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NEW CLASSES OF TOPOLOGICAL QUANTUM CODES ASSOCIATED WITH SELF-DUAL, QUASI SELF-DUAL AND DENSER TESSELLATIONS*

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In this paper we present six classes of topological quantum codes (TQC) on compact surfaces with genus $g \geq 2$. These codes are derived from self-dual, quasi self-dual and denser tessellations associated with embeddings of self-dual complete graphs and complete bipartite graphs on the corresponding compact surfaces. The majority of the new classes has the self-dual tessellations as their algebraic and geometric supporting mathematical structures. Every code achieves minimum distance 3 and its encoding rate is such that $\frac{k}{n} \to 1$ as $n \to \infty$, except for the one case where $\frac{k}{n} \to \frac{1}{3}$ as $n \to \infty$.

Keywords: Topological quantum codes, Quantum error-correcting codes, Surface codes, Tessellations.
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1. General

Topological quantum computation is an alternative model of quantum computation with the advantage of being naturally fault-tolerant due to the topological properties of the physical system.

The topological quantum codes (TQC) introduced by Kitaev, [1], are a subclass of the stabilizer codes, which in turn form a class of quantum error correcting codes whose construction is based on the structure of linear codes. In Kitaev's construction, a qubit is associated, in a one-to-one correspondence, with each edge of a tessellation of a compact surface (in this case, a torus), whereas the stabilizer operators, defining the code, are associated with each vertex

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and each face of the tessellation. The latter are local operators, constituting a Hamiltonian with local interactions, whose ground state coincides with the protected space of the code. The operations described by the Hamiltonian control an intrinsic mechanism of protection of the encoded quantum states. One advantage of these codes is related to the locality property of its operators which may facilitate the physical implementation of these systems.

In [2] Kitaev's construction is generalized by considering compact surfaces with genus $g \geq 2$. The geometry associated with such surfaces is the hyperbolic geometry, and consequently, the corresponding tessellations are hyperbolic, and for their determination it is necessary to consider the polygons that generate such surfaces and the corresponding tilling. The remaining steps of the construction is similar to the Kitaev model, taking into consideration the associated geometry, for more details we refer the reader to [2].

The aim of this paper is to present classes of topological quantum codes from the construction proposed in [2] derived from self-dual ($\{p, p\}$), quasi self-dual ($\{q-1, q\}$) and denser tessellations ($\{p, 3\}$). Such codes are associated with embeddings of self-dual complete graphs and complete bipartite graphs on orientable compact surfaces. The new codes can correct an arbitrary quantum error and their encoding rate is such that $\frac{k}{n} \rightarrow 1$ as $n \rightarrow \infty$, except for the one class where $\frac{k}{n} \rightarrow \frac{1}{3}$ as $n \rightarrow \infty$.

This paper is organized as follows. In Section II, we review the relevant aspects of the TQC codes. In Section III, a brief review on embedding of graphs is presented. In Section IV, classes of codes derived from self-dual tessellations are presented. Two of these classes of TQC with distance 3 are associated with self-dual embeddings of complete graphs and the remaining classes are associated with embeddings of complete bipartite graphs. In Section V, another class achieving the same distance as in the previous cases, however derived from quasi self-dual tessellations is considered, where this class is associated with embeddings of complete bipartite graph. In Section VI a class of TQC derived from denser tessellations are considered. Finally, in Section VII the conclusions are drawn.

2. Topological Quantum Codes

A quantum error-correcting code (QEC) is a mapping from the 2^k-dimensional Hilbert space, $\mathcal{H}^k$, to the 2^n-dimensional Hilbert space, $\mathcal{H}^n$, where $k < n$. A QEC code $\mathcal{C}$ with codeword length n, dimension k, and minimum distance d is denoted by $[[n, k, d]]$. Such a code is able to correct t arbitrary quantum errors which may occur in the qubits of a codeword, where $t = \lfloor \frac{d-1}{2} \rfloor$.

A stabilizer code $\mathcal{C}$ is the simultaneous eigenspace, with eigenvalue $+1$, of all the elements of an Abelian subgroup $\mathcal{S}$ of the Pauli group P_n, called *stabilizer group*. The elements of $\mathcal{S}$ are called *stabilizer operators*. The Pauli group P_n consists of all the n tensor products of the elements of the set $P_1 = \pm\{I, \sigma_x, \sigma_y, \sigma_z\}$, where

$$I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, the stabilizer code $\mathcal{C}$ is defined as $\mathcal{C} = \{|\psi\rangle : M|\psi\rangle = |\psi\rangle, \ \forall \ M \in \mathcal{S}\}$, [3].

Kitaev's toric codes are defined on the tessellation $\{4, 4\}$, Fig. 1, with parameters $[[2l^2, 2, l]]$, where the codeword length is defined as the number of edges of the $l \times l$ square grid, that is, $n = 2l^2$; the number of encoded qubits depends on the genus of the surface and it is given by $k = 2g$ (in the particular case of the torus, $g = 1$); and the code distance is given by the minimum number of edges contained either in the shortest homologically nontrivial cycle of the $l \times l$ square grid or in the shortest homologically nontrivial cycle of the dual of the $l \times l$ square grid. Since the tessellation $\{4, 4\}$ is self-dual, and that the homologically nontrivial cycle is an edge path in the lattice that can not be shrunk to a face, it follows that the shortest homologically nontrivial cycle corresponds exactly to either the orthogonal axis of the square grid or the orthogonal axis of its dual square grid. Consequently, $d = l$.

