are reduced to

$$|\eta = 1\rangle_\pm = \frac{1}{\sqrt{2}} \left((|N\rangle_N)_x \pm (|0\rangle_N)_x \right), \tag{41}$$

which are the GHZ states in the x direction. Eq. (41) is obtained from the relationship that $(|N\rangle_N)_x = 2^{-N/2} \sum_{n=0}^N (C_N^n)^{1/2} (|n\rangle_N)_z$ [29]. For a GHZ state, it can produce phase estimation at the Heisenberg limit as shown in Eq. (14). Furthermore, when it is performed by a unitary rotation, χ^2 keeps invariant. Thus for the states $|\eta = 1\rangle_\pm$, we also obtain

$\Delta\phi_{QCR} = 1/N$, which is a little better than the superposition state $|\psi_D\rangle$. Additionally, in the limit $\eta = 0$, the even SCS $|\eta = 0\rangle_+ = |0\rangle_N$ is a separable state, and

$$\chi_+^2|_{\eta=0} = 1. \tag{42}$$

However, for the odd SCS, we find

$$\chi_-^2|_{\eta=0} = \frac{1}{3 - \frac{2}{N}}, \tag{43}$$

which approaches 1/3 when $N \gg 1$, as shown in Fig. 2.

3.3 A general state

Finally, we consider a general superposition of the SCSs $|\pm\eta\rangle$ like

$$|\eta\rangle_g = C_\eta \left(\cos\theta|\eta\rangle + e^{i\varphi}\sin\theta|-\eta\rangle\right), \tag{44}$$

with the normalization coefficient $C_\eta = 1/\sqrt{1+\gamma^N \sin 2\theta \cos\varphi}$. The angle $\theta \in [0, \pi)$, and the phase $\varphi \in [0, 2\pi)$. Its mean spin $\langle\vec{J}\rangle$ is not always along z direction, since the expectations of the three components of $\vec{J}$ are

$$\begin{aligned}
\langle J_x\rangle &= N\eta C_\eta^2[\cos 2\theta \left(1+|\eta|^2\right)^{-1} - i\sin 2\theta \sin\varphi\gamma^N \left(1-|\eta|^2\right)^{-1}], \\
\langle J_y\rangle &= 0, \\
\langle J_z\rangle &= \frac{N}{2}C_\eta^2\left[\gamma + \sin 2\theta \cos\varphi\gamma^N\right].
\end{aligned} \tag{45}$$

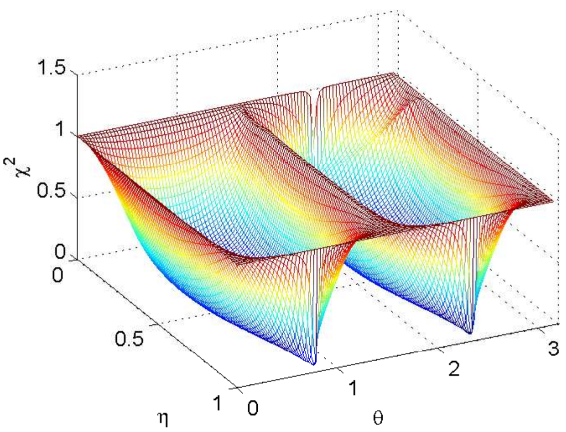

Fig. 3. χ^2 of the general superposition state $|\eta\rangle_g$ versus the parameter η and θ for the spin number $N = 50$. Here we assume η is real and $\varphi = 0$.

Thus we should appeal to Eq. (20) to calculate its χ^2. From Fig. 3, we see that when $\theta = 0$ or $\pi/2$, $\chi^2 = 1$, which means that a pure spin coherent state has no contribution to the enhancement of phase sensitivity. However, the superpositions of spin states are generally useful for phase estimation. Interestingly, it seems that χ^2 is always equal to one when $\eta = 1$, but when at $\theta = \pi/4$ and $3\pi/4$, χ^2 reaches its minimum, which indeed correspond to the even and odd SCSs as shown in Eq. (36). This means that the even and odd SCSs are the preferable superpositions of the SCSs $|\pm\eta\rangle$ in terms of phase estimation.

4 Conclusion

In terms of the quantity χ^2 based on QFI, we examine the entanglement properties and the efficiency for phase estimation of a class of pure states with exchange symmetry. For the states with the mean spin along z direction, we derive an expression of χ^2, and generalize it to the states with arbitrary mean spin direction Z. We find that the entangled symmetric states are helpful for sub-shot-noise phase sensitivity.

Using the general expressions of χ^2, we study a series of superpositions of spin states, and take out the special states for higher phase sensitivity. We find some of them almost saturate the phase sensitivity at the Heisenberg limit. For a single Dicke state, we find $|\frac{N}{2}\rangle_N$ (for even N) and $|\frac{N}{2} \pm \frac{1}{2}\rangle_N$ (for odd N) give the highest sensitivity $\Delta\phi_{QCR} \sim \sqrt{2}/N$ when the system size $N \gg 1$. For the superposition of Dicke states $|\psi_D\rangle$, we show that the best superpositions for phase estimation are $|\psi_D\rangle = \frac{1}{\sqrt{2}}\left(|\frac{N}{2}-1\rangle_N \pm e^{i\varphi}|\frac{N}{2}+1\rangle_N\right)$ (for even N) and $|\psi_D^{(1)}\rangle = \cos\theta'|\frac{N}{2}-\frac{3}{2}\rangle + e^{i\varphi}\sin\theta'|\frac{N}{2}+\frac{1}{2}\rangle, |\psi_D^{(2)}\rangle = \cos\theta'|\frac{N}{2}-\frac{1}{2}\rangle + e^{i\varphi}\sin\theta'|\frac{N}{2}+\frac{3}{2}\rangle$ (for odd N), which are able to achieve a phase sensitivity $\Delta\phi_{QCR} \sim 2/\left(\sqrt{3}N\right)$, slightly higher than the Dicke states themselves. For the superposition of SCSs, we find that, although the SCSs themselves are useless for sub-shot-noise phase estimation, their superpositions are generally useful for higher phase estimation. For even and odd SCSs, $|\eta\rangle_\pm = \frac{1}{\sqrt{2(1\pm\gamma^N)}}\left(|\eta\rangle \pm |-\eta\rangle\right)$, when $\eta \to 1$, the obtainable highest phase sensitivity is $\Delta\phi_{QCR} \sim 1/N$.

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