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INDIRECT QUANTUM CONTROL FOR FINITE-DIMENSIONAL COUPLED SYSTEMS

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We present a new analysis on the quantum control for a quantum system coupled to a quantum probe. This analysis is based on the coherent control for the quantum system and a hypothesis that the probe can be prepared in specified initial states. The results show that a quantum system can be manipulated by probe state-dependent coherent control. In this sense, the present analysis provides a new control scheme which combines the coherent control and state preparation technology.

Keywords: quantum information, quantum control

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1 Introduction

Controlling the time evolution [1, 2, 3, 4, 5, 6, 7, 8] of a quantum system is a major task required for quantum information processing. Several approaches to the control of a quantum system have been proposed in the past decade, which can be divided into coherent (unitary) and incoherent (non-unitary) control, according to how the controls enter the dynamics. In the coherent control scheme, the controls enter the dynamics through the system Hamiltonian. It affects the time evolution of the system state, but not its spectrum, i.e., the eigenvalues of the target density matrix ρ_f remain unchanged in the dynamics, due to the unitarity of the evolution. In the incoherent control scheme [9, 10, 11, 12], an auxiliary system, called probe, is introduced to manipulate the target system through their mutual interaction. This incoherent control scheme is of relevance whenever the system dynamics can not be directly accessed, and it provides a non-unitary evolution which is capable for transferring all initial states (pure or mixed) into an arbitrary pure or mixed state. This breaks the limitation for the coherent control mentioned above.

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To be specific, Romano and colleagues [9, 10] have investigated accessibility and controllability of a quantum bit (or a two-level system) coupled to a quantum probe, under the condition that the external control affects only the probe. This analysis is based on the Cartan decomposition [13, 14, 15] of the dynamics, and hence it is involved for high-dimensional systems. In Ref.[16] the authors propose a leaning control with a non-equilibrium environment. The results show that by tailoring the dissipative dynamics we can control the quantum system form a given state to a limited set of states (reachable states).

In this paper, we first examine the controllability for two-dimensional systems, then extend the approach to finite-dimensional quantum systems. This analysis is based on the quantum coherent control scheme and the hypothesis that the probe can be prepared in a specified initial state. The advantages of this scheme are threefold. Firstly, it overcomes the difficulty of the Cartan decomposition based analysis. Secondly, it brings a connection between the coherent control and incoherent control for quantum systems, hence it is easy to be generalized to finite-dimensional systems. Finally, this analysis provides a new control scheme, namely quantum states can be manipulated by probe state-dependent coherent control.

Throughout this paper, we describe the state of the controlled quantum system s by a density matrix ρ_s, a positive, unit trace operator on the Hilbert space $\mathcal{H}_s$ of the system. The convex set of all possible states is represented by $\mathcal{P}_s$ [9]. Its boundary $\partial\mathcal{P}_s$ is a set of pure states satisfying $\rho_s^2 = \rho_s$. By the definition [9], the system s is controllable if and only if for all pairs $(\rho_i, \rho_f) \in \mathcal{P}_s \times \mathcal{P}_s$, there exists a set of controls $\vec{g}$ such that $\rho_s(t=0) = \rho_i$ and $\rho_s(t, \vec{g} = \vec{g}_{\text{fixed}}) = \rho_f$ for some $t \geq 0$. Suppose that the quantum system s interacts with an initially unentangled probe p, whose density matrix is denoted by ρ_p on its Hilbert space $\mathcal{H}_p$. The time evolution of the composite system of the controlled system and the probe is governed by $H = H_s + H_p + H_I$, where H_s and H_p are the free Hamiltonian for the controlled system and the probe, respectively, while H_I denotes the coupling of the system and the probe. Since local transformations do not affect the controllability of the system [9], H_s and H_p can be ignored safely in the later analysis.

The paper is organized as follows. In Sec. II, we study the controllability of a two-dimensional system in the indirect control scheme, then we extend the approach to arbitrary finite-dimensional systems in Sec. III. Conclusions and discussions are presented in Sec. IV.