The stabilizer operators of this class of codes are associated with each vertex and each face of the $l \times l$ square grid, Fig. 1. Given a vertex $v \in V$, the vertex operator A_v is defined as the tensor product of σ_x corresponding to each one of the edges having v as the the common vertex and the identity operator acting on the remaining qubits. Analogously, given a face $f \in F$, the face operator B_f is defined as the tensor product σ_z corresponding to each one of the four edges forming the border of the face f and the identity operator acting on the remaining qubits. Equivalently,

$$A_v = \bigotimes_{j \in E} \sigma_x^{\delta(j \in E_v)} \qquad B_f = \bigotimes_{j \in E} \sigma_z^{\delta(j \in E_f)},$$

where δ is the Kronecker delta.

The toric code $\mathcal{C}$ consists of the space fixed by the A_v and B_f operators, or equivalently, $\mathcal{C} = \{|\psi\rangle : A_v|\psi\rangle = |\psi\rangle,\ B_f|\psi\rangle = |\psi\rangle \ \ \forall\ v, f\}$. The dimension of $\mathcal{C}$ is 4, that is, $\mathcal{C}$ encodes $k = 2$ qubits.

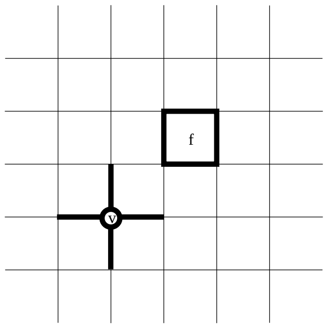

Fig. 1. Square lattice of the torus.

In general, we have

Definition 1 *Let* $\mathbf{M}$ *be a compact surface and* $\{p, q\}$ *a tessellation (see Section IV) of* $\mathbf{M}$ *with* E *edges,* V *vertices and* F *faces. Given a vertex* $v \in V$ *and a face* $f \in F$*, we define the operators* A_v *as the tensor product of* σ_x *corresponding to each edge having* v *as the common vertex and the operators* B_f *as the tensor product of* σ_z *corresponding to each edge forming the border of the face* f*. A topological quantum code* $\mathcal{C}$ *with length* $n = |E|$*, with stabilizer* $\mathcal{S} = \{A_v|\ v \in V\} \cup \{B_f|\ f \in F\}$*, encodes* $k = 2g$ *qubits (if the surface has no border) and its distance is* $d = min\{\delta, \delta^*\}$*, where* δ *denotes the code distance in the tessellation* $\{p, q\}$*, whereas* δ^* *denotes the code distance in the dual tessellation* $\{q, p\}$*.*

As previously mentioned, in [2] a generalization of Kitaev's codes is presented for orientable compact surfaces with genus $g \geq 2$. In this construction the geometry to be considered is the hyperbolic geometry. For more information on hyperbolic geometry we refer the reader to [4, 5, 6, 7].

This construction consists in selecting a regular hyperbolic polygon (plane model P' of the surface) and its possible tilling $\{p, q\}$. Next, we briefly review such a construction.

Remember that a *hyperbolic polygon* P' *with* p' *edges*, or a p'-gon, is a convex closed set consisting of p' hyperbolic geodesic segments. The intersection of two adjacent geodesics is called *vertex* of the polygon. A p'-gon whose edges have the same length and the internal angles are equal, is called a *regular* p'*-gon.* Furthermore, a *regular tessellation* of the Euclidean or hyperbolic plane, is a covering of the whole plane by regular polygons, all with the same number of edges, without superposition of such polygons, meeting completely only on edges or vertices. We denote a regular tessellation by $\{p, q\}$, where q regular polygons with p edges meet in each vertex. In particular, if $p = q$ the tessellation is said to be *self-dual*. Note that the notation P' to regular hyperbolic polygon (plane model of the surface), means the polygon is associated with the fundamental region of the tessellation $\{p', q'\}$, that is, P' is a polygon with p' edges where q' polygons with p' edges meet in each vertex.

Any compact surface can be realized geometrically. In particular, all compact surface with genus $g \geq 2$ can be realized geometrically as hyperbolic surfaces.

A compact topological surface $\mathbf{M}$ may be obtained from a polygon P' by pairwise edge-identifications. An oriented edge-pairing transformation of a hyperbolic polygon P', with equal length edges, is an isometry $\gamma \neq Id$ of an orientation preserving isometry group Γ, taking an edge s of P' to another edge $\gamma(s) = s'$ of P'. Furthermore, $\gamma^{-1} \in \Gamma \setminus \{Id\}$ takes $\gamma(s) = s'$ to s. Thus, we say that the edges s and s' are paired. If s is identified with s', and s' is identified with s'', then s is identified with s''. Such a chain of identifications may also occur with vertices, and so we call a maximal set $\{v_1, v_2, \ldots v_k\}$ of identified vertices a *vertex cycle*.

An edge-pairing of P' defines an *identification space* $S_{P'}$ making it a hyperbolic surface if the angles of each vertex cycle adds up to 2π. $S_{P'}$ in turn can be identified with a complete and connected hyperbolic surface $\mathbf{H}^2/\Gamma$, where Γ is a Fuchsian group, since P' is compact, [6].

