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ESTIMATION OF THE HEATING RATE OF IONS DUE TO LASER FLUCTUATIONS WHEN IMPLEMENTING QUANTUM ALGORITHMS

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We analyze numerically the heating of trapped ions due to laser intensity and phase fluctuations when implementing Grover's algorithm and the Quantum Fourier Transform. For a simpler analysis we assume that the stochastic processes are white noise processes and average over each noise as in [Phys. Rev. A. **57**, 3748, (1998)]. We investigate the fidelity and the heating rate for these algorithms using parameters estimated from experiments, and we can see the order of magnitude difference in the heating rate depending on the quantum algorithms.

Keywords: Ion-trap quantum computation, decoherence, quantum algorithm
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1 INTRODUCTION

Intense theoretical and experimental research in quantum computation has been performed since Shor discovered the fast algorithm for factorization[1]. Quantum computation exploits quantum-mechanical two-level systems (qubits) for information processing and several physical systems for implementing a quantum computer (QC) have been suggested up to date. Especially the ion-trap quantum computation scheme, which was first proposed by Cirac and Zoller in 1995[2], is promising because the superposition of quantum states has long coherence time and there is the possibility of expanding the number of qubits. Other types of ion-trap quantum computation schemes have been suggested [3, 4, 5, 6, 7] and in parallel many experiments have also been realized: the Deutsch-Jozsa algorithm[8], the Cirac-Zoller controlled-NOT gate[9, 10], the robust high-fidelity geometric two ion-qubit phase gate[11]. The ultimate challenge now is the development of scalable ion-trap QC for practical calculations.

However, we need to overcome the problem of decoherence for implementing a reliable QC [12]. Laser and magnetic- field fluctuations are known to be large sources of decoherence in recent experiments[13, 14]. Moreover, a considerable amount of research has been dedicated to analyzing the decoherence, especially heating of ions: analysis of (i) the Hamiltonian which includes the fluctuating terms[15, 16], (ii) the interaction between the ion-trap system and the surrounding environment[17, 18, 19, 20, 21, 22, 23, 24, 25], (iii) the estimation of the accuracy of quantum algorithms[26, 27, 28, 29], and (iv) the relation between experimental

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and theoretical data [12, 30, 31, 32, 33]. In general the maintenance of the trapped ions at very low temperature is very important for accurate quantum computation. Heating mainly occurs due to the coupling between noisy classical electromagnetic fields and ions. However, the imperfect transfer between the internal states is also leading to the heating of ions when implementing quantum algorithms through sideband transitions.

The simulation of quantum algorithms taking into consideration decoherence has been performed in [34, 35]. The authors proposed a general method for implementing a practical controlled-Z gate by adjusting the laser phase based on the Cirac-Zoller proposal and simulated Grover's algorithm with the laser fluctuations in [36]. In real experiments there are limitations not only due to the inevitable physical process but also due to the technical problems. Thus the reduction of the latter effect is critical for realizing reliable quantum computation in future experiments.

In this paper we focus on an inevitable source of decoherence, laser intensity and phase fluctuations and present a numerical analysis of the heating of trapped ions due to these fluctuations when implementing benchmark algorithms, like Grover's algorithm[39] and the Quantum Fourier Transform (QFT) using parameters estimated from experiments[9]. When comparing the results of the fidelity and the distribution of the collective motion of ions between these different algorithms, we set the same number of qubits in the quantum networks. Finally we make the comparison of the normalized heating rate. For the simple analysis, we assume that the stochastic processes are white noise processes and take the average over each noise as in [15, 16].

In Sec.II we first present important transitions in ion-trap QC, and the way of implementing Grover's algorithm and the QFT in the ion-trap system. In Sec. III we show the derivation of the master equation including the effects of both laser intensity and phase fluctuations and Sec.IV is devoted to the investigation of these effects when implementing Grover's algorithm and the QFT. In Sec.V we discuss the results and finally draw the conclusions in Sec.VI.

