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ASYMMETRIC QUANTUM TELECLONING OF MULTIQUBIT STATES

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A scheme of 1→2 optimal universal asymmetric quantum telecloning for pure multiqubit states is proposed. We first investigate the telecloning of arbitrary 2-qubit states and then extend it to the case of multiqubit system. We discuss the scheme in terms of the quantum channels and fidelities of clones, as well as the entanglement of states in the telecloning.

Keywords: telecloning, multiqubit state, Bell projection
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1 Introduction

One of the most essential differences between classical and quantum information theory (QIT) originates from the no cloning theorem [1, 2]. It forbids the perfect cloning of arbitrarily given quantum state, in both pure and mixed cases. It is therefore natural to ask how well one can copy quantum states, i.e., with the highest fidelity. This problem was firstly addressed by Buzek and Hillery [3], whose scheme was then proved to be optimal by [4]. The Buzek-Hillery theory actually exhibits a universal symmetric 1→2 quantum cloning machine (QCM), which exports two identical clones closest to the input pure qubit state with a constant fidelity. The related work in past years has established the $N \to M$ universal symmetric QCM for both qubits [5] and qudits [6, 7] (transforming N identical input states into $M > N$ identical output copies), as well as the continuous-variable systems [8]. Correspondingly, the $N \to M$ asymmetric QCM generates M output states with different fidelities from N input copies [9, 10, 11, 12]. Some experimental progress on quantum cloning has also been made [13].

The essentiality of quantum cloning is to broadcast information to certain distributed objects, so it is regarded as a widely useful quantum-information transmission, e.g., the eavesdropping on implementation of quantum key distribution [14]. It is well-known that quantum teleportation [15, 16] is the most effective technique for remotely broadcasting information. Murao *et al.* [17] has advanced the $1 \to M$ quantum telecloning which combines the tricks of both quantum teleportation and cloning. In this scheme, the sender Alice holds an unknown input state and she previously shares an entangled channel with M receivers, which resembles the scenario of quantum teleportation. The object is to duplicate the input at the location of every receiver as well as possible, since the no-cloning theorem precludes the faithful copy of unknown quantum state. Similarly, there exist symmetric and asymmetric telecloning with identical and different fidelities of the clones respectively. Unlike the broadcasting of entangled states [18], quantum telecloning is optimal if it achieves the best fidelity as those of universal QCMs. The technique of symmetric telecloning has been extended to the case of $N \to M$ for qubit states [19] and $1 \to M$ for qudits [7], while the $1 \to 2$ universal optimal asymmetric telecloning was realized by [7, 20]. Of all these traditional schemes, the input state is local, so the sender can perform any unitary operation on its system.

On the other hand, the situation is different when the input states are entangled, and some primary investigation for entanglement cloning has been made recently [21, 22]. In particular, they have found out the condition on which a universal QCM can be optimal for the maximally entangled inputs, while for other cases the problem proves exceedingly difficult. Recently, [23] proposed the scheme of telecloning for the entangled inputs $|\psi\rangle = \alpha |00\rangle + \beta |11\rangle$, which is a small set of two-qubit states (we refer to $|j\rangle, j = 0, 1, ...$ as the computational basis in this paper, see below). It is then interesting to extend this scheme to the case of general two-qubit inputs. However, we doubt that whether such a scheme can be universal for any input, namely with a constant fidelity. If so, can it reach the optimal fidelity of the universal QCMs such as Werner's bound [6] ? Furthermore, as the extensive use of multipartite entanglement, it is important to explore the telecloning of multiqubit states.

Here we propose a scheme of 1→2 optimal universal asymmetric quantum telecloning of pure multiqubit states, by virtue of the Heisenberg QCM in [10, 20]. We firstly investigate the asymmetric telecloning of arbitrary two-qubit states and the properties of its quantum channel. Our scheme may be implemented by the generalization of recent experiments. The required entanglement in the presented scheme is shown to be optimal for the 4-dimensional input states. We compare the achievable fidelity with the existing universal QCMs, and also explicitly prove that the telecloner never creates more entanglement than that contained in the input qubits. Furthermore, we extend the above scheme to the case of multiqubit inputs. Theoretically, it is the first time to realize the universal telecloning of nonlocal multiqubit states. As a $d-$level system can be written in terms of qubits, one can optimally teleclone any entangled state by our scheme.

The paper is organized as follows. In Sec. II we present the explicit protocol for the case of 2-qubit input states, and investigate the properties of the telecloning in terms of its channels, e.g., the genuine entanglement and necessary cost, and fidelities of the clones. In Sec. III we extend it to the case of multiqubit input states. We present our conclusion in Sec. IV.

2 Optimal universal 1→2 telecloning of 2-qubit states

As shown in [15, 7], either of quantum teleportation and telecloning requires an unknown input state, which is to be reconstructed in several remotely distributed places. In the present situation, the input state shared by two parties A_1, A_2 has the following form,

$$|\psi\rangle_{A_1A_2} = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle, \quad (1)$$

where $\alpha_i \in \mathcal{C}, \forall i$, and $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$ by the normalization condition. The aim of the telecloning is to respectively transimit two copies of this state to two groups of receivers B_1, B_2 and C_1, C_2 with the highest fidelity. In addition, every participant can only operate on their states locally with the help of classical communication. To find out the appropriate quantum channel between senders and receivers, we recall the optimal universal asymmetric Heisenberg QCM [10, 20],

$$U|j\rangle_B|00\rangle_{C,Anc} = \frac{1}{\sqrt{1+(d-1)(p^2+q^2)}} \times \left(p|j\rangle_B \left|\Phi_d^+\right\rangle_{C,Anc} + q|j\rangle_C \left|\Phi_d^+\right\rangle_{B,Anc}\right), \quad (2)$$

where $\left|\Phi_d^+\right\rangle$ denotes the maximally entangled state of two qudits,

$$\left|\Phi_d^+\right\rangle = \sum_{j=0}^{d-1} |jj\rangle\,. \quad (3)$$

For simplicity, we set $|\eta_j\rangle \equiv U|j\rangle_B|00\rangle_{C,Anc}, j = 0, 1, ...$

The gates $U|j\rangle_B|00\rangle_{C,Anc}$ represent series of QCMs by altering j. Here, $\overline{j+r} = j + r$ modulo d and d is the dimension of the input state. The real constants p, q satisfy $p + q = 1$ and their concrete meaning is to generate a universal QCM and to keep the optimality of it, so they can be properly defined previously. By superposition of the QCMs in expression (2), one can set an arbitrary state of system B as input to obtain two clones at systems B, C respectively, and the third qudit is the ancilla. The cloner is optimal in the sense that the fidelity of the second clone is maximal if that of the first one is fixed, and the explicit form of the fidelity will appear in the discussion of our scheme. Clearly, the fidelity of Heisenberg cloner is also the upper bound for the universal asymmetric telecloning. A $1 \to 2$ scheme reaching the bound has been given in [20], in which the input states are supposed to be local. This scheme employed the technique of Heisenberg cloner, which is also proven to be useful here for quantum telecloning with nonlocal inputs.