Now, $PSL(2, \mathbf{R})$ is the multiplicative group of Möbius transformations $T : \mathbf{C} \to \mathbf{C}$ defined by $T(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbf{R}$ such that $ad - bc = 1$, and a Fuchsian group Γ is a discrete subgroup of $PSL(2, \mathbf{R})$. In this case, Γ is an orientation preserving isometry group whose elements are edge-pairing transformations γ.

On the other hand, a compact hyperbolic surface $\mathbf{M} \equiv \mathbf{H}^2/\Gamma$ is the identification space of a polygon P' if P' is the fundamental region for Γ, that is, a closed subset of a metric space X which Γ acts, with non-empty interior, such that $\bigcup_{\gamma \in \Gamma} \gamma(P') = X$ and

$$int(P') \bigcap \gamma(int(P')) = \emptyset, \quad \forall\ \gamma \in \Gamma - \{Id\}.$$

This holds if the following conditions are satisfied:
Edge and Angle conditions [6]: *If a compact polygon P' is the fundamental region for an orientation preserving isometry group Γ of $\mathbf{S^2}$ (sphere surface), $\mathbf{R^2}$ (Euclidean plane), or $\mathbf{H^2}$ (hyperbolic plane), then*

(i) For each edge s of P' there exists a unique edge s' of P' such that $s' = \gamma(s)$, for $\gamma \in \Gamma$;

(ii) Given edge-pairings of P', for each set of the identified vertices, the sum of the angles has to be equal to 2π. This set is a vertex cycle.

Theorem 1 *(Poincaré), [6] A compact polygon P' satisfying the edge and angle conditions is a fundamental region for the group Γ generated by the edge-pairing transformations of P', and Γ is a Fuchsian group.*

The procedure proposed in [2] takes into consideration polygons P' of the type $4g$-gon (fundamental region of the self-dual tessellation $\{4g, 4g\}$) as the plane models of the corresponding surfaces. In these polygons the edge-pairing transformations are defined by, $\gamma : S \to S; \gamma(s_i) = s_{i+2g}$, where $S = \{s_1, \dots, s_{4g}\}$ is the set of edges of P', $i = 1, 2, \dots, 4g$, and the sum of the subscripts of s is realized modulo $4g$. Such isometry γ realizes the pairings of opposite edges of P', see Fig. 2. The selection of these edge-pairing transformations leads to a code distance having the greatest hyperbolic distance between the identified edges of P'. Since $p' = q' = 4g$, the unique cycle of vertices obtained from these edge-pairing transformations has the sum of the internal angles equal to $(p'/q')(2\pi) = 2\pi$, and so satisfying the necessary and sufficient conditions for P' to be a fundamental region of the group of these edge-pairing transformations Γ.

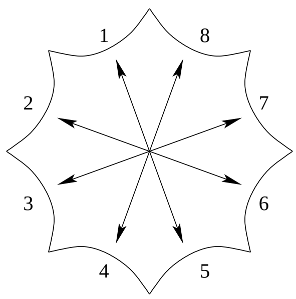

Fig. 2. Edge-pairing transformation $\gamma(s_i) = s_{i+2g}$ in 8-gon ($g = 2$).

We call the attention to the fact that, for each fixed value of g, polygons with a different number of edges from $4g$ (associated with the tessellation $\{4g, 4g\}$), for example $4g + 2$ (associated with the tessellation $\{4g + 2, 2g + 1\}$), $8g - 4$ (associated with the tessellation $\{8g - 4, 4\}$), and $12g - 6$ ($\{12g - 6, 3\}$), among others, generate surfaces with the same genus. Theoretically, any polygon which generates a compact surface can be employed in the construction of such codes. Nevertheless, one of the reasons in selecting the model $\{4g, 4g\}$, is that all the pairings are from opposite edges, and so achieving the greatest minimum distance of the code.

Every possible tiling $\{p, q\}$ of the polygon P' satisfies the following equation:

$$\mu(P') = n_f \mu(P), \tag{1}$$

in addition to the following constraint $(p - 2)(q - 2) > 4$. In (1) $\mu(P')$ denotes the area of the polygon P', $\mu(P)$ denotes the area of the polygon with p edges associated with the tiling $\{p, q\}$, and n_f is a positive integer which denotes the number of faces of the tessellation $\{p, q\}$. Note that, given a tessellation $\{p, q\}$, the dual tessellation $\{q, p\}$ has to satisfy the same previous conditions.

The tessellations obtained as the solutions of equation (1) are in fact all the possible tessellations of P' because they satisfy the following theorem.

Theorem 2 *[8] Let $\mathbf{M}$ be a closed surface and let p, q, V, E, F be positive integers such that*

$$V - E + F = \chi(\mathbf{M}), \tag{2}$$

$$pF = 2E = qV. \tag{3}$$

Then the following hold:

- *(Existence):There exist a $\{p,q\}$-pattern on $\mathbf{M}$ consisting of F p-sided faces, E edges and V vertices each of valence q; except when $\mathbf{M}$ is the projective plane, $\{p,q\} = \{3,3\}$, $V = F = 2$, and $E = 3$;*
- *(Geometrization): A $\{p,q\}$-pattern on $\mathbf{M}$ can be made geometric;*
- *(Classification): A $\{p,q\}$-pattern on the sphere or projective plane is unique. For all other closed surfaces $\mathbf{M}$ the $\{p,q\}$-patterns on $\mathbf{M}$ are classified by conjugate classes of subgroups isomorphic to the fundamental group of $\mathbf{M}$ in the extended $(p,q,2)$-triangle groups of Schwarz.*