2 ION-TRAP QC AND QUANTUM ALGORITHMS

We consider the situation that n two-level ions (the mass of each ion is m and the atomic transition frequency is ω_A) are bound in a harmonic trap and constrained to move in one direction of the trap axis at harmonic frequency ν. The ground and excited states are described as $|0\rangle$ and $|1\rangle$. We assume that a single laser with wave vector k_L and frequency ω_L irradiates the kth ($k = 1, 2, \cdots, n$) ion for causing the laser-ion interaction. In this condition, the interaction Hamiltonian in rotating-wave approximation is

$$H_k = \hbar(\frac{\Omega_0}{2})|1_k\rangle\langle 0_k| \exp(i\eta(ae^{-i\nu t} + a^\dagger e^{i\nu t}) + i(\phi_0 - \delta t)) + \text{H.c.}, \tag{1}$$

where the subscript k refers to the target ion, Ω_0 is the Rabi frequency, $\eta = k_L(\hbar/2m\nu)^{1/2}$ is the Lamb-Dicke parameter, a and $a^\dagger$ are the annihilation and creation operators of the collective motion, ϕ_0 is the laser phase, and $\delta = \omega_A - \omega_L$ is the detuning. Moreover, we can expand the interaction Hamiltonian to the lowest order of η when assuming that the amplitude of the ions' motion in the direction of the laser field is much less than a wavelength as in [15, 16].

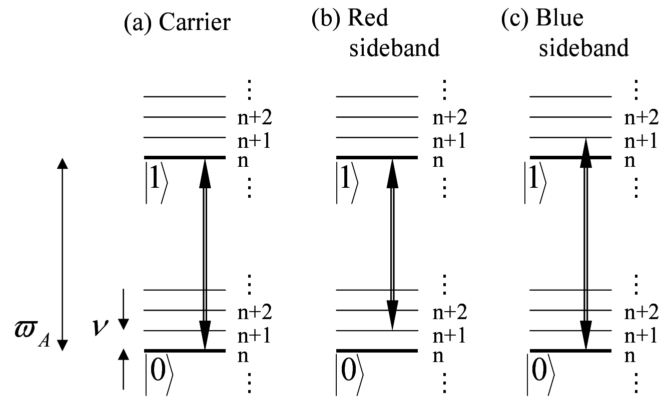

Fig. 1. Three kinds of transitions, (a) carrier: $|0\rangle|n\rangle \leftrightarrow |1\rangle|n\rangle$, (b) red sideband: $|0\rangle|n+1\rangle \leftrightarrow |1\rangle|n\rangle$, and (c) blue sideband: $|0\rangle|n\rangle \leftrightarrow |1\rangle|n+1\rangle$.

In the Cirac-Zoller proposal[2] and the controlled-NOT gate experiment[9, 10], one needs the following three kinds of transitions (Fig.1), carrier ($\delta = 0$), red sideband ($\delta = -\nu$) and blue sideband ($\delta = \nu$):

$$H_k^{(c)} = \hbar(\frac{\Omega_0}{2})(1 - \eta^2 a^\dagger a)(|1_k\rangle\langle 0_k|e^{i\phi_0} + \text{H.c.}), \tag{2}$$

$$H_k^{(r)} = \hbar(\frac{\Omega_0}{2})(\frac{\eta}{\sqrt{n}})(a|1_k\rangle\langle 0_k|e^{i\phi_0} + \text{H.c.}), \tag{3}$$

$$H_k^{(b)} = \hbar(\frac{\Omega_0}{2})(\frac{\eta}{\sqrt{n}})(a^\dagger|1_k\rangle\langle 0_k|e^{i\phi_0} + \text{H.c.}). \tag{4}$$