2 Two-dimensional systems

Consider a two-dimensional system s coupled to a probe p modeled as another two-dimensional system. The interaction Hamiltonian describing such a composite system takes the form

$$H_I(\vec{g}) = (g_1\sigma_s^z + g_2\sigma_s^+ + g_2^*\sigma_s^-) \otimes (g_3\sigma_p^x + g_4\sigma_p^z), \tag{1}$$

where $\vec{g} = (g_1, g_2, g_3, g_4)$ are coupling constants, $\sigma_s^{+,-,z}$ and $\sigma_p^{x,z}$ denote the Pauli matrices acting on Hilbert space $\mathcal{H}_s$ and $\mathcal{H}_p$, respectively.

We now show that the two-dimensional system is controllable governed by H_I. The complete controllability requires that one can steer the system s from *arbitrary* initial states to an arbitrary pure or mixed target state. This requirement on the initial states can be partly lifted by requiring that the interaction Hamiltonian H_I is unchanged up to $\vec{g}$ under the local unitary transformation

$$\begin{aligned} F(\theta,\phi) &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\phi}\sin\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}_s \otimes I_p \\ &\equiv f_s \otimes I_p, \end{aligned} \tag{2}$$

where $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$ and I_p is the identity operator on $\mathcal{H}_p$. By unchanged we mean $H_I(\vec{g'}) = FH_I(\vec{g})F^\dagger$, namely the transformation $F(\theta,\phi)$ changes the coupling constants (or the controls) in the Hamiltonian only. The proof is straightforward. Suppose the composite system is initially prepared on a state $\rho(0) = \rho_s(0) \otimes \rho_p(0)$. The final state of the quantum system s reads ($F = F(\theta,\phi)$)

$$\begin{aligned} \rho_s(t,\vec{g}) &= \mathrm{Tr}_p(u\rho(0)u^\dagger) \\ &= \mathrm{Tr}_p(uF^\dagger F\rho(0)F^\dagger F u^\dagger) \\ &= f_s^\dagger \mathrm{Tr}_p(u'\rho'(0)u'^\dagger) f_s, \end{aligned} \tag{3}$$

where the trace is taken over the probe, $u = e^{-iH_I(\vec{g})t}$, $u' = e^{-iH_I(\vec{g'})t}$, and $\rho'(0) = F\rho(0)F^\dagger$ represents a set of states which have the same spectrums as $\rho(0)$. Since the two sets $\{f_s\rho_s(t,\vec{g})f_s^\dagger\}$ and $\{\rho_s(t,\vec{g})\}$ are in one-to-one correspondence, thus the quantum system s is controllable initially in $\{\mathrm{Tr}_p\rho'(0)\}$ if and only if it is controllable initially in $\{\mathrm{Tr}_p\rho(0)\}$ [17]. As by proper choice of the unitary transformation f_s, the initial state can be always transformed to the form

$$\rho_s(0) = \rho_i = p_s(0)|0\rangle_s\langle 0| + (1 - p_s(0))|1\rangle_s\langle 1|, \tag{4}$$

(*where* $|0\rangle_s$ *and* $|1\rangle_s$ *denote the ground and excited states of the quantum system, respectively, and* $0 \le p_s(0) \le 1$) the system is completely controllable if we can steer the system to arbitrary target states from the initial state in the form (4).