The area of a hyperbolic polygon is given by, [5, 6],

$$\mu(P') = 4\pi(g-1), \tag{4}$$

where g is the genus of the surface. Moreover, the Gauss-Bonnet Theorem, [5, 6], shows that the hyperbolic area of a hyperbolic triangle Δ depends only on its angles, α, β, γ,

$$\mu(\Delta) = \pi - \alpha - \beta - \gamma.$$

Given a tessellation, by triangulation of its fundamental region the internal angles of such a triangle are $\frac{2\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{q}$. Thus, by using the Gauss-Bonnet Theorem we have

$$\mu(P) = p[\pi - \frac{2\pi}{p} - 2\frac{\pi}{q}] = (p-2)\pi - \frac{2p\pi}{q}.$$

Thus equation (1) may be rewritten as:

$$4\pi(g-1) \quad = \quad n_f\left[(p-2)\pi - \frac{2p\pi}{q}\right]. \tag{5}$$

Hence, the number of faces, n_f, associated with the tiling $\{p,q\}$ of P' is given by

$$n_f = \frac{4q(g-1)}{pq - 2p - 2q}. \tag{6}$$

Note that the tessellation $\{p,q\}$ tiles the polygon P' for those values of p and q such that (6) is a positive integer.

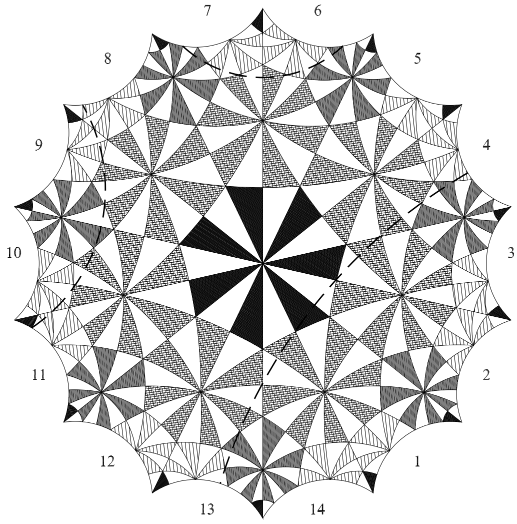

Fig. 3. Klein group - a 14-gon tiled by the tessellation $\{7, 3\}$.

Example 1 *The Klein group [6], is a surface of genus 3, obtained by the edge-pairing transformations of a 14-gon (more specifically $P' = \{p', q'\} = \{14, 7\}$) in the hyperbolic plane where the edges are paired by the relation $s_{2i+1} \mapsto s_{2i+6}$, and the sum of the subscripts of s is realized modulo 14, Fig. 3. Note that, since this 14-gon satisfies the edge and angle conditions, it is a fundamental region for Γ, where Γ is the group consisting of the edge-pairing transformations.*

In the Klein group, we can see that the 14-gon is tiled by a set of 24 identical regular heptagons. Note that the condition $(p-2)(q-2) > 4$ is satisfied by the tessellations $\{7, 3\}$ and its dual $\{3, 7\}$. These two tessellations are dual to each other, in the sense that the vertices of one tessellation correspond to the faces of the other. That is, the 14-gon may be tiled by the tessellation $\{7, 3\}$ and by its dual tessellation $\{3, 7\}$. Observe that the area of the 14-gon is equal to the area of the 24 heptagons or the area of the 56 equilateral triangles.

In fact,

$$\mu(P') = 4\pi(g-1) = 8\pi,$$

and, considering the tessellation $\{7, 3\}$, we have

$$\mu(P) = (p-2)\pi - \frac{2p\pi}{q} = \frac{(pq - 2q - 2p)\pi}{q} = \frac{\pi}{3}.$$

Thus,

$$n_f = \frac{8\pi}{\frac{\pi}{3}} = 24.$$

Considering the tessellation $\{3, 7\}$, we arrive at $\mu(P') = 4\pi(g-1) = \pi/7$, and $n_f = 56$.

Note that the number of vertices n_v of the tessellation $\{7, 3\}$ is equal to the number of faces of the tessellation $\{3, 7\}$, and vice-versa.

2.1. *Operators and parameters*

As in Kitaev's construction, given a vertex v of the tessellation, the vertex operator acts non-trivially on the q qubits having v as the common vertex and the identity operator acts on the remaining qubits, that is, $A_v = \bigotimes_{j \in E_v} \sigma_x^j$, where E_v denotes the set of edges having v as the common vertex. Similarly, given a face f of the tessellation, the face operator acts non-trivially on the p qubits forming the border of this face, and the identity operator acts on the remaining qubits of the tessellation, that is, $B_f = \bigotimes_{j \in E_f} \sigma_z^j$, where E_f denotes the set of edges forming the border of f. Therefore, the code is given by $\mathcal{C} = \{|\psi\rangle : A_v|\psi\rangle = |\psi\rangle,\ B_f|\psi\rangle = |\psi\rangle\ \ \forall\ v, f\}$. The operators A_v and B_f are the stabilizer operators of this code.

We have that n_f is the number of faces of the tessellation $\{p,q\}$ tiling P', since each edge of this tessellation belongs simultaneously to two faces, then the length of the code is $n = n_f p/2$ edges, or qubits.