The initial condition of the collective motion needs to be cooled down to $|v = 0\rangle$ (the Lamb-Dicke region is satisfied). Then we can derive the time-evolution operator of the carrier, red and blue sideband transitions from Eq.(2), (3) and (4) by ignoring the η terms :

$$\hat{V}_k^{\sigma+}(\theta, \phi) = \exp[-i\frac{\theta}{2}\left(|1_k\rangle\langle 0_k|e^{i\phi_0} + \text{H.c.}\right)], \tag{5}$$

$$\hat{U}_k^{\sigma+}(\theta, \phi) = \exp[-i\frac{\theta}{2}\left(a|1_k\rangle\langle 0_k|ae^{i\phi_0} + \text{H.c.}\right)], \tag{6}$$

$$\hat{R}_k^{\sigma+}(\theta, \phi) = \exp[-i\frac{\theta}{2}\left(a^\dagger|1_k\rangle\langle 0_k|ae^{i\phi_0} + \text{H.c.}\right)], \tag{7}$$

where $\theta = l\pi$ with l depending on the laser radiation time t (carrier: $t = l\pi/\Omega_0$ and red and blue sideband: $t = l\pi/\left(\Omega_0\eta/\sqrt{n}\right)$), and σ_+ is the laser polarization. One can utilize an auxiliary state $|aux\rangle$ (degenerated from $|1\rangle$) by using the σ^- laser polarization instead of the normal σ^+ polarization in order to create a new transition between $|0\rangle$ and $|aux\rangle$. One can implement any quantum gate with the proper combination of the above transitions and the radiation time of the lasers.

Next, we briefly review Grover's algorithm and the QFT, and their implementation method in the ion-trap QC. Grover's algorithm is a fast database-search algorithm utilizing the amplitude amplification. The steps of this algorithm consist of three kinds of operations:

(i)preparation of a superposition of quantum states with equal amplitude, (ii)repetition of the following two operations: (a)inversion of the amplitude of the desired state and (b)inversion about average of the amplitude of all states , and at last (iii)measurement of the quantum state. When simulating Grover's algorithm, we use the same method as in [36].

The QFT is the quantum circuit which performs the discrete Fourier transform. For an orthonormal state vector $|a\rangle (a = 0, 1, \cdots, 2^n - 1)$, we have

$$\mathrm{QFT}(|a\rangle) = \frac{1}{\sqrt{2^n}} \exp(2\pi ac/2^n)|c\rangle. \tag{8}$$

The QFT circuit consists of two kinds of quantum gates: the Hadamard gate and the conditional phase-shift gate. This conditional rotation of α is implemented by the following sequence of the laser manipulations:

$$\hat{U}_c^{\sigma_+}(\pi, 0)\, \hat{U}_t^{\sigma_-}(\pi, 0)\, \hat{U}_t^{\sigma_-}(\pi, -(\pi+\alpha))\, \hat{U}_c^{\sigma_+}(\pi, -(\pi+\alpha))\,, \tag{9}$$

where c denotes the control bit and t denotes the target bit.

3 LASER FLUCTUATIONS

In this paper we consider the decoherence due to the laser intensity and phase fluctuations because they cause large errors in the controlled-NOT gate experiment[9]. We need to change Ω_o and ϕ_o in Eq.(2), (3) and (4) to $\Omega(t)$ and $\phi(t)$ depending on time t and describe these fluctuations using white-noise as in [15]:

$$\Omega_o \quad \rightarrow \quad \Omega(t)dt = \Omega_o[dt + \sqrt{\Gamma} dW(t)], \tag{10}$$

$$\phi_o \quad \rightarrow \quad \phi(t) = \phi_o + \sqrt{\gamma} W(t), \tag{11}$$

where $W(t)$ is a Wiener process, $dW(t)$ is its increment, and the parameters Γ and γ scales the intensity and phase noise. The physical meaning of these parameters is explained in [15, 40]; Γ is the ratio of the rms fluctuations in the pulse area to the deterministic pulse area and γ is the band-width (HWHM).