We then show that the Hamiltonian H_I can drive the quantum system from the initial state $\rho_s(0)$ to an arbitrary final state. The time evolution operator for the composite system governed by H_I can be written as

$$U(t) = U_+^s(t)|+\rangle_p\langle +| + U_-^s(t)|-\rangle_p\langle -|, \tag{5}$$

where $|+\rangle_p = \cos\frac{\theta}{2}|1\rangle_p + \sin\frac{\theta}{2}|0\rangle_p$, $|-\rangle_p = \sin\frac{\theta}{2}|1\rangle_p - \cos\frac{\theta}{2}|0\rangle_p$, and $U_\pm^s(t)$ satisfy

$$i\hbar\frac{\partial}{\partial t}U_\pm^s(t) = H_\pm^s U_\pm^s(t), \tag{6}$$

with

$$\begin{aligned} H_\pm^s &= \pm\sqrt{g_3^2 + g_4^2}(g_1\sigma_s^z + g_2\sigma_s^+ + g_2^*\sigma_s^-), \\ \sin\theta &= \frac{g_3}{\sqrt{g_3^2 + g_4^2}}. \end{aligned} \tag{7}$$

We assume that the probe is initially prepared on the state

$$\rho_p(0) = p_p(0)|0\rangle_p\langle 0| + (1 - p_p(0))|1\rangle_p\langle 1|, \tag{8}$$

in terms of $|+\rangle_p$ and $|-\rangle_p$, $\rho_p(0)$ can be rewritten as,

$$\begin{aligned}\rho_p(0) &= \rho_p^{++}|+\rangle_p\langle+| + \rho_p^{--}|-\rangle_p\langle-| \\ &+ \rho_p^{+-}|+\rangle_p\langle-| + \rho_p^{-+}|-\rangle_p\langle+| \end{aligned} \tag{9}$$

where $0 \le p_p(0) \le 1$, and $\rho_p^{++} = \cos^2\frac{\theta}{2} - p_p\cos\theta$, $\rho_p^{--} = 1 - \rho_p^{++}$, $\rho_p^{+-} = \frac{1}{2}\sin\theta - p_p\sin\theta$, as well as $\rho_p^{-+} = \rho_p^{+-}$. We shall discuss how to prepare such a state for the probe at the end of this section. After some algebras, we obtain the density matrix for the quantum system at time t

$$\begin{aligned}\rho_s(t) &= p_s\rho_p^{++}U_+^s(t)|0\rangle_s\langle 0|U_+^{s\dagger}(t) \\ &+ p_s\rho_p^{--}U_-^s(t)|0\rangle_s\langle 0|U_-^{s\dagger}(t) \\ &+ (1-p_s)\rho_p^{++}U_+^s(t)|1\rangle_s\langle 1|U_+^{s\dagger}(t) \\ &+ (1-p_s)\rho_p^{--}U_-^s(t)|1\rangle_s\langle 1|U_-^{s\dagger}(t), \end{aligned} \tag{10}$$

define

$$\begin{aligned}&|\psi_{+0}\rangle = U_+^s|0\rangle_s, \quad |\psi_{-0}\rangle = U_-^s|0\rangle_s, \\ &|\psi_{-1}\rangle = U_-^s|1\rangle_s, \quad |\psi_{+0}^{\perp}\rangle = U_+^s|1\rangle_s, \end{aligned} \tag{11}$$

clearly $|\psi_{+0}\rangle$ and $|\psi_{+0}^{\perp}\rangle$ are normalized and orthogonal. Rewrite the density matrix for the quantum system in the basis spanned by $\{|\psi_{+0}\rangle, |\psi_{+0}^{\perp}\rangle\}$, we obtain

$$\rho_s(t) = \begin{pmatrix} \rho_s^{00}(t) & \rho_s^{01}(t) \\ \rho_s^{10}(t) & \rho_s^{11}(t) \end{pmatrix}, \tag{12}$$

where (setting $p_p(0) = p_p$, and $p_s(0) = p_s$)

$$\begin{aligned}\rho_s^{10}(t) &= (\rho_s^{01})^* = \frac{\rho_p^{--}}{2}\sin 2\alpha e^{-i\beta}(2p_s - 1), \\ \rho_s^{00}(t) &= p_s\rho_p^{++} + (1-p_s)\rho_p^{--}\sin^2\alpha + p_s\rho_p^{--}\cos^2\alpha, \\ \rho_s^{11}(t) &= (1-p_s)\rho_p^{++} + p_s\rho_p^{--}\sin\alpha^2 \\ &+ (1-p_s)\rho_p^{--}\cos^2\alpha. \end{aligned} \tag{13}$$