The number of encoded qubits is $k = n - n_f - n_v + 2$, where $n_v = n_f p/q$. From (5), it can be shown that $k = 2g$, and so the dimension of the code $\mathcal{C}$ is $2^{2g} = 4^g$.

For TQC on surfaces of genus $g \geq 2$, the procedure used to obtain the code minimum distance is similar to the one used when considering a toric code. We are looking for the shortest homologically nontrivial cycle either on the tessellation or on the dual tessellation.

We call the attention to the fact that the shortest homologically nontrivial cycle in a p'-gon is given by the geodesics of least length that connect the edge-pairing of P'. In terms of the edges of the tessellation of P', the shortest homologically nontrivial cycle is an edge path that is closest to the geodesic with shortest length. Thus, the code distance is the minimum number of edges between the nearest path of the shortest homologically nontrivial cycle of the tessellation and the nearest path to the shortest homologically nontrivial cycle of the dual tessellation.

Thus, the minimum distance of these codes, d_{TQC}, is the lower bound on the ratio $\frac{d_h}{l(p,q)}$, where d_h is the distance between the edge-pairings of P', and $l(p,q)$ is the edge-length of the tessellation $\{p,q\}$. The distance d_h is the hyperbolic length of the orthogonal geodesic common to two opposite edges, and it is given by, [2],

$$d_h = 2a = 2\ \text{arccosh}\left[\frac{\cos(\pi/4g)}{\sin(\pi/4g)}\right], \tag{7}$$

and the edge-length of the tessellation $\{p,q\}$ is given by, [2],

$$l(p,q) \quad = \quad \text{arccosh}\left[\frac{\cos^2(\pi/q) + \cos(2\pi/p)}{\sin^2(\pi/q)}\right]. \tag{8}$$

Thus,

$$d_{TQC} > \frac{2\ \text{arccosh}\left[\frac{\cos(\frac{\pi}{4g})}{\sin(\frac{\pi}{4g})}\right]}{\text{arccosh}\left[\frac{\cos^2(\frac{\pi}{q})+\cos(\frac{2\pi}{p})}{\sin^2(\frac{\pi}{q})}\right]}, \tag{9}$$

3. Review of Embedding of Graphs

A finite graph G consists of a finite set V_G of vertices, a finite set E_G of edges and an incidence function I_G that associates to each edge e an endpoint set $V_G(e)$ containing either one or two elements of the vertex set V_G. A graph is called *simplicial* if it has no self-loops or multiple edges. A simplicial graph is called *complete* if every pair of vertices is adjacent, that is, they are the endpoints of the same edge. A complete graph on s vertices is denoted by K_s.

A *bipartite graph* is a graph where its vertex set can be partitioned into two subsets U and W such that the vertices in each one of theses subsets are mutually nonadjacent. If every vertex of U is adjacent to every vertex of W, then the graph is called *complete bipartite* on the sets U and W. The complete bipartite graph on sets U and W with m' and n' vertices, respectively, is denoted by $K_{m',n'}$.

A graph is said to be *embeddable* on a surface $\mathbf{M}$ if it can be drawn on $\mathbf{M}$ without crossing edges. The genus of a graph is the lowest genus of any surface on which the graph can be embedded, and it is denoted by $\gamma(G)$. The embedding of a graph is said to be orientable if $\mathbf{M}$ is an orientable surface.

If a graph G has a minimal embedding on an orientable compact surface $\mathbf{M}$ with genus g, then

$$V - E + F = 2 - 2g,$$

where, V, E and F are the number of vertices, edges and faces of $\mathbf{M}$, respectively.

For $m', n' \geq 2$, the Eüler characteristic of the complete bipartite graph $K_{m',n'}$ is given by

$$\chi(K_{m',n'}) = 2[(m' + n' - \frac{m'n'}{2})/2], \tag{10}$$

where $[a]$ denotes the greatest integer less than or equal to the real number a.

If there is no restriction of the embedding to be a 2-cell embedding (a region homeomorphic to an open disc), then the embedding may be realized on every orientable compact surface with characteristic greater than or equal to the Eüler characteristic of the given surface.

The dual of an embedding of a graph on a surface is obtained by considering the interior of each face of the original embedding as a vertex of a new embedded graph. If two faces are adjacent along an edge in the original graph, we join the two faces with a new edge crossing the old edge along which the two faces are adjacent. The resulting embedding of the new graph on the same surface is called the dual of the original embedding, and if the dual graph is isomorphic to the original one, then the original and the new embedding are said to be *self-dual*.

For more information on graphs we refer the reader to [9].

4. Classes of TQC Derived from Self-dual Tessellations

From the tables shown in [2], we took into consideration the codes with the greatest distances d_{TQC}. Among these cases, we analyze the self-dual tessellations due to the lesser complexity involved in a possible code implementation.

4.1. *Codes associated with self-dual embeddings of complete graphs*

Let V, E and F denote respectively the number of vertices, edges and faces in a self-dual orientable embedding of complete graphs, K_s. For such graphs, the parameters can take on the following values:

$$V = F = s, \quad E = \frac{s(s-1)}{2} = \binom{s}{2}.$$

Thus, the Eüler characteristic of these graphs is given by $\gamma(K_s) = s - \frac{s.(s-1)}{2} + s$. If the minimal embedding is on a surface with genus g, then $\gamma(K_s) = \chi(\mathbf{M}) = 2 - 2g$, yielding $g = \frac{(s-1)(s-4)}{4}$. Therefore, such an embedding can exist only if $s \equiv 0$ or 1 *mod* 4, [10].