We first use the stochastic Liouville-von Neumann equation with considering the effect of laser intensity fluctuations. We thus set $\gamma = 0$ and average over the noise terms, therefore obtaining the following master equation:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho] - \frac{\Gamma}{2\hbar^2}[H, [H, \rho]], \tag{12}$$

with H as described in Eqs.(5),(6), and (7).

Next, we consider the phase fluctuations, however this case includes some difficulties in deriving the master equation. Therefore, we take the same process as in [15] and obtain the following equations:

$$\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar}[H, \tilde{\rho}] - \frac{\gamma}{\hbar^2}[\sigma_+\sigma_-, [\sigma_+\sigma_-, \tilde{\rho}]], \tag{13}$$

where

$$\phi(t) \quad \rightarrow \quad \tilde{\phi}(t) = \phi(t) - \phi_0 \tag{14}$$

$$\rho \quad \rightarrow \quad \tilde{\rho} = \exp[-i\tilde{\phi}(t)\sigma_+\sigma_-]\ \rho\ \exp[i\tilde{\phi}\sigma_+\sigma_-], \tag{15}$$

$$H \quad \rightarrow \quad H_0 = H - \tilde{\phi}(t)\sigma_+\sigma_-/2. \tag{16}$$

The time evolution of the populations in three types of transitions in the instantaneous transformed frame are the same as those in the original frame. Therefore, it is valid to calculate the fidelity and the distribution of the collective motion of ions when implementing quantum algorithms by using Eq.(15).

In addition, we can derive the following equation for the case in which both intensity and phase fluctuations exist at the same time by using the frame $\tilde{\rho}$:

$$\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar}[H, \tilde{\rho}] - \frac{\Gamma}{2\hbar^2}[H, [H, \tilde{\rho}]] - \frac{\gamma}{\hbar^2}[\sigma_+\sigma_-, [\sigma_+\sigma_-, \tilde{\rho}]], \tag{17}$$

Here we ignore the correlation between the intensity and phase fluctuations, which means that the product of the increment of a Weiner process for the intensity and phase fluctuations is equal to zero.

4 Results

In our simulation we set the Lamb-Dicke parameter as $\eta = 0.052$, the experimental value used in [9, 10]. Next, we take into consideration the gate operation time and obtain the Rabi frequency as $\Omega_0 = 121.3$[kHz]. Moreover, for simulating the quantum algorithms with decoherence, we estimate the parameters Γ and γ with respect to the error budget in [9]. Concretely speaking, the effect of the laser intensity fluctuations contributes to the error budget by 1%, and the contribution of the laser phase noise is 10%. We simulate the controlled-NOT gate by solving Eqs.(12) and (17) with the fourth-order Runge Kutta and set $\Gamma = 5.3 \times 10^{-8}$ and $\gamma = 9.7 \times 10^2$. Here we assume that the laser radiation time is well controlled and the perfect initialization and measurement are realized. Moreover, we ignore the effects of spontaneous emissions because the coherence time in an ion-trap QC with small number of qubits is long. Thus, we simulate the quantum networks for Grover's algorithm and the QFT taking into consideration the effects of the intensity and phase fluctuations by solving Eq.(12) and (17).

First, we show the relationship between the number of amplitude amplifications in Grover's algorithm (four-qubit case) and the fidelity in Fig.2(a), and the probability of finding the desired state in Fig.2(b). We set the desired state as $|2^n - 1\rangle$ in the n-qubit case and investigate the effects of laser intensity and phase fluctuations. We obtain the fidelity from the following equation:

$$F = \sqrt{\langle \psi_{ideal} | \rho | \psi_{ideal} \rangle}, \tag{18}$$

where $|\psi_{ideal}\rangle$ is the ideal quantum state and ρ is the density matrix in the case of considering the effect of the decoherence.