Here α and β is defined by,

$$|\psi_{-0}\rangle = \cos\alpha|\psi_{+0}\rangle + \sin\alpha e^{i\beta}|\psi_{+0}^{\perp}\rangle,$$

equivalently,

$$|\psi_{-1}\rangle = \sin\alpha e^{-i\beta}|\psi_{+0}\rangle + \cos\alpha|\psi_{+0}^{\perp}\rangle.$$

Namely,

$$\begin{aligned}\cos\alpha &= |\langle\psi_{+0}|\psi_{-0}\rangle|, \\ \beta &= \mathrm{Arg}\langle\psi_{+0}^{\perp}|\psi_{-0}\rangle - \mathrm{Arg}\langle\psi_{+0}|\psi_{-0}\rangle, \end{aligned} \tag{14}$$

where Arg denotes a function that extracts the angular component (sometimes called the phase angle) of a complex number.

For a given target state, we need to determine the control parameters $\vec{g}$ and initial state parameter p_p from α and β as well as the conditions that guarantee $|\psi_{+0}\rangle$ is an arbitrary pure state of the system. Dependence of $\vec{g}$ on α and β is so complicated that explicit expressions cannot be found for general cases. We will present an example to show how to calculate this dependence at the end of this paragraph. Two observations can be made from Eq.(11). (1) $|\psi_{+0}\rangle$ can be prepared for the quantum system to be an arbitrary pure state (unnormalized), this can be done by control g_1 and g_2 in $H^s_{\pm}$. The reason is that $|\psi_{+0}\rangle$ is a solution to the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\psi_{+0}\rangle = H^s_+|\psi_{+0}\rangle$ with the initial state $|0\rangle_s$. Since $i\sigma^z_s \in su(2)$ and $i\sigma^{\pm}_s \in su(2)$ are generators of the Lie algebra $SU(2)$, $|\psi_{+0}\rangle$ then can be manipulated to be any pure state in $\partial\mathcal{P}_s$ by varying g_1 and g_2; (2) α (or β) are controllable by g_3 and g_4 resulting in the change of the overlap between $|\psi_{-0}\rangle$ and $|\psi_{+0}\rangle$. As a result of these observations, we conclude that $\rho^{10}_s(t)$ can be manipulated to be zero for any p_s and p_p, leading to

$$\rho_s(t) = \rho^{00}_s(t)|\psi_{+0}\rangle\langle\psi_{+0}| + \rho^{11}_s(t)|\psi^{\perp}_{+0}\rangle\langle\psi^{\perp}_{+0}|. \tag{15}$$

Obviously, $(\rho^{00}_s + \rho^{11}_s) = 1$, and ρ^{00}_s can be arbitrarily controlled in the interval $[0, 1]$ by changing p_p. Therefore, the quantum system is controllable with H_I. To be specific, from the condition $\rho^{10}_s(t) = 0$ we can easily find that

$$\begin{aligned} \cos\theta &= \frac{1}{1-2p_p}, \\ \alpha &= n\pi, n = 0, \pm 1, \pm 2, ..., \end{aligned} \tag{16}$$

Suppose that the target state is $\rho^{00}_s(t) = q, (0 \le q \le 1)$, we find

$$p_p = \frac{q - p_s\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\sin^2\alpha - p_s\sin^2\frac{\theta}{2}\cos 2\alpha}{\cos\theta\sin^2\alpha + p_s\cos\theta\cos 2\alpha - p_s\cos\theta}. \tag{17}$$

Eqs.(16) and (17) together determine the parameters g_3, g_4 and p_p.