Since $g = \frac{(s-1)(s-4)}{4}$, it follows that the number of qubits to be encoded is

$$\begin{aligned} k = 2g &= \frac{(s-1)(s-4)}{2} \\ &= \frac{s^2 - 5s + 4}{2} \\ &= \frac{s^2 - s}{2} + \frac{(-4s+4)}{2} \\ &= \frac{s(s-1)}{2} - 2(s-1) \\ &= \left(\begin{array}{c} s \\ 2 \end{array} \right) - 2(s-1). \end{aligned} \tag{11}$$

On the other hand, the number of edges of the polygon resulting from the embedding of the complete graph K_s on a compact surface with genus g, is equal to the number of edges of the graph. Hence, $n = \left(\begin{array}{c} s \\ 2 \end{array} \right)$.

In order to know the tessellations which give rise to codes with parameters k and n given above, first note that $n = s(s-1)/2$, and that $n = n_f p/2$. By considering $n_f = s$ and $p = n_f - 1 = s - 1$, and substituting these values in (6) yields $q = s - 1$. Therefore, the resulting tessellations are self-dual $\{s-1, s-1\}$ tessellations.

Independent of the case in consideration, that is, either $s \equiv 0$ *mod* 4 or $s \equiv 1$ *mod* 4, the code distance using (9), is given by

$$d_{TQC} > \frac{2 \text{ arccosh} \left[\frac{\cos(\frac{\pi}{(s-1)(s-4)})}{\sin(\frac{\pi}{(s-1)(s-4)})} \right]}{\text{arccosh} \left[\frac{\cos^2(\frac{\pi}{s-1}) + \cos(\frac{2\pi}{s-1})}{\sin^2(\frac{\pi}{s-1})} \right]}, \tag{12}$$

where $g = \frac{(s-1)(s-4)}{4}$, $p = q = s - 1$. It can be shown that the right-hand side of (12) is the infimum of the code distance and for $s \to \infty$ leads to 2.

Therefore, the topological quantum codes have parameters

$$[[n,\ k,\ d]] = \left[\left[\left(\begin{array}{c} s \\ 2 \end{array} \right),\ \left(\begin{array}{c} s \\ 2 \end{array} \right) - 2(s-1),\ 3 \right]\right],$$

for $s \equiv 0$ or 1 *mod* 4.

When $s \equiv 1$ *mod* 4, such a class of codes coincides with the class shown in [11].

Tables 1 and 2 illustrate some examples of self-dual tessellations and the corresponding TQC codes.

Observe that the encoding rate $\frac{k}{n} = 1 - \frac{4}{s} \to 1$ as $s \to \infty$.

4.2. *Codes associated with embeddings of complete bipartite graphs*

According to [10], self-dual orientable minimal embedding of complete graphs K_s can exist only if $s \equiv 0$ or 1 *mod* 4. Nevertheless, there are self-dual tessellations such as $\{5, 5\}$, $\{6, 6\}$, $\{9, 9\}$,

Table 1. Codes derived from embeddings of complete graphs K_s, $s \equiv 0 \; mod \, 4$, $s \geq 8$

s	g	$\{p,q\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
8	7	{7,7}	5.75	2.72	2.11	[[28,14,3]]
12	22	{11,11}	8.05	3.79	2.12	[[66,44,3]]
16	45	{15,15}	9.48	4.46	2.13	[[120,90,3]]
20	76	{19,19}	10.53	4.95	2.13	[[190,152,3]]

Table 2. Codes derived from embeddings of complete graphs K_s, $s \equiv 1 \; mod \, 4$, $s \geq 9$

s	g	$\{p,q\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
9	10	{8,8}	6.47	3.06	2.12	[[36,20,3]]
13	27	{12,12}	8.46	3.98	2.12	[[78,54,3]]
17	52	{16,16}	9.77	4.60	2.13	[[136,104,3]]
21	85	{20,20}	10.75	5.06	2.13	[[210,170,3]]

$\{10,10\}, \{13,13\}, \{14,14\}, \ldots$ which are not the embedding of complete graphs on compact surfaces. In the next subsections we show that these tessellations generate two classes of TQC, one class with $s \equiv 0 \; mod \; 4$, and the other one with $s \equiv 0 \; mod \; 2$. Such codes are associated with embeddings of complete bipartite graphs $K_{m',n'}$.

4.2.1. *Class of TQC derived from embeddings of graphs of the type* $K_{\frac{s}{2},s-3}$

Consider $n_f = s$ and $p = q = s-3$. Since $n = n_f \frac{p}{2}$, it follows that the code length is $n = \frac{s(s-3)}{2}$. Substituting the values of n_f, p and q in (6) yields $g = \frac{s(s-3)}{4} - s + 1$, and therefore the code dimension is $k = 2g = \frac{s(s-3)}{2} - 2(s-1)$. As can be seen from Table 3 and by substituting the values of g, p and q for each case in consideration in (9) leads to $d_{TQC} = 3$. Hence, this class of topological quantum codes has parameters

$$[[n,\; k\; d_{TQC}]] = \left[\left[\frac{s(s-3)}{2},\; \frac{s(s-3)}{2} - 2(s-1),\; 3\right]\right].$$

Table 3 illustrates some examples of codes belonging to this class. Note that $s \equiv 0 \; mod \, 4$ and $s \geq 8$.