As Fig.2(a) and (b) show, the intensity fluctuations are dominant factors in the decay of the fidelity and probability of finding the desired state. From Fig.2(b) we can see that the optimal number of amplitude amplifications is one because both effects exist at the same time in a real laser and this result is the same as in [36]. This optimal number is always one in the case of the different number of qubits. Therefore, the accurate control of the intensity fluctuations is necessary for overcoming the decay of the fidelity and the probability of searching the desired state.

In Fig.3(a) we show the relationship between the number of qubits in Grover's algorithm and the fidelity. The desired state is $|2^n - 1\rangle$ in each n-qubit case. Fig.3(b) describes the case

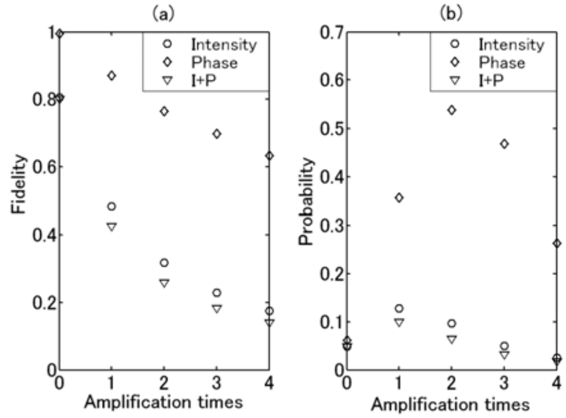

Fig. 2. (a)The relationship between the number of amplitude amplifications in Grover's algorithm (four-qubit case) and the fidelity. (b)The relationship between the number of amplitude amplifications and the probability of finding the desired state $|15\rangle$.

of the QFT networks and we set the initial condition as $|0\rangle$. In Grover's case we utilize the results after one iteration of Grover's operation.

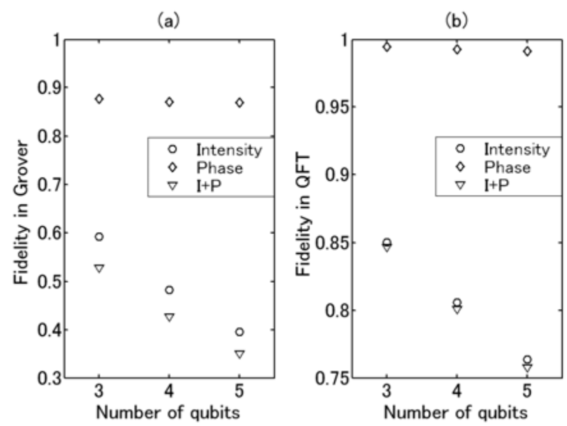

Fig. 3. (a)The relationship between the number of qubits and the fidelity after implementing Grover's algorithm (one iteration of Grover's operation). The desired state is $|2^n - 1\rangle$ in each n-qubit case. (b)The relationship between the number of qubits and the fidelity after implementing the QFT networks. The initial condition is $|0\rangle$ in each n-qubit case.

From this result we can see that intensity fluctuations strongly affect the decay of the fidelity in both results, and the effect of laser fluctuations in Grover's algorithm is much larger than in the QFT networks.

In order to calculate the heating rate, we also obtain the distribution of the collective motion of ions after implementing Grover's algorithm and the QFT. In this analysis the probability remaining in $|v = 0\rangle$ is almost one even when considering both types of fluctuations in the QFT networks. Moreover, we can see that the intensity fluctuations cause slightly larger effects to the leakage of the probability from $|v = 0\rangle$. We need to treat the distribution quantitatively and then show the total heating after implementing (a)Grover's algorithm and (b)the QFT networks as in Fig.4. The value of the heating $d\langle n\rangle$ is obtained from the probability of the distribution of the collective motion $|v\rangle$ and the number of the collective motions. In the case of Grover's algorithm, we utilize the distribution after one iteration of Grover's operation. It is indicated that the result considering both the intensity and phase fluctuations is larger than the sum of that of each effect from Fig.4. Especially this tendency appears most clearly in the QFT result (Fig.4 (b)). This is not because the algorithms are different but because the effects of decoherence are different in these algorithms.