In comparison with earlier work [9, 10], it seems that our proposal needs more real parameters to control a two-level system. This is not true. In fact, $\rho^{10}_s(t) = 0$ is not necessary to prove the controllability. The reason is as follows. Diagonalizing the density matrix Eq.(12), we can write the state of the quantum system as

$$\rho_s(t) = E_+(t)|\psi_+(t)\rangle\langle\psi_+(t)| + E_-(t)|\psi_-(t)\rangle\langle\psi_-(t)|, \tag{18}$$

with the eigenvalues and the corresponding eigenvectors as follows

$$\begin{aligned} E_{\pm}(t) &= \frac{1}{4}\left[(\rho^{00}+\rho^{11}) \pm \sqrt{(\rho^{00}-\rho^{11})^2 + 4|\rho^{01}|^2}\right], \\ |\psi_+(t)\rangle &= \cos\frac{\Gamma}{2}|\psi_{+0}\rangle + \sin\frac{\Gamma}{2}e^{i\gamma}|\psi^{\perp}_{+0}\rangle, \end{aligned}$$

where $|\psi_-(t)\rangle$ is orthogonal to $|\psi_+(t)\rangle$, $\gamma = \text{Arg}(\rho^{01})$, and the mixing angle Γ are defined by $\cos\Gamma = (\rho^{00}-\rho^{11})\left[(\rho^{00}-\rho^{11})^2 + 4|\rho^{01}|^2\right]^{1/2}$. Note that $|\psi_+(t)\rangle$ (or $|\psi_-(t)\rangle$) can be controlled to be an arbitrary pure state regardless of Γ and γ, since $|\psi_{+0}\rangle$ is controllable (in the sense of pure states). Hence, two real parameters in H are enough to control $|\psi_+(t)\rangle$. This together with the parameter to control E_+ (from zero to 1/2) [18], only three real parameters are required to control the two-level system. Therefore, Hamiltonian Eq.(1) can be rewritten as

$$H_I(\vec{g}) = (g_1\sigma_s^z + g_2\sigma_s^x) \otimes (g_3\sigma_p^x + g_3\sigma_p^z), \tag{19}$$

where g_1, g_2, g_3 are real. This Hamiltonian can be realized in solid-state NMR experiment as follows[19]. Consider two coupled heteronuclear spins I and S under magic angle spinning. The spins are irradiated with rf fields at their Larmor frequencies along the x direction. In a doubly rotating Zeeman frame, rotating with both the spins at their Larmor frequency, the Hamiltonian of the system takes the form

$$\begin{aligned} H(t) &= \omega_I(t)I_z + \omega_S(t)S_z + \omega_{IS}(t)2I_zS_z \\ &+ \omega_{rf}^I(t)I_x + \omega_{rf}^S(t)S_x, \end{aligned} \tag{20}$$

where $\omega_I(t)$, $\omega_S(t)$, and $\omega_{IS}(t)$ represent time-dependent chemical shifts for the two spins I and S and the coupling between them, respectively. When the rf field strengths on the two spins is chosen to be integral (or half integral) multiples of spinning frequency, i.e., $\omega_{rf}^I = p\omega_r$ and $\omega_{rf}^S = q\omega_r$, the Hamiltonian for the dipole-dipole coupling in the interaction frame of the rf irradiation averages over a rotor period to [19]

$$\bar{H}_{IS} = (AZ^+ + BY^+) + (CZ^- + DY^-), \tag{21}$$

where A, B, C, D are parameters, $Z^\pm = I_zS_z \mp I_yS_y$ and $Y^\pm = I_yS_z \pm I_zS_y$. This is exactly the Hamiltonian in Eq.(19)(the role played by S_x and S_y are similar).

These discussions implies that a two-level system can be controlled by the Hamiltonian Eq.(1) together with the initial state preparation of the probe. The probe is not required to be prepared in an arbitrary initial state, rather it is a specified state determined according to the target state. In fact, this requirement on the probe can be removed by adding a control in the Hamiltonian (see Eq.(17)). This holds true for arbitrary finite dimensional systems we will discuss in the next section.