In a $d-$dimension Hilbert space, the computational basis can be expressed as a composition of the qubits, e.g., when $d = 4$ we can denote that $|0\rangle \to |00\rangle$, $|1\rangle \to |01\rangle$, $|2\rangle \to |10\rangle$ and $|3\rangle \to |11\rangle$. As far as the state $|\eta_j\rangle_{B,C,Anc}$ is concerned, either of the system B, C and the ancilla should be a composite system of two separated qubits, whose dimension is at most $d = 4$. Concretely, we write out $|\eta_j\rangle$'s in terms of expression (2),

$$\begin{aligned}
|\eta_0\rangle &= [1+3(p^2+q^2)]^{-1/2}\big(|00\rangle|00\rangle|00\rangle \\
&+ p|00\rangle|01\rangle|01\rangle + p|00\rangle|10\rangle|10\rangle + p|00\rangle|11\rangle|11\rangle \\
&+ q|01\rangle|00\rangle|01\rangle + q|10\rangle|00\rangle|10\rangle + q|11\rangle|00\rangle|11\rangle\big), \\
|\eta_1\rangle &= [1+3(p^2+q^2)]^{-1/2}\big(|01\rangle|01\rangle|01\rangle \\
&+ p|01\rangle|00\rangle|00\rangle + p|01\rangle|10\rangle|10\rangle + p|01\rangle|11\rangle|11\rangle \\
&+ q|00\rangle|01\rangle|00\rangle + q|10\rangle|01\rangle|10\rangle + q|11\rangle|01\rangle|11\rangle\big), \\
|\eta_2\rangle &= [1+3(p^2+q^2)]^{-1/2}\big(|10\rangle|10\rangle|10\rangle \\
&+ p|10\rangle|00\rangle|00\rangle + p|10\rangle|01\rangle|01\rangle + p|10\rangle|11\rangle|11\rangle \\
&+ q|00\rangle|10\rangle|00\rangle + q|01\rangle|10\rangle|01\rangle + q|11\rangle|10\rangle|11\rangle\big), \\
|\eta_3\rangle &= [1+3(p^2+q^2)]^{-1/2}\big(|11\rangle|11\rangle|11\rangle \\
&+ p|11\rangle|00\rangle|00\rangle + p|11\rangle|01\rangle|01\rangle + p|11\rangle|10\rangle|10\rangle \\
&+ q|00\rangle|11\rangle|00\rangle + q|01\rangle|11\rangle|01\rangle + q|10\rangle|11\rangle|10\rangle\big).
\end{aligned} \quad (4)$$

In particular, for N-qubit state $d = 2^N$, we rewrite the state $\left|\Phi_d^+\right\rangle$ as a tensor product of N two-qubit Bell states

$$\left|\Phi_d^+\right\rangle_{B,C} = \otimes_{k=1}^N \left|\Phi^+\right\rangle_{Bk,Ck}, \tag{5}$$

where $|\Phi^+\rangle = |00\rangle + |11\rangle$, the parties $B_1, ..., B_N$ are in the system B and similarly for $C_1, ..., C_N$. Then we propose that the quantum channel shared by all parties is

$$|\Omega\rangle_{A_1'A_2'B_1B_2C_1C_2a_1a_2} = \frac{1}{2\sqrt{1+3(p^2+q^2)}} \times \\ (p\left|\Phi_4^+\right\rangle_{AB}\left|\Phi_4^+\right\rangle_{Ca} + q\left|\Phi_4^+\right\rangle_{AC}\left|\Phi_4^+\right\rangle_{Ba}), \tag{6}$$

here A_1' and A_2' are two particles belonging to the senders A_1 and A_2 respectively. The two ancillas a_1, a_2 are held by some separated observers. The ancilla particles are necessary for the Heisenberg QCM, otherwise it cannot reach the optimal fidelity [10]. Although the ancillas do not play the role of clones, they actually join the realization of optimal telecloning of entanglement. For example, there are some useful relations with respect to the states $|\eta_i\rangle$'s, which involves all the participants in the system

$$\sigma_{zB_1} \otimes \sigma_{zC_1} \otimes \sigma_{za_1} |\eta_i\rangle = |\eta_i\rangle, i = 0, 1 \tag{7}$$

$$\sigma_{zB_1} \otimes \sigma_{zC_1} \otimes \sigma_{za_1} |\eta_i\rangle = -|\eta_i\rangle, i = 2, 3 \tag{8}$$

$$\sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{za_2} |\eta_i\rangle = |\eta_i\rangle, i = 0, 2 \tag{9}$$

$$\sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{za_2} |\eta_i\rangle = -|\eta_i\rangle, i = 1, 3 \tag{10}$$

$$\sigma_{xB_1} \otimes \sigma_{xC_1} \otimes \sigma_{xa_1} |\eta_0\rangle = |\eta_2\rangle, \tag{11}$$

$$\sigma_{xB_1} \otimes \sigma_{xC_1} \otimes \sigma_{xa_1} |\eta_1\rangle = |\eta_3\rangle, \tag{12}$$

$$\sigma_{xB_2} \otimes \sigma_{xC_2} \otimes \sigma_{xa_2} |\eta_0\rangle = |\eta_1\rangle, \tag{13}$$

$$\sigma_{xB_2} \otimes \sigma_{xC_2} \otimes \sigma_{xa_2} |\eta_2\rangle = |\eta_3\rangle. \tag{14}$$

The equations follow from the following invariance property of the Bell state: $U \otimes U^* |\Phi^+\rangle = |\Phi^+\rangle$, where U is an arbitrary single-qubit unitary transformation. Specially, the first four equations represent the changes of the state signs while the last four represent the changes between the states $|\eta_i\rangle$'s. We thus call them parity-transformation and state-transformation respectively.