Since the number of edges of a complete bipartite graph $K_{m',n'}$ is $m'n'$, it follows that $n = m'n'$. Thus, we may consider $m' = \frac{s}{2}$ and $n' = s-3$. Therefore, this class of codes is associated with the embedding of the graph $K_{\frac{s}{2},s-3}$. Such embeddings are possible since the genus $g = \frac{s(s-3)}{4} - s + 1$ is greater than or equal to the minimum value of the Eüler characteristic, (10).

Table 3. Codes derived from embeddings of complete bipartite graphs $K_{\frac{s}{2},s-3}$, $s \equiv 0 \; mod \, 4$

s	g	$\{p,q\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
8	3	{5,5}	3.98	1.68	2.36	[[20,6,3]]
12	16	{9,9}	7.41	3.34	2.22	[[54,32,3]]
16	37	{13,13}	9.09	4.16	2.19	[[104,74,3]]
20	66	{17,17}	10.25	4.72	2.17	[[170,132,3]]

Observe that the encoding rate $\frac{k}{n} = 1 - \frac{2(s-1)}{s(s-3)} \to 1$ as $s \to \infty$.

4.2.2. *Class of TQC derived from embeddings of graphs of the type $K_{s-5,s}$*

Consider $n_f = s$ and $p = q = 2(s-5)$. Thus, the code length is $n = s(s-5)$. Substituting the values of n_f, p and q in (6) we obtain $g = \frac{s(s-5)}{2} - s + 1$, and so the code dimension is $k = 2g = s(s-5) - 2(s-1)$.

Following the same procedure as in the previous cases, it can be shown that the code distance is $d_{TQC} = 3$. Therefore, the resulting class of TQC codes has parameters

$$[[n,\ k,\ d_{TQC}]] = [[s(s-5),\ s(s-5) - 2(s-1),\ 3]]\,.$$

Table 4 illustrates some examples of codes belonging to this class. Note that $s \equiv 0\ mod\ 2$ and $s \geq 8$.

Again, since the number of edges of a complete bipartite graph $K_{m',n'}$ is $m'n'$, and since $n = s(s-5)$, we may consider $m' = s-5$ and $n' = s$. Therefore, this class of codes is associated with the embedding of the graph $K_{s-5,s}$. Observe that the genus $g = \frac{s(s-5)}{2} - s+1$, is greater than or equal to the minimum value of g obtained from (10), and so it is possible to have such embeddings.

Table 4. Codes derived from embeddings of complete bipartite graphs $K_{s-5,s}$, $s \equiv 0\ mod\ 2$

s	g	$\{p,q\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
8	5	{6,6}	5.06	2.29	2.21	[[24,10,3]]
10	16	{10,10}	7.41	3.58	2.07	[[50,32,3]]
12	31	{14,14}	8.74	4.31	2.02	[[84,62,3]]
14	50	{18,18}	9.69	4.84	2.002	[[126,100,3]]

Again, $\frac{k}{n} = 1 - \frac{2(s-1)}{s(s-5)} \to 1$ as $s \to \infty$.

5. Classes of TQC Derived from Quasi Self-dual Tessellations

Besides the self-dual codes, we find a class of TQC codes derived from non self-dual tessellations. In particular, we consider the case where $\{p,q\} = \{q-1,q\}$ because the code minimum distance associated with the $\{q-1,q\}$ tessellation equals the code minimum distance of its dual tessellation, [2].

Let $n_f = s, q = \frac{s}{2}$ and $p = \frac{s}{2} - 1$. Thus, the code length is $n = \frac{s(s-2)}{4}$. Once the number of vertices, n_v, of the tessellation $\{p,q\}$ is known, the Eüler characteristic may be determined. However, the number of vertices of the tessellation $\{p,q\}$ equals the number of faces of the dual tessellation $\{q,p\}$. Thus, $n_v = \frac{sp}{q} = s-2$. Substituting the values of n_f, n_v and n in the Eüler characteristic, yields $2g = \frac{s(s-2)}{4} - 2s + 4$, that is, the code dimension is $k = 2g = \frac{s(s-2)}{4} - 2(s-2)$.

Following the same procedure as in the previous cases, it can be shown that the code distance is $d_{TQC} = 3$. Therefore, the parameters of this class of codes are

$$[[n,\ k,\ d_{TQC}]] = \left[\left[\frac{s(s-2)}{4},\ \frac{s(s-2)}{4} - 2(s-2),\ 3\right]\right].$$

Note that if we consider the dual tessellation the results are the same.

Table 5 illustrates some examples of codes belonging to this class. Note that $s \equiv 0\ mod\ 2$ for $s \geq 10$.

Since $n = \frac{s(s-2)}{4}$, it follows that $m' = \frac{s-2}{2}$ and $n' = \frac{s}{2}$. Therefore, this class of codes is associated with the embedding of the graph $K_{\frac{s-2}{2},\frac{s}{2}}$. Note that the genus $g = \frac{s(s-2)}{8} - s + 2$ satisfies condition (10) for the existence of such embeddings.