Before closing this section, we propose a possible way to prepare the probe on the required mixed state, $\rho_p(0) = p_p(0)|0\rangle_p\langle 0| + (1 - p_p(0))|1\rangle_p\langle 1|$. For this purpose, we place the probe in a thermal environment at temperature T. At equilibrium, the density matrix of the probe would take the above form with $p_p = e^{-\beta E_0}/(e^{-\beta E_0} + e^{-\beta E_1})$, where $\beta = 1/k_BT$, and $E_0(E_1)$ is the energy corresponding to state $|0\rangle$ ($|1\rangle$). By varying the energy spacing $|E_1 - E_0|$, we can in principle obtain the required p_p. Note that p_p is not required to vary between 0 and 1 [18], we conclude that the mixed state prepared in this way meets requirement in this proposal.

3 Finite-dimensional system

In order to extend the approach to finite-dimensional systems, we recall that the controllability in this case can be expressed in the following way. For any state ρ_s in $\mathcal{P}_s$, the quantum system is controllable if and only if there exists a Kraus map Φ_{ρ_s} such that $\Phi_{\rho_s}(\rho_s) = \rho_f$ for all state ρ_s in $\mathcal{P}_s$. In this section, we shall show that the following Hamiltonian

$$H_I = \left(\sum_{j\neq i=1}^{N} g_{ij}|i\rangle_s\langle j|\right) \otimes \left(\sum_{m\neq n}^{N} f_{mn}|m\rangle_p\langle n|\right) = h_s \otimes h_p, \tag{22}$$

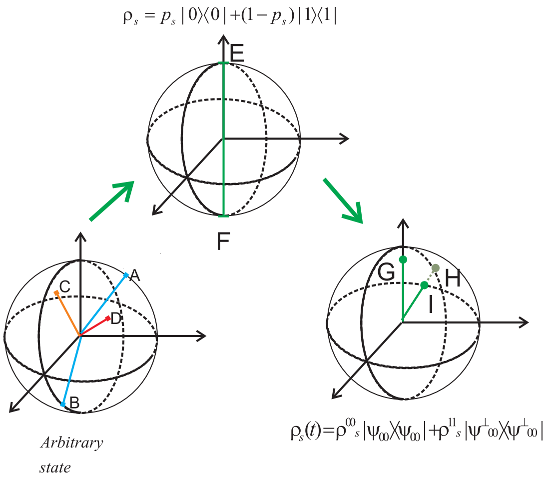

Fig. 1. (Color online) Schematic illustration of the proof for controllability. A, B, C and D in the left Bloch ball represent arbitrary pure and mixed states, in which A and B denote arbitrary pure states, while C and D denote arbitrary mixed states. The controllability requires that the quantum system can be controlled from A to B and C to D, from C to A, B and D as well as from A to C, B and D. We first showed that it is not necessary to require the quantum system to be controllable starting from an arbitrary initial state (i.e., an arbitrary point in the Bloch ball), provided the interacting Hamiltonian H_I is unchanged under $F(\theta, \phi)$, rather it is enough to control the system initially from a state on the green line in the middle of Bloch ball. The actual control can be done by first rotating the initial state G determined by the target state to H, and then changing the purity of the state G to reach the target state I.

which describes interaction between the system s and the probe p, would give rise to the Kraus map Φ_{ρ_s} required for the controllability. Here the probe p is modeled as a finite dimensional system with the same dimension as the system. Assume that the composite system is initially on an uncorrelated state $\rho(0) = \rho_s(0) \otimes \rho_p(0)$, and the target state of the quantum system reads

$$\rho_s(t) = Tr_p[\mathcal{U}(t)\rho_s(0) \otimes \rho_p(0)\mathcal{U}^\dagger(t)], \tag{23}$$

where $\mathcal{U}(t)$ denotes the time evolution operator of the whole system. Preparing initially the probe on state ($\sum_n P_n = 1$)