In what follows we show how to carry out the universal optimal $1 \to 2$ telecloning of the two-qubit state $|\psi\rangle_{A_1A_2} = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$. The whole system is in the state

$$|\Psi\rangle_{tot} = |\psi\rangle_{A_1A_2} \otimes |\Omega\rangle_{A_1'A_2'B_1B_2C_1C_2a_1a_2} = \\ \frac{\alpha_0}{2}(|00\rangle_{A_1A_1'}|00\rangle_{A_2A_2'}|\eta_0\rangle + |00\rangle_{A_1A_1'}|01\rangle_{A_2A_2'}|\eta_1\rangle + \\ |01\rangle_{A_1A_1'}|00\rangle_{A_2A_2'}|\eta_2\rangle + |01\rangle_{A_1A_1'}|01\rangle_{A_2A_2'}|\eta_3\rangle) \\ +\frac{\alpha_1}{2}(|00\rangle_{A_1A_1'}|10\rangle_{A_2A_2'}|\eta_0\rangle + |00\rangle_{A_1A_1'}|11\rangle_{A_2A_2'}|\eta_1\rangle + \\ |01\rangle_{A_1A_1'}|10\rangle_{A_2A_2'}|\eta_2\rangle + |01\rangle_{A_1A_1'}|11\rangle_{A_2A_2'}|\eta_3\rangle) \\ +\frac{\alpha_2}{2}(|10\rangle_{A_1A_1'}|00\rangle_{A_2A_2'}|\eta_0\rangle + |10\rangle_{A_1A_1'}|01\rangle_{A_2A_2'}|\eta_1\rangle +$$

$$|11\rangle_{A_1A_1'}|00\rangle_{A_2A_2'}|\eta_2\rangle + |11\rangle_{A_1A_1'}|01\rangle_{A_2A_2'}|\eta_3\rangle)$$
$$+\frac{\alpha_3}{2}(|10\rangle_{A_1A_1'}|10\rangle_{A_2A_2'}|\eta_0\rangle + |10\rangle_{A_1A_1'}|11\rangle_{A_2A_2'}|\eta_1\rangle +$$
$$|11\rangle_{A_1A_1'}|10\rangle_{A_2A_2'}|\eta_2\rangle + |11\rangle_{A_1A_1'}|11\rangle_{A_2A_2'}|\eta_3\rangle). \quad (15)$$

To saturate the optimal fidelity of Heisenberg cloner, the target state has the form

$$|\omega\rangle_{B_1B_2C_1C_2a_1a_2} \equiv \sum_{j=0}^{3}\alpha_j|\eta_j\rangle, \quad (16)$$

which contains the optimal two clones of system B_1B_2 and C_1C_2 respectively, as well as one ancilla of system a_1a_2. Since either of the senders A_1 and A_2 holds two particles being in the state $|\Psi\rangle_{tot}$, they can individually perform a joint measurement on their 2-qubit systems in the Bell basis

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (17)$$

Evidently, the resulting state is $\langle\Phi^{\pm}|_{A_1A_1'}\langle\Phi^{\pm}|_{A_2A_2'}|\Psi\rangle_{tot}$, etc, and there are in all 16 cases here. To simplify the situation, we call the superscript "+" or "−" of the Bell basis the parity of it. It is easy to show that any Bell projection can be turned into one of the cases $\langle\Phi^{\pm}|_{A_1A_1'}\langle\Phi^{\pm}|_{A_2A_2'}|\Psi\rangle_{tot}$ with the same parity as the former one, by using of the state-transformations (11)-(14) and their joint operations, e.g., $\sigma_{xB_1}\otimes\sigma_{xC_1}\otimes\sigma_{xa_1}\otimes\sigma_{xB_2}\otimes\sigma_{xC_2}\otimes\sigma_{xa_2}$.

For example, if the measurement is taken in $\{|\Phi^-\rangle_{A_1A_1'}, |\Psi^-\rangle_{A_2A_2'}\}$, the resulting state is

$$|\Psi\rangle \quad = \quad \alpha_0|\eta_1\rangle - \alpha_1|\eta_0\rangle - \alpha_2|\eta_3\rangle + \alpha_3|\eta_2\rangle. \quad (18)$$

By using of the state-transformation $|\eta_0\rangle \leftrightarrow |\eta_1\rangle$ and $|\eta_2\rangle \leftrightarrow |\eta_3\rangle$ (it requires the classical communication between the participants), one can obtain

$$|\Psi\rangle_{res} \quad = \quad \alpha_0|\eta_0\rangle - \alpha_1|\eta_1\rangle - \alpha_2|\eta_2\rangle + \alpha_3|\eta_3\rangle, \quad (19)$$

which is the resulting state by measuring $|\Psi\rangle_{tot}$ in $\{|\Phi^-\rangle_{A_1A_1'}, |\Phi^-\rangle_{A_2A_2'}\}$, and its parity is unchanged. Similarly, one can check that the resulting states derived from other Bell measurements can be turned with the same parity by the state-transformation operators and their joint operators. For example, if the outcome is $\langle\Psi^-|_{A_1A_1'}\langle\Psi^-|_{A_2A_2'}|\Psi\rangle_{tot}$, one should perform $\sigma_{xB_1}\otimes\sigma_{xC_1}\otimes\sigma_{xa_1}\otimes\sigma_{xB_2}\otimes\sigma_{xC_2}\otimes\sigma_{xa_2}$ on the resulting state. In this step, the sign of the state $|\eta\rangle_i, i = 0, 1, 2, 3$ do not change.

