GENERAL CLASSES OF IMPOSSIBLE OPERATIONS
THROUGH THE EXISTENCE OF INCOMPARABLE STATES

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In this work we show that the most general class of anti-unitary operators are nonphysical in nature through the existence of incomparable pure bipartite entangled states. It is also shown that a large class of inner-product-preserving operations defined only on the three qubits having spin-directions along x,y and z are impossible. If we perform such an operation locally on a particular pure bipartite state then it will exactly transform to another pure bipartite state that is incomparable with the original one. As subcases of the above results we find the nonphysical nature of universal exact flipping operation and existence of universal Hadamard gate. Beyond the information conservation in terms of entanglement, this work shows how an impossible local operation evolve with the joint system in a nonphysical way.

Keywords: Incomparability, LOCC, Entanglement

Communicated by: R Jozsa & B Terhal

1   Introduction

Quantum systems allow physical operations to perform some tasks that seems to be impossible in classical domain [1, 2, 3]. However with the nature of the operations performed it restricts correctness or exact behavior of the operations to act for the whole class of states of the quantum system. Possibilities or impossibilities of various kind of such operations acting on some specified system is then one of the basic tasks of quantum information processing. In case of cloning and deleting the input states must be orthogonal to each other for the exactness of the operation performed [4, 5, 6, 7, 8]. Rather if the operation considered is spin-flipping [9, 10, 11] or Hadamard type then the input set of states enhanced to a great circle of the Bloch sphere [12, 13, 14]. It indicates that any angle preserving operation has some restriction on the allowable input set of states. The unitary nature of all physical evolution [15] raised the question that whether the non-physical nature of the anti-unitary operations is a natural constraint over the system or not. In other words, it is nice to show how an impossible operation like anti-unitary, evolve with the physical systems concerned.

First part of this paper concerns with a connection between general anti-unitary operations and evolution of a joint system through local operations together with classical communications, in short LOCC. Some constraint over the system are always imposed by the condition

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that the system is evolved under LOCC. For example, performing any kind of LOCC on a joint system shared between distinct parties, the amount of entanglement between some spatially separated subsystems can not be increased. If we further assume that the concerned system is pure bipartite, then by Nielsen's criteria [16, 17] it is possible to determine whether a pure bipartite state can be transformed to another pure bipartite state with certainty by LOCC or not. Consequently we find that there are pairs of pure bipartite states, denoted by incomparable states which are not interconvertible under LOCC with certainty. The existence of such class of states prove that the amount of entanglement does not always determine the possibility of exact transformation of a joint system by applying LOCC. Now we first pose the problem that would be discussed in this paper.

Suppose $\rho_{ABCD...}$ be a state shared between distinct parties situated at distant locations. They are allowed to do local operations on their subsystems and also they may communicate any amount of classical information among themselves. But they do not know whether their local operations are valid physical operations or not. By valid physical operation we mean a completely positive map (may be trace-preserving or not) acting on the physical system. Sometimes an operation is confusing in the sense that it works as a valid physical operation for a certain class of states but not as a whole. Therefore they want to judge their local operations using quantum formalism or other physical principles, may be along with quantum formalism or may not be. No-signalling, non-increase of entanglement by LOCC are some of the good detectors of nonphysical operations [18, 19, 20, 21, 22, 23]. In this paper we want to establish another good detector for a large number of nonphysical operations. The existence of incomparable states enables us to find that detector. Suppose $L_A \otimes L_B \otimes L_C \otimes L_D \otimes \cdots$ be an operation acting on the physical system represented by $\rho_{ABCD...}$ and $\rho'_{ABCD...}$ be the transformed state. Now it is known that the states $\rho_{ABCD...}$ and $\rho'_{ABCD...}$ are incomparable by the action of any deterministic LOCC, then we could certainly say that at least one of the operations $L_A, L_B, L_C, L_D, \cdots$ are nonphysical. Therefore if somehow we find two states that are incomparable and by an operation acting on any party (or a number of parties) one state is transformed to another then we certainly claim that the operation is a nonphysical one. We find several classes of nonphysical operations through this procedure and it is our main motivation in this work. The paper is organized as follows: in section 2 we describe what we actually mean by a physical operation and its relation with LOCC. In section 3 we describe the notion of incomparability for pure bipartite entangled states. In section 4 we show the nonphysical nature of the most general class of universal exact anti-unitary operators through the impossibility of inter-converting two incomparable states by deterministic LOCC. Lastly, in section 5 we show a large class of inner-product preserving operations are also non physical in nature, including the Hadamard operation. As a subcase of the above operations we reproduce the nonexistence of exact universal flipping machine [26]. In all the above cases we have tried to use minimum number of qubits (only on three spin directions along x, y, z) and the quantum system considered as simple as possible. Also the states considered here to prove the impossibilities are pure entangled states.