$$\begin{aligned} \rho_p(t=0) &= \sum_n P_n |n\rangle_p \langle n| \\ &= \sum_{M,N} p_{MN} |M\rangle_p \langle N|, \end{aligned} \tag{24}$$

we obtain

$$\rho_s(t) = \sum_M p_{MM} U_M(t)^s \rho_s(0) U_M^{s\dagger}(t). \tag{25}$$

Here, $|M\rangle_p$ is an eigenstate of h_p, i.e., $h_p|M\rangle_p = E_p|M\rangle_p$. It is easy to find that $U_M^s(t)$ satisfies

$$i\hbar \frac{\partial U_M^s(t)}{\partial t} = H_M^s U_M^s(t),$$

$$H_M^s = E_M \sum_{i \neq j=1}^{N} g_{ij} |i\rangle_s \langle j|. \tag{26}$$

The above results can be readily derived by writing $\mathcal{U}(t) = \sum_M U_M^s(t)|M\rangle_p\langle M|$, and $\rho_s(t) = \mathrm{T}r\mathcal{U}(t)\rho_p(0) \otimes \rho_s(0)\mathcal{U}^\dagger(t)$. With these results, the Kraus map Φ_{ρ_s} is

$$\Phi_{\rho_s}(\rho_s(0)) = \sum_M K_M(t)\rho_s(0)K_M^\dagger(t), \tag{27}$$

where $K_M(t) = \sqrt{p_{MM}}\, U_M^s(t)$. It has been proved that $U_M^s(t)$ can drive all initial pure states of the quantum system s into an arbitrary pure state at some time $t \geq 0$ [8, 7]. So $U_M^s(t)$ can be written as

$$U_M = |\phi_M(t)\rangle\langle\varphi_M(0)|, \tag{28}$$

where both $|\phi_M(t)\rangle$ and $|\varphi_M(0)\rangle$ are arbitrary and in $\partial\mathcal{P}_s$. This together with the hypothesis that the probe can be initially prepared in an arbitrary state, show controllability of the quantum system s. In contrast to the earlier study [20], in which the authors presented a constructive proof of complete kinematic state controllability for finite-dimensional open systems via the Kraus map, we here not only provide the details to realize such a Kraus map but also derive the equations for parameters to control the system from an arbitrary initial state to an arbitrary target state. It is worth pointing out that the realization of the Kraus map is not a trivial task, many works in literature are devoted to this task, in particular not all Kraus maps allowed in quantum mechanics can be used to control a quantum system. This problem was shown in Ref. [21] and has been realized in Ref. [16].

Now we present a general formulism to show that the indirect control can be represented as a combination of coherent control for the quantum system s and preparation of the initial state of the probe p. Consider an initial state of the composite system $\rho(0) = \rho_s(0) \otimes \rho_p(0)$, and a Hamiltonian $H = H_s \otimes H_p$ that governs the evolution of the composite system. Suppose that both the quantum system and the probe are N-dimensional, and the quantum system is coherently controllable driven by H_s. The latter assumption means H_s can drive the quantum system from an arbitrary pure state to any other pure state, i.e., $\{\,|\phi_i\rangle_s \equiv e^{-iH_st}|i\rangle_s \,|\, i = 1, 2, ..., N\}$ spans a basis for the quantum system, and $|\phi_i\rangle_s$ is an arbitrary pure state in $\partial\mathcal{P}_s$ for any i. The density matrix for the quantum system at time t reads

$$\rho_s(t) = \sum_m {}_p\langle m|e^{-i(H_s\otimes H_p)t}\rho_s(0) \otimes \rho_p(0)e^{i(H_s\otimes H_p)t}|m\rangle_p.$$