Hence, it suffices to merely consider the cases of measurements in $\{|\Phi^{\pm}\rangle_{A_1A_1'}, |\Phi^{\pm}\rangle_{A_2A_2'}\}$. In particular, there are four subcases such that $\{|\Phi^+\rangle_{A_1A_1'}, |\Phi^+\rangle_{A_2A_2'}\}$, $\{|\Phi^+\rangle_{A_1A_1'}, |\Phi^-\rangle_{A_2A_2'}\}$, $\{|\Phi^-\rangle_{A_1A_1'}, |\Phi^+\rangle_{A_2A_2'}\}$ and $\{|\Phi^-\rangle_{A_1A_1'}, |\Phi^-\rangle_{A_2A_2'}\}$ here. Besides, the senders need broadcast the results of the measurement to the receivers and ancillas so that they can perform the unitary operations to modify the shared states locally. The result by the measurement $\{|\Phi^+\rangle_{A_1A_1'}, |\Phi^+\rangle_{A_2A_2'}\}$ is precisely $|\omega\rangle_{B_1B_2C_1C_2a_1a_2}$. For the second and third cases, by using of the parity-transformations $\sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{za_2}$ and $\sigma_{zB_1} \otimes \sigma_{zC_1} \otimes \sigma_{za_1}$ respectively, it is known from the relations (7)-(10) that the receivers recover the correct state again. In case of the final situation, it requires the rotations $\sigma_{zB_1} \otimes \sigma_{zC_1} \otimes \sigma_{za_1} \otimes \sigma_{zB_2} \otimes \sigma_{zC_2} \otimes \sigma_{za_2}$ by all parties. Hence, one can always recover the target state and thereby explicitly realize the optimal universal asymmetric $1 \rightarrow 2$ telecloning of arbitrary two-qubit state by LOCC.

We investigate the scheme in terms of some figures of merit. First, the required entanglement between senders and receivers is $E(|\Omega\rangle_{A'_1A'_2B_1B_2C_1C_2a_1a_2}) = 2$ ebits. Besides, the classical cost informing the receivers and ancillas is 4 cbits in all. The quantum channel $|\Omega\rangle_{A'_1A'_2B_1B_2C_1C_2a_1a_2}$ is always a fully entangled state if $p, q \neq 0$. To account for this fact, we first regard the state as $|\Omega\rangle_{A'BCa}$, where each group contains the two parties respectively, e.g., A' consists of A'_1 and A'_2, etc. We rewrite it as

$$\begin{aligned} |\Omega\rangle_{A'BCa} &= \frac{1}{2\sqrt{1+3(p^2+q^2)}} \sum_{k,r=0}^{3} \\ &\times \ (p\,|kkrr\rangle + q\,|krkr\rangle)_{A'BCa}. \end{aligned} \tag{20}$$

This form contains some symmetry, e.g., one can see that $|\Omega\rangle_{aBCA'}$ has the same expression as $|\Omega\rangle_{A'BCa}$ under an exchange $p \leftrightarrow q$. So the ancillas are maximally entangled with other parties. It is easy to check this result also holds for the partition $B - aCA'$ and $C - aBA'$. As the channel state is evidently entangled under the partition $A'a - BC$ and $A'B - aC$, we get that $|\Omega\rangle_{A'BCa}$ is fully entangled. This fact implies that if the channel state is separable, it must have the form

$$\begin{aligned} |\Omega\rangle_{A'BCa} &= (a_0\,|0\rangle_x\,|\phi_0\rangle + a_1\,|1\rangle_x\,|\phi_1\rangle) \\ &\times \ (b_0\,|0\rangle_y\,|\psi_0\rangle + b_1\,|1\rangle_y\,|\psi_1\rangle) \end{aligned} \tag{21}$$

with two particles x, y belonging to the same group. Compare to equation (6), it means every state $|\eta_j\rangle, j = 0,1,2,3$ is separable. However, one can easily check these states are fully entangled if $p, q \neq 0$. Hence the state $|\Omega\rangle_{A'_1A'_2B_1B_2C_1C_2a_1a_2}$ has genuine multipartite entanglement. Despite so, the parties in every group are separable, e.g., $\rho_{A'_1A'_2} = \frac{1}{2}I_{A'_1} \otimes \frac{1}{2}I_{A'_1}$, etc. So the parties in the same group cannot affect each other. Furthermore, one can similarly extend these properties to the general channel state, where each group consists of more parties (see next section). Practically, one may prepare the channel state by a maximal entangled state between the senders and receivers, or by the Heisenberg QCM [10, 20], based on the fact that

$$\begin{aligned} |\Omega\rangle_{A'_1A'_2B_1B_2C_1C_2a_1a_2} &= \frac{1}{2} I_{A'_1A'_2} \otimes U_{B_1B_2C_1C_2a_1a_2} \\ (|00\rangle + |11\rangle)_{A'_1B_1} &(|00\rangle + |11\rangle)_{A'_2B_2}\,|0000\rangle_{C_1C_2a_1a_2}, \end{aligned} \tag{22}$$

where we employ the equations $|\eta_j\rangle = U\,|j\rangle_B\,|00\rangle_{C,Anc}, j = 0,1,2,3$. So the previous condition for the scheme is to realize the universal Heisenberg cloner. Recently it is shown that the Pauli cloning [10], which is a special case of Heisenberg cloner, has been experimentally implemented via partial teleportation [24]. There is also the conditional implementation of an asymmetrical cloner for a polarization state of photon [25]. These results are beneficial to the realization of the presented scheme. There are more demonstrations for symmetric cloning machines [26], since the Heisenberg cloner is also optimal when it is symmetric.

Although the protocol in our paper is sufficient to treat the optimal $1 \to 2$ asymmetric telecloning of any 2-qubit input, the quantum cost here is not always necessary for it has turned out that by using of only 1 ebit one can complete the optimal telecloning of a special family of two-qubit states as described in the introduction [23]. We readily prove that for the case of genuine 4-dimensional space, namely $\alpha_0\alpha_1\alpha_2\alpha_3 \neq 0$, the cost of 2 ebits is also necessary for the telecloning scheme. Suppose that the input state is maximally entangled with another qudit:

$$|\psi'\rangle_{A_1A_2A_3} = \frac{1}{2}\big(\,|000\rangle + |011\rangle + |102\rangle + |113\rangle\,\big). \tag{23}$$

Following the formal procedure in the scheme, one can obtain the resulting state

$$|\omega'\rangle_{A_3B_1B_2C_1C_2a_1a_2} = \sum_{j=0}^{3}\frac{1}{2}\,|j\rangle\,|\eta\rangle_j\,. \tag{24}$$

That is, the universal telecloning QCM of arbitrary 2-qubit state can always create 2 ebits between the uncorrelated parties A_3 and the receivers. Since the entanglement cannot be increased on average under LOCC [27], we then assert that the cost of 2 ebits is always necessary and sufficient for this case. However, for the case of $d = 3$ namely there is a vanishing number among $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, it is difficult to show that $\log_2 3$ ebits is the necessary amount of entanglement, e.g., by a way similar to our scheme. A potentially feasible way can be the $1 \to M$ telecloning in [7], but it is necessary to find the decomposition of the unitary transformations collectively performed on the system.