Table 5. Codes derived from embeddings of complete bipartite graphs $K_{\frac{s-2}{2},\frac{s}{2}}$, $s \equiv 0 \ mod\ 2$

s	g	$\{p,q\}$	$\{q,p\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$l(q,p)$	$d_h/l(q,p)$	$[[n,k,d_{TQC}]]$
10	2	{4,5}	{5,4}	3.06	1.25	2.44	1.06	2.88	[[20,4,3]]
12	5	{5,6}	{6,5}	5.06	2.12	2.38	1.88	2.70	[[30,10,3]]
14	9	{6,7}	{7,6}	6.25	2.63	2.38	2.39	2.62	[[42,18,3]]
16	14	{7,8}	{8,7}	7.14	3.002	2.38	2.78	2.57	[[56,28,3]]

Note that the encoding rate $\frac{k}{n} = 1 - \frac{8}{s} \to 1$ as $s \to \infty$.

6. Classes of TQC Derived from Densest Tessellations

It is known that the tessellations $\{p, 3\}$ are denser than the corresponding previous tessellations. Hence, based on this fact we consider the codes obtained from these tessellations in order to find a class of TQC codes derived from embeddings of complete bipartite graphs.

In this case, $n_v = s, p = \frac{s}{4}$ and $q = 3$. Since $n_v = n_f \frac{p}{q}$, it follows that $n_f = n_v \frac{q}{p} = 12$. Thus, the code length is $n = \frac{3s}{2}$. Substituting the values of n_f, n_v and n in the Eüler characteristic, the code dimension is $k = 2g = \frac{s}{2} - 10$. For the cases considered in Table 6 and from (9) the code distance is $d_{TQC} = 3$.

In this case, there is a large class of codes with parameters

$$[[n,\ k,\ d_{TQC}]] = \left[\left[\frac{3s}{2},\ \frac{s}{2} - 10,\ 3\right]\right].$$

Note from (9) as s goes to infinity the code distance goes to one.

Table 6 illustrates some examples of codes belonging to this class. Note that $s \equiv 0 \ mod\ 4$ for $s \geq 28$. Since $n = \frac{3s}{2}$, it follows that $m' = 3$ and $n' = \frac{s}{2}$. Therefore, this class of codes is associated with the embedding of the graph $K_{3,\frac{s}{2}}$. Note that, the genus $g = \frac{s}{4} - 5$ satisfies the condition for the existence of such embeddings, that is, it is greater than or equal to the minimum value of g, (10).

Table 6. Codes derived from embeddings of complete bipartite graphs $K_{3,\frac{s}{2}}$, $s \equiv 0 \ mod\ 4$

s	g	$\{p,q\}$	$\{q,p\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$l(q,p)$	$d_h/l(q,p)$	$[[n,k,\delta]]$	$[[n,k,\delta^*]]$
28	2	{7,3}	{3,7}	3.06	0.57	5.37	1.09	2.81	[[42,4,6]]	[[42,4,3]]
32	3	{8,3}	{3,8}	3.98	0.73	5.45	1.53	2.60	[[48,6,6]]	[[48,6,3]]
36	4	{9,3}	{3,9}	4.60	0.82	5.60	1.86	2.47	[[54,8,6]]	[[54,8,3]]
40	5	{10,3}	{3,10}	5.06	0.88	5.75	2.12	2.39	[[60,10,6]]	[[60,10,3]]
44	6	{11,3}	{3,11}	5.43	0.92	5.90	2.35	2.31	[[66,12,6]]	[[66,12,3]]
48	7	{12,3}	{3,12}	5.75	0.95	6.05	2.55	2.25	[[72,14,7]]	[[72,14,3]]

Note that the encoding rate $\frac{k}{n} = \frac{1}{3} - \frac{20}{3s} \to \frac{1}{3}$ as $s \to \infty$.

As can be seen from the last two columns in Table 6, the codes derived from the complete bipartite graphs $K_{3,\frac{s}{2}}$, $s \equiv 0 \ mod\ 4$, inherit an unequal error protection. This property has its advantages in applications where the interferences in the channel act in a nonhomogeneous way on the qubits, or equivalently, act unequally on each homologically nontrivial cycle.

7. Conclusions

In this paper we have presented six classes of topological quantum codes (TQC). What is relevant in the study of such classes of codes is the fact that they are associated with the self-dual, quasi-self-dual and denser tessellations, providing in this way a global view from what is possible to obtain in terms of TQC codes. These classes are associated with embeddings of self-dual complete graphs and complete bipartite graphs on the corresponding compact surfaces.

It is known that codes defined by self-dual tessellations demand less computational effort, moreover as shown by Kitaev, [1], these codes may be implemented by use of *anyons*. Whereas codes defined by non-self-dual tessellation, for example, cases where $q = 3$, besides being denser tessellations, they may be of use in channels where the unequal error protection is essential. This is due to the fact that the code distances associated with the tessellation and its dual are different.

Four classes are derived from self-dual tessellations whereas two classes are derived from non-self-dual tessellations. These tessellations were chosen due to their good properties such as the distance d_{TQC} and by the fact of being self-dual and denser tessellations. Every code achieves minimum distance 3 and its encoding rate is such that $\frac{k}{n} \to 1$ as $n \to \infty$, except for the latter class whose asymptotic rate is $\frac{k}{n} \to \frac{1}{3}$. However, this latter class contains the case of denser tessellations than the previous ones.

These constructions could be applied to other situations in the future, as suggested by the referee, like: a) TQCs for qudits (generalization of qubits); b) Surfaces with boundaries; and c) Topological color codes.

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