Choosing $H_p|m\rangle_p = E_m|m\rangle_p$, we obtain

$$\rho_s(t) = \sum_j p_j \sum_m {}_p\langle m|\rho_p(0)|m\rangle_p |\phi_j(E_mt)\rangle\langle\phi_j(E_mt)|, \tag{29}$$

where we have set the initial state of the system as $\rho_s(0) = \sum_j p_j|j\rangle_s\langle j|$, and used the relation $|\phi_j(E_mt)\rangle = e^{-iH_sE_mt}|j\rangle_s$. Suppose the target state is ($\sum_j q_j = 1$)

$$\rho_s(T) = \sum_m^N q_m|\phi_m(T)\rangle\langle\phi_m(T)|, \tag{30}$$

we find the condition to determine the controls in H_s and H_p

$$q_\alpha = \sum_j p_j \sum_m {}_p\langle m|\rho_p(0)|m\rangle_p |c_\alpha^{jm}|^2,$$
$$\sum_j p_j \sum_m {}_p\langle m|\rho_p(0)|m\rangle_p c_\beta^{jm} (c_\gamma^{jm})^* = 0,$$
$$\alpha, \beta, \gamma = 1, 2, ..., N, \tag{31}$$

where c_α^{jm} is defined by $|\phi_j(E_m t)\rangle = \sum_\alpha c_\alpha^{jm}|\phi_\alpha(T)\rangle$. These results suggest that the manipulation of a quantum system may be realized by coherent controls for the quantum system conditioned on the spectrum of the initial density matrix of the probe.

Before summarize our results, we present a discussion on the limitation of this scheme. This limitation resulting from the Hamiltonian in the form $H_I = h_s \otimes h_p$ is that the completely mixed states of the system remains completely mixed. We take the two-dimensional system as an example to show this drawback. Consider an initial state of the whole system

$$\rho(t=0) = \frac{1}{2} I \otimes \rho_p,$$

where I is a identity operator. Following the same procedure in Sec.2, we obtain the density matrix of the system at time t, $\rho_s(t) = 1/2I$, i.e., the completely mixed state remains completely mixed. Nevertheless, the presented results could be useful in the less demanding objective of purification[22], which is already explored in the NMR experiment.

4 conclusion

In this paper we have presented a scheme for indirect control of a quantum system coupled to a probe. This control scheme is actually a combination of coherent control conditioned on the initial state of the probe. For a two-level system, we have proved that the restriction on the initial state for the quantum system can be partly removed. This simplifies the formulation for the controllability. The scheme has been generalized to arbitrary finite dimensional systems and equations to determine the controls are given. We would like to note that the probe is not required to be initially prepared in an arbitrary state but with a specified spectrum. This requirement can be lifted by adding more controls in the composite systems.

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17. This also can be understood as follows. Consider the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle = H_I|\varphi(t)\rangle$, where $|\varphi(t)\rangle$ is the wavefunction of the composite system. By the time-independent transformation $F(\theta,\phi)$, $|\varphi(t)\rangle \rightarrow F^\dagger|\phi'(t)\rangle$ we find $i\hbar\frac{\partial}{\partial t}(F^\dagger|\phi'(t)\rangle) = H_I(F^\dagger|\phi'(t)\rangle)$, and $i\hbar\frac{\partial}{\partial t}|\phi'(t)\rangle = FH_IF^\dagger|\phi'(t)\rangle$. Since $H_I(\vec{g}') = FH_I(\vec{g})F^\dagger$, we claim that there exists a one-to-one correspondence in sets $\{Tr_p(|\phi'(t)\rangle\langle\phi'(t)|)\}$ and $\{Tr_p(|\phi(t)\rangle\langle\phi(t)|\}$. Therefore, if $\{Tr_p(|\phi'(t)\rangle\langle\phi'(t)|)\}$ covers all states in $\mathcal{P}_s$, $\{\mathrm{Tr}_p(|\phi(t)\rangle\langle\phi(t)|\}$ is a convex set of all possible states for the two dimensional system.
18. We can obtain the state with $E_+ \in (1/2, 2]$ by controlling $|\psi_+(t)\rangle$, because of $\langle\psi_+(t)|\psi_-(t)\rangle = 0$.
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