Second, the fidelity of the scheme is optimal. Due to the Heisenberg QCM [10, 20], for a d-level input state $|\psi\rangle$ the clones have the form

$$\rho_B = [1 + (d-1)(p^2 + q^2)]^{-1}\big\{[1 - q^2 + (d-1)p^2]|\psi\rangle\langle\psi| + q^2 I\big\}, \tag{25}$$

and

$$\rho_C = [1 + (d-1)(p^2 + q^2)]^{-1}\big\{[1 - p^2 + (d-1)q^2]|\psi\rangle\langle\psi| + p^2 I\big\}. \tag{26}$$

Then we can obtain the corresponding fidelities

$$F_B(\rho_\psi, \rho_B) = \frac{1 + (d-1)p^2}{1 + (d-1)(p^2 + q^2)}, \tag{27}$$

$$F_C(\rho_\psi, \rho_C) = \frac{1 + (d-1)q^2}{1 + (d-1)(p^2 + q^2)}, \tag{28}$$

which saturates Werner's fidelity bound [6] when $p = q = 1/2$. Our protocol employs the technique of Heisenberg QCM, so it is not dependent on what the input state is. Recently, an optimal universal $1 \to 2$ QCM for maximally entangled inputs has been proposed [21]. Their fidelity is a little higher than Werner's bound, since the set of maximal entanglement is a small part of the whole d-dimensional states. One can thus expect to get a more efficient scheme of telecloning by following [21], as well as other special QCMs such as the phase covariant cloning [28] and real cloner [29]. The main difficulty lies in the restriction of local operations, which makes it difficult to find out an appropriate quantum channel for the corresponding telecloning schemes.

Finally, we prove that our scheme does not create more entanglement than that contained in the input state. The case of maximally entangled input by the optimal QCM has been checked in [21], i.e., when $\mu \equiv |\alpha_0\alpha_3 - \alpha_1\alpha_2| = 1/2$. Here, we show that this is a universal result for any μ of entangled input. Due to the normalization condition of $|\psi\rangle_{A_1A_2}$, we have $\mu \in [0, 1/2]$. Let

$$\begin{aligned} H(x) \;&\equiv\; -(\frac{1}{2}+\frac{1}{2}\sqrt{1-x^2})\log_2(\frac{1}{2}+\frac{1}{2}\sqrt{1-x^2}) \\ &-\; (\frac{1}{2}-\frac{1}{2}\sqrt{1-x^2})\log_2(\frac{1}{2}-\frac{1}{2}\sqrt{1-x^2}), \end{aligned} \tag{29}$$

which is monotonically increasing with $x \in [0, 1]$. One can simply obtain the entanglement of the input state is $E(|\psi\rangle_{A_1A_2}) = H(2\mu)$. We employ the entanglement of formation $E = H(C)$ [30], where $C = C_B(p)$ or $C_C(p)$ is the concurrence [31], to calculate the entanglement of the clones. Replace $|\psi\rangle$ in ρ_B with $\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$, and calculate the eigenvalues λ_i's of $\rho_B(\sigma_y \otimes \sigma_y)\rho_B^*(\sigma_y \otimes \sigma_y)$. Notice $F_B(\rho_\psi, \rho_B) = F_B(p) = \frac{1+3p^2}{1+3(p^2+q^2)}$, some simple algebra leads to

$$\begin{aligned} C_B(p) \;&=\; \max\{0, \sqrt{\lambda_0} - \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3}\} \\ &=\; \max\{0, (\frac{8}{3}F_B - \frac{2}{3})\mu - \frac{2}{3}(1 - F_B)\}, \end{aligned} \tag{30}$$

where λ_i's are decreasingly ordered. Similarly, let $F_C(\rho_\psi, \rho_C) = F_C(p) = \frac{1+3q^2}{1+3(p^2+q^2)}$, thus we obtain

$$C_C(p) = \max\{0, (\frac{8}{3}F_C - \frac{2}{3})\mu - \frac{2}{3}(1 - F_C)\}. \tag{31}$$

Suppose $\Delta(\mu) = H(2\mu) - H(C_B(p)) - H(C_C(p))$, where the entanglement of the initial state is $H(2\mu)$ and that of the clones are $H(C_B(p))$ and $H(C_C(p))$. To show our cloning scheme cannot create more entanglement than contained in the input state, it must hold that $\Delta(\mu) \geq 0, \mu \in [0, 1/2]$. We have analytically proven it in appendix, so our scheme will never create more entanglement than that contained in the original state. In addition, F_B and F_C are not less than 1/4 from their expressions. When one of them reaches this lowest value, the other must be explicitly unit. For the symmetric case namely $p = q = 1/2$, we have $F_B = F_C = 7/10$, which reaches Werner's bound. Hence, $C_B(1/2) = C_C(1/2) = \max\{0, \frac{6}{5}\mu - \frac{1}{5}\}$ and the maximal amount of entanglement created in either of the clones is $H(C(\mu = 1/2)) = 0.250225$ ebits. This is less than that in [21], which is a special set of the two-qubit states. Generally, the relation between entanglements created in the clones constitute a teeterboard due to the monotonicity of $H(C_B(p))$ and $H(C_C(p))$, i.e., if one of them decreases then the other must increases, and vice versa.