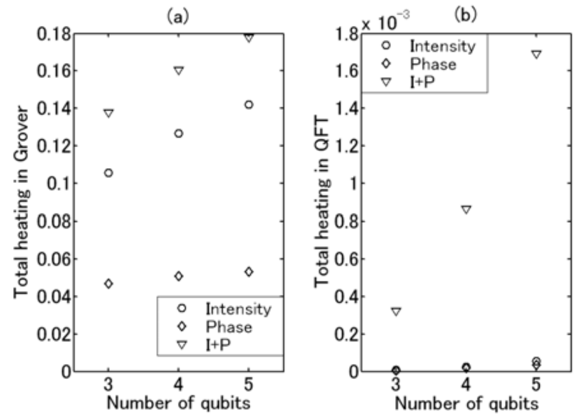

Fig. 4. (a)The total heating of ions ($d\langle n\rangle$) after implementing Grover's algorithm and (b)after implementing the QFT networks.

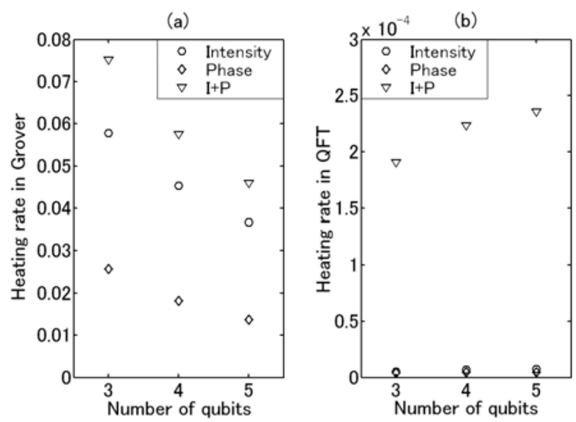

Fig. 5. (a)The heating rate of ions ($d\langle n\rangle/dt(/ms)$) after implementing Grover's algorithm and (b)after implementing the QFT networks.

Figure 5 shows the heating rate after implementing (a)Grover's algorithm and (b)the QFT networks. The heating rate is $d\langle n\rangle/dt$ and we obtain this value from dividing $d\langle n\rangle$ (the value in Fig.4) by the gate operation time of each qubit case. From Fig.5 we can see the different tendencies in Grover's algorithm and the QFT networks. We consider that the reason is as follows: the effect of the laser fluctuations on the total heating is slightly larger in Grover's case even if the number of qubits is small, therefore this leading to a smaller increase rate of the total heating in Grover's algorithm than that in the QFT as in Fig.4. Additionally we can see the order of magnitude difference in the heating rate between Grover's algorithm and the QFT: $O(10^{-2})$ and $O(10^{-4})$. As far as the results for each effect, the order for Grover and the QFT is $O(10^{-2})$ and $O(10^{-6})$ respectively, and in Fig.5 there is the same tendency as in Fig.4.

5 DISCUSSION

First, we discuss the relationship between our calculation and experimental results. Grover's algorithm was implemented in [41] with the fidelity 0.6. In this scheme the conditional-phase shift gate was realized with the entangled gate([5]), however approximately the same fidelity will be implemented in the two-qubit case if using our conditional-phase shift gate. In our scheme we utilize combinations of sideband transitions, therefore the reduction of the fidelity will be proportional to the number of these transitions. It is natural that the reduction of the fidelity in the experimental realization of the entangled gate of [5] occurs when the number of ions increases.