3 Optimal universal 1→2 telecloning of n-qubit states

In this section we extend the 1→2 telecloning to the case of n-qubit pure states, and many properties of the above scheme works here. For convenience, we define the n-bit binary form of integer N. Let $N = 2^{n-1} \cdot c_{n-1} + \cdots + 2^1 \cdot c_1 + 2^0 \cdot c_0$, where $2^n > N$ and $c_i = 0$ or 1,

$\forall i$. Then the unique binary form is $\overline{N} = c_{n-1}\cdots c_1c_0$ (we also write $N = \overline{c_{n-1}\cdots c_1c_0}$). The present situation is that n separated senders $A_1, A_2, ..., A_n$ share an arbitrary multiqubit state

$$|\psi\rangle_{A_1,A_2,...,A_n} = \sum_{k=0}^{2^n-1} \alpha_k | \; \overline{k} \; \rangle_{A_1,A_2,...,A_n}, \tag{32}$$

where the coefficients α_i's satisfy $\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$, and the senders know nothing about the state. They plan to optimally teleclone this state at two remote locations, where two groups of uncorrelated receivers $B_1, B_2, ..., B_n$ and $C_1, C_2, ..., C_n$ make up the system in the clone respectively. Again, either of the participants in the whole system can only operate locally and they can communicate with each other. Considering the state $|\eta_j\rangle_{BC,anc}$ in the last section, we write j in its n-bit binary form and each of the bit represents a party B_i or C_i, namely

$$|\eta_j\rangle_{BC,anc} \sim |\eta_j\rangle_{B_1,B_2,...,B_n,C_1,C_2,...,C_n,a_1,a_2,...,a_n}. \tag{33}$$

Having explained the form of $|\eta_j\rangle_{BC,anc}$, we can propose the feasible quantum channel for the telecloning as follows

$$|\Omega\rangle_{A'BC,anc} = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} |\overline{k}\rangle_{A_1',A_2',...,A_n'} |\eta_k\rangle_{BC,anc}. \tag{34}$$

Here, the particle A_i' belongs to the sender A_i. So the total system is in the state

$$\begin{aligned}
|\Psi\rangle_{tot} &= |\psi\rangle_{A_1,A_2,...,A_n} \otimes |\Omega\rangle_{A'BC,anc} \\
&= \sum_{k=0}^{2^n-1} \alpha_k |\overline{k}\rangle_{A_1A_2,...A_n} \\
&\otimes \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} |\overline{k}\rangle_{A_1'A_2'...A_n'} |\eta_k\rangle_{BC,anc} \\
&= \frac{\alpha_0}{2^{n/2}} \Big(\sum_{k=0}^{2^n-1} |\overline{0}\rangle_{A_1A_2...A_n} |\overline{k}\rangle_{A_1'A_2'...A_n'} |\eta_k\rangle_{BC,anc} \Big) \\
&+ \frac{\alpha_1}{2^{n/2}} \Big(\sum_{k=0}^{2^n-1} |\overline{1}\rangle_{A_1A_2...A_n} |\overline{k}\rangle_{A_1'A_2'...A_n'} |\eta_k\rangle_{BC,anc} \Big) \\
\cdots & \\
&+ \frac{\alpha_{2^n-1}}{2^{n/2}} \Big(\sum_{k=0}^{2^n-1} |\overline{2^n-1}\rangle_{A_1A_2...A_n} |\overline{k}\rangle_{A_1'A_2'...A_n'} \\
&\otimes |\eta_k\rangle_{BC,anc} \Big).
\end{aligned} \tag{35}$$

Next, the senders take measurement in the Bell basis on their two-qubit systems respectively, and it is similar to that in the telecloning of two-qubit states. The target state in the present protocol is

$$|\Omega\rangle_{BC,anc} \equiv \sum_{j=0}^{2^n-1} \alpha_j |\eta_j\rangle, \tag{36}$$

which contains two optimal asymmetric clones of system B and C [10, 20]. However, we face a more complicated situation here, and there are two main steps required for reaching the target state, which also resembles two-qubit's case. First, we prove that any Bell measurement can be turned into one of the cases $\langle\Phi^{\pm}|_{A_1A_1'}\langle\Phi^{\pm}|_{A_2A_2'}...\langle\Phi^{\pm}|_{A_nA_n'}|\Psi\rangle_{tot}$ with the same parity as the former one by certain collective unitary operations. That is, one can always obtain the state

$$|\Phi\rangle_{res} = \sum_{j=0}^{2^n-1} \alpha_j(-1)^{n_j}\,|\eta_j\rangle\,, \tag{37}$$

where the sign $(-1)^{n_j}$ originates from the parity of Bell projection. In order to get this result, we notice that every term in $|\Psi\rangle_{tot}$ can be written as (the coefficient is omitted)

$$T_{lk} = |a_1, a_1'\rangle_{A_1,A_1'}\,|a_2, a_2'\rangle_{A_2,A_2'}\cdots|a_n, a_n'\rangle_{A_n,A_n'}\,|\eta_k\rangle\,, \tag{38}$$

where $k = \overline{a_1'a_2'\cdots a_n'}$, and $l = \overline{a_1a_2\cdots a_n}$ denotes the sequence number of α_l out of the bracket including this term. We call the factor $|a_m, a_m'\rangle_{A_m,A_m'}\,, \forall m$ the secondary term of T_{lk}. Moreover, if $a_m = a_m'$, then this secondary term is *even*, otherwise it is *odd*. Evidently, the even secondary term is the sum (or subtraction) of the Bell basis $|\Phi^{\pm}\rangle$, while the odd one is the sum (or subtraction) of the Bell basis $|\Psi^{\pm}\rangle$. This implies that a single Bell projection only operates on an even or odd secondary term.

Observe the terms in the l'th bracket in $|\Psi\rangle_{tot}\,, T_{l0}, T_{l1}, ...$ One can find that no two terms contain completely the same secondary terms with respect to the position of every secondary term, since the sequence number l is unchanged. Hence, there must be a uniquely residual term in every bracket after the Bell-measurement by the senders. Denote e_i the parity " $+$ " or " $-$ ", and suppose that the measurements are taken in the sequence $\{|\Psi^{e_{b_1}}\rangle_{A_{b_1}A_{b_1}'}\,, \cdots, |\Psi^{e_{bs}}\rangle_{A_{bs}A_{bs}'}\}$, and other two-qubit systems are projected onto the basis $\{|\Phi^{e_m}\rangle_{A_mA_m'}\}$. Such a projection eliminates all but one term in every bracket, which has s odd secondary terms. Concretely for the l'th bracket, the residual term is

$$\begin{aligned} T_{lk} \;\; &= \;\; |a_1, a_1\rangle_{A_1,A_1'}\cdots|a_{b_1}, \widetilde{a_{b_1}}\rangle_{A_{b_1},A_{b_1}'}\cdots|a_{b_2}, \widetilde{a_{b_2}}\rangle_{A_{b_2},A_{b_2}'} \\ \cdots \;\; & \;\; |a_{b_s}, \widetilde{a_{b_s}}\rangle_{A_{bs},A_{bs}'}\cdots|a_n, a_n\rangle_{A_n,A_n'}\,|\eta_k\rangle\,, \end{aligned} \tag{39}$$

where the tilde means the bit-shift, $\widetilde{0} = 1, \widetilde{1} = 0$. Thus $k = \overline{a_1\cdots\widetilde{a_{b_1}}\cdots\widetilde{a_{b_2}}\cdots\widetilde{a_{b_s}}\cdots a_n}$, and we must transform the term $|\eta_k\rangle$ into $|\eta_l\rangle$ for the l'th bracket simultaneously. We can realize it by virtue of a collective operation $\prod_{i=1}^{s}\sigma_{xB_{bi}}\otimes\prod_{i=1}^{s}\sigma_{xC_{bi}}\otimes\prod_{i=1}^{s}\sigma_{xa_{bi}}$ acting on the state $|\eta_j\rangle_{B_1,B_2,...,B_n,C_1,C_2,...,C_n,a_1,a_2,...,a_n}$, with the maximal value of $s = n$, namely all parties perform the local operations on their particles. This is similar to the state-transformation for two-qubit's case.

Therefore, the only term that operate in the l'th bracket is $|\eta_l\rangle$, whose sign originates from

$$\begin{aligned} &\prod\nolimits_{\substack{i=1,\\ i\neq b_1,b_2,...,bs}}^{n} \langle\Phi^{e_i}|a_i, a_i\rangle_{A_i,A_i'} \times \prod_{i=1}^{s}\langle\Psi^{e_{b_i}}|a_{b_i}, \widetilde{a_{b_i}}\rangle_{A_{b_i},A_{b_i}'} \\ &= \prod_{i=1}^{n}\langle\Phi^{e_i}|a_i, a_i\rangle_{A_i,A_i'}\,, \end{aligned} \tag{40}$$

where we have used the equation $\langle\Psi^{e_{b_i}}|a_{b_i},\widetilde{a_{b_i}}\rangle_{A_{b_i},A'_{b_i}} = \langle\Phi^{e_{b_i}}|a_{b_i},a_{b_i}\rangle_{A_{b_i},A'_{b_i}}$. This means that an arbitrary Bell measurement on the state $|\Psi\rangle_{tot}$ can be turned into the projection $\langle\Phi^{e_1}|_{A_1A'_1}\langle\Phi^{e_2}|_{A_2A'_2}...\langle\Phi^{e_n}|_{A_nA'_n}|\Psi\rangle_{tot}$, while the parity of each Bell projection is unchanged. In this process, the senders need broadcast $2n$ classical bits to inform the receivers of the results of measurement, so that the latter can carry out the state-transformations. This step is illustrated in Fig. 1.

Subsequently, we focus on the sign of every term in $|\Phi\rangle_{res}$. This step is relatively simpler. Suppose the sign solely originates in the projection of some secondary term $|1,1\rangle_{A_k,A'_k}$ (the term $|0,0\rangle_{A_k,A'_k}$ never contributes the sign), then it suffices to perform the operation $\sigma_{zB_k}\otimes\sigma_{zC_k}\otimes\sigma_{za_k}$ on $|\Phi\rangle_{res}$ to get the target state $|\Omega\rangle_{BC,anc}$ (the classical communication in the first step is available for this course). Generally, for the sign produced by several projections by the senders $A_{b_1},...,A_{b_s}$, one can recover the target state explicitly by performing the unitary operation $\prod_{i=1}^{s}\sigma_{zB_{bi}}\otimes\prod_{i=1}^{s}\sigma_{zC_{bi}}\otimes\prod_{i=1}^{s}\sigma_{za_{bi}}$. The above operation hence can be regarded as a universal parity-transformation.

In this scheme, the resource required is n ebits and $2n$ cbits in all between senders and receivers. Due to the property of Heisenberg QCM, our scheme realizes the optimal unversal asymmetric $1\rightarrow 2$ telecloning. As any pure state can always be composed of a certain number of qubits, we thus have proposed a method to the telecloning of an arbitrary multipartite state, while the expectant fidelity is also optimal due to Werner's bound.

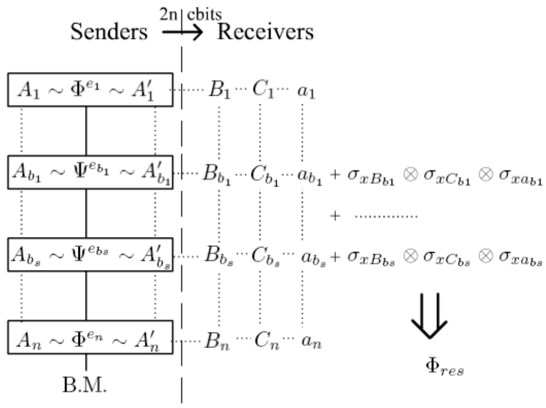

Fig. 1. Quantum telecloning of multiqubit state $|\psi\rangle_{A_1,A_2,...,A_n}$. The quantum channel between the senders and receivers is $|\Omega\rangle_{A'BC,anc}$, where each group B, C and ancilla consists of n parties, respectively. The dotted line indicates the genuine multipartite entanglement of the system. The solid line implies that simultaneously, each pair of senders A_i, A'_i (or A_{b_i}, A'_{b_i}) performs the Bell measurement (B.M.) in $\{|\Phi^{e_i}\rangle_{A_iA'_i}\}$ (or $\{\left|\Psi^{e_{b_i}}\right\rangle_{A_iA'_i}\}$). Then the senders broadcast the result to the receivers, which requires a cost of $2n$ cbits. According to the result, the state-transformation operations are implemented to obtain the state $|\Phi\rangle_{res}$.

4 Conclusions

In this paper, we addressed the problem of asymmetric quantum telecloning of arbitrary multipartite states in a universal case. Our 1→2 optimal scheme employed the Heisenberg cloning machine, which explicitly reaches Werner's fidelity bound. It is possible to realize our scheme by recent experiments. The presented scheme does not create more entanglement than that of the original state. An open problem is that how our scheme can be extended to the case of 1→ M telecloning. In addition, it is also possible to apply our scheme to the cloning of mixed states [32].