Next, we investigate the heating rate which we obtained numerically by comparing it with the value estimated from the experiment[30]. In [30], the heating (motional decoherence) is investigated as the following processes: after sideband cooling, the system is left alone to interact with the environment for a delay time t. After that, the measurement is done by looking at the Rabi-flopping signal with the blue sideband laser. From this experiment the heating rate is estimated as $d\langle n\rangle/dt = 0.0053ms^{-1}$, but this is caused mainly by the interaction with the environment. However, we consider the heating by the laser intensity and phase fluctuations during the operation of quantum algorithms. We indicated that the imperfect transitions between the internal states of the ions due to the laser fluctuations lead to heating during the operations of some quantum algorithms, therefore, we should take into consideration the magnitude of this heating in a real experiment.

We also consider the rapid decrease of the fidelity in Grover's algorithm. From Fig.2 we can see the dramatic reduction of the fidelity after the one amplitude amplification, and this value becomes lower than 0.5. However, we can confirm that the leakage probability from $|v = 0\rangle$ is about 0.15, the remaining probability in the auxiliary states is approximately 0.38 as in [36]. Therefore, we can say that these low fidelity values are valid. Moreover, from Fig.2(a) the effect of the intensity fluctuations in the Hadamard gate is much larger than that of the phase fluctuations. This might be the reason for the fact that the intensity fluctuations cause the dramatic decrease of the fidelity in Grover's algorithm.

Finally, we discuss the range in which the parameters Γ and γ should be found so that the quantum algorithms are realized with high fidelity. This is very important for experimentalists who wish to compare to their available laser capabilities. Also of great use would be a prediction of how narrow a laser is required to be for the benchmark algorithms with different numbers of qubits. From Fig.3.(b) the effect of the fluctuations of the laser phase is much smaller than that of the laser intensity in the fidelity result, therefore, in Fig.6 we show the relationship between the scale of Γ and the fidelity for the QFT case. Moreover, from the analysis of the heating rate, we found that the heating rate is decreased to $O(10^{-3})$ by

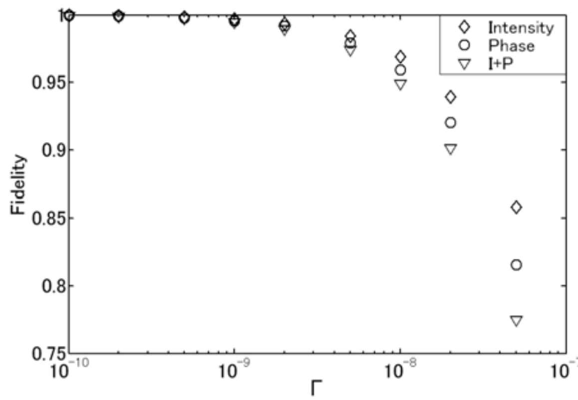

Fig. 6. The relationship between the scale of Γ and the fidelity in the QFT case.

reducing the order of Γ and γ by $O(10^{-1})$. On the other hand, we also calculate the case of Grover's algorithm with five qubits for the specific case: $\Gamma = 0.05 \times 10^{-8}$ and $\gamma = 10$ (we reduce orders by $O(10^{-2})$ from the estimated value). Consequently, the number of iterations for the Grover operator comes to be same as in the ideal case. The searching probability, fidelity and heating rate become 0.946, 0.973 and 3.75×10^{-3}(/ms), for intensity fluctuations, and 0.988, 0.994 and 4.14×10^{-4}(/ms) for phase fluctuations. Therefore, we can show that the implementation of the QFT with the high fidelity is comparatively easier than that of Grover by improving the laser fluctuations.

6 CONCLUSION

In summary, we analyzed the heating of trapped ions due to the laser intensity and phase fluctuations using the parameters which are estimated from experiments. Moreover, we investigated the effects of each decoherence when implementing Grover's algorithm and Quantum Fourier Transform. As result, we can see the different characteristics of the fidelity, distribution of the collective motion, and heating rate between these two algorithms. We think that further analysis of many kinds of decoherence in each quantum algorithm and in each physical system is needed for implementing reliable QC.

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