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APPENDIX The proof of nonincreasing entanglement

Here we show that $\Delta(\mu) \geq 0, \mu \in [0, 1/2]$. For the case of $\mu = 1/2$, $\Delta(1/2) = H(1) - H(\max\{0, 2F_B - 1\}) - H(\max\{0, 2F_C - 1\})$. When either of $2F_B - 1$ and $2F_C - 1$ is less than zero, the monotonicity of $H(x)$ makes $\Delta(\mu) \geq 0$. When both of them are positive, it is easy to recover the assertion by plotting the function $\Delta(1/2)$, whose independent variable is $p \in [1/3, 2/3]$. Moreover, when $\mu \in [0, 1/6]$, there is at least one zero in $C_B(p)$ and $C_C(p)$, and the monotonicity of $H(x)$ makes $\Delta(\mu) \geq 0$ again. This implies that for the inputs with $\mu \leq 1/6$, it is impossible to create entanglement in both of the clones simultaneously by our scheme. So it suffices to investigate the case of $\mu \in (1/6, 1/2)$, where both $C_B(p)$ and $C_C(p)$ must be positive. Recall that $q = 1 - p$, we have

$$F_B(p) = \frac{1 + 3p^2}{4 - 6p + 6p^2} > \frac{\mu + 1}{4\mu + 1},$$
$$F_C(p) = \frac{4 - 6p + 3p^2}{4 - 6p + 6p^2} > \frac{\mu + 1}{4\mu + 1},$$

namely

$$p \in \Big(\frac{1 + \mu - \sqrt{4\mu + \mu^2}}{1 - 2\mu}, \frac{-3\mu + \sqrt{4\mu + \mu^2}}{1 - 2\mu}\Big).$$

Mathematically, we only need calculate the derivative of $\Delta(\mu)$ with respect to p, but it is difficult to do it in this way because of the confused deduction. Notice that $F_B(p) = F_C(1-p)$, so $H(C_B(p)) = H(C_C(1 - p))$, i.e., they are symmetric and the symmetry axis is $p = 1/2$. Thus we focus on the property of $H(C_B(p))$. As $C_B(p)$ is monotonically increasing with p, $H(C_B(p))$ is also monotonically increasing with p. Calculate the second derivative of $H(C_B(p))$ with respect to p,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}p^2} H(C_B(p)) &= \frac{\mathrm{d}}{\mathrm{d}p}\Big(\frac{\mathrm{d}H}{\mathrm{d}C}\frac{\mathrm{d}C}{\mathrm{d}p}\Big) \\
&= \frac{\mathrm{d}^2 H}{\mathrm{d}C^2}\Big(\frac{\mathrm{d}C}{\mathrm{d}p}\Big)^2 + \frac{\mathrm{d}H}{\mathrm{d}C}\frac{\mathrm{d}^2 C}{\mathrm{d}p^2} \\
&= \lambda\Big[\Big(\frac{8}{3}\mu + \frac{2}{3}\Big)\frac{2\sqrt{1 - C^2} + \log_e \frac{1-\sqrt{1-C^2}}{1+\sqrt{1-C^2}}}{C(1 - C^2)\log_e \frac{1-\sqrt{1-C^2}}{1+\sqrt{1-C^2}}} \\
&+ \frac{4(2 - 3p + 3p^2)(5 - 9p - 9p^2 + 9p^3)}{3(-1 - 2p + 3p^2)^2}\Big],
\end{aligned}$$

where $\lambda = \left(\frac{8}{3}\mu + \frac{2}{3}\right)\left(\frac{\mathrm{d}F_B}{\mathrm{d}p}\right)^2 \frac{\mathrm{d}H}{\mathrm{d}C}$ is positive. As the first part and second part in the square bracket are monotonically decreasing with $C = C_B(p)$ and p respectively, $\frac{\mathrm{d}^2}{\mathrm{d}p^2} H(C_B(p))$ is monotonically decreasing with p. By virtue of plotting it is easy to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} H(C_B(p))\bigg|_{p=0.56} > 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}p} H(C_B(p))\bigg|_{p=2/3} > \frac{\mathrm{d}}{\mathrm{d}p} H(C_B(p))\bigg|_{p=0.44}.$$

Although the point $p = 2/3$ is usually not in the physical region $\left(\frac{1+\mu-\sqrt{4\mu+\mu^2}}{1-2\mu}, \frac{-3\mu+\sqrt{4\mu+\mu^2}}{1-2\mu}\right)$, the above argument mathematically applies to the region $\left(\frac{1+\mu-\sqrt{4\mu+\mu^2}}{1-2\mu}, \frac{2}{3}\right]$. Thus the inflection point of $H(C_B(p))$ is $p_{in} > 0.56$. Consider the sum of the $H(C_B(p))$ and $H(C_C(p))$, where $H(C_B(p)) = H(C_C(1-p))$. When $p \in \left[p_{in}, \frac{-3\mu+\sqrt{4\mu+\mu^2}}{1-2\mu}\right]$,

$$\frac{\mathrm{d}}{\mathrm{d}p}[H(C_B(p)) + H(C_C(p))] >$$

$$\frac{\mathrm{d}}{\mathrm{d}p} H(C_B(p))\bigg|_{p=2/3} - \frac{\mathrm{d}}{\mathrm{d}p} H(C_B(p))\bigg|_{p=0.44} > 0,$$

and when $p \in \left[1/2, p_{in}\right]$, one readily obtains $\frac{\mathrm{d}}{\mathrm{d}p}[H(C_B(p)) + H(C_C(p))] > 0$ as the reflection point $p_{in} > 0.56$. So $H(C_B(p)) + H(C_C(p))$ is monotonically increasing when $p \in [1/2, \frac{-3\mu+\sqrt{4\mu+\mu^2}}{1-2\mu}]$. Due to the symmetry of $H(C_B(p))$ and $H(C_C(p))$, the maximum of $\Delta(\mu)$ is in the bound $p = \frac{1+\mu-\sqrt{4\mu+\mu^2}}{1-2\mu}$ or $\frac{-3\mu+\sqrt{4\mu+\mu^2}}{1-2\mu}$. This is just the case where $C_B(p)$ or $C_C(p)$ vanishes, and thus $\Delta(\mu) \geq 0$. So we conclude that our scheme of 2-qubit telecloning will never create more entanglement than that contained in the original state.