2   Physical Operations and LOCC

In this section we first describe the notion of a physical operation in the sense of Kraus [15]. Suppose a physical system is described by a state ρ. By a physical operation on ρ we mean

a completely positive map $\mathcal{E}$ acting on the system and described by

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \tag{1}$$

where each A_k is positive linear operator that satisfies the relation $\sum_k A_k^\dagger A_k \leq I$. If $\sum_k A_k^\dagger A_k = I$, then the operation is trace preserving. When the state is shared between a number of parties, say, A, B, C, D,. .... and each A_k has the form $A_k = L_k^A \otimes L_k^B \otimes L_k^C \otimes L_k^D \otimes \cdots$ with all the $L_k^A, L_k^B, L_k^C, L_k^D, \cdots$ are linear positive operators, the operator is then called a separable superoperator. In this context we would like to mention an interesting result concerned with LOCC. Every LOCC is a separable superoperator but it is unknown to us whether the converse is also true or not. It is further affirmed that there are separable superoperators which cannot be expressed by finite LOCC [24]. Now if a physical system evolved under LOCC (may be deterministic or stochastic) then quantum mechanics does not allow the system to behave arbitrarily. More precisely, under the action of any LOCC one could find some fundamental constraints over any entangled system. The content of entanglement will not increase under LOCC. This is usually known as the principle of non-increase of entanglement under LOCC. Further for any closed system as unitarity is the only possible evolution, the constraint is then: the entanglement content will not change under LOCC. So if we find some violation of these principles under the action of any local operation, then we certainly claim that the operation is not a physical one. No-cloning, no-deleting, no-flipping, all those theorems are already established with these principles, basically with the principles of non-increase of entanglement [22, 23]. These kind of proof for those important no-go theorems will always give us a more powerful physically intuitive approaches for quantum information processing apart from the mathematical proofs that the dynamics should be linear as well as unitary. Linearity and unitarity are the building blocks of every physical operation [15, 25]. But within the quantum formalism we always search for better physical situations that are more useful and intuitive for quantum information processing. Existence of incomparable states in pure bipartite entangled systems allow us to use it as a new detector. We have already proved three impossibilities, viz., exact universal cloning, deleting and flipping operations by the existence of incomparable states under LOCC [26, 27] and we would provide some further classes of nonphysical operations in this paper.

3　Notion of Incomparability

To present our work we need to define the condition for a pair of states to be incomparable with each other. The notion of incomparability of a pair of bipartite pure states directly follows from the necessary and sufficient condition for conversion of a pure bipartite entangled state to another by deterministic LOCC, i.e., with probability one. It is prescribed by M. A. Nielsen [16, 17]. Suppose we want to convert the pure bipartite state $|\Psi\rangle$ of $d \times d$ system to another state $|\Phi\rangle$ shared between two parties, say, Alice and Bob by deterministic LOCC. Consider $|\Psi\rangle$, $|\Phi\rangle$ in their Schmidt bases $\{|i_A\rangle, |i_B\rangle\}$ with decreasing order of Schmidt coefficients: $|\Psi\rangle = \sum_{i=1}^d \sqrt{\alpha_i}|i_A i_B\rangle$, $|\Phi\rangle = \sum_{i=1}^d \sqrt{\beta_i}|i_A i_B\rangle$, where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$, for $i = 1, 2, \cdots, d-1$, and $\sum_{i=1}^d \alpha_i = 1 = \sum_{i=1}^d \beta_i$. The Schmidt vectors corresponding to the states $|\Psi\rangle$ and $|\Phi\rangle$ are $\lambda_\Psi \equiv (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $\lambda_\Phi \equiv (\beta_1, \beta_2, \cdots, \beta_d)$. Then Nielsen's criterion says $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible with certainty under LOCC if and only if λ_Ψ is majorized by λ_Φ,

denoted by $\lambda_\Psi \prec \lambda_\Phi$ and described as,

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \ \ \forall \ \ k = 1, 2, \cdots, d \tag{2}$$

It is interesting to note that however majorization [28] criteria is an algebraic tool, it shows great applicability in different context of quantum information processing [29, 30, 31, 32]. Now, as a consequence of non-increase of entanglement by LOCC, if $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible under LOCC with certainty, then $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ [where $E(\cdot)$ denote the von-Neumann entropy of the reduced density operator of any subsystem and known as the entropy of entanglement]. If the above criterion (2) does not hold, then it is usually denoted by $|\Psi\rangle \nrightarrow |\Phi\rangle$. Though it may happen that $|\Phi\rangle \rightarrow |\Psi\rangle$ under LOCC. If it happens that $|\Psi\rangle \nrightarrow |\Phi\rangle$ and $|\Phi\rangle \nrightarrow |\Psi\rangle$ then we denote it as $|\Psi\rangle \nleftrightarrow |\Phi\rangle$ and describe $(|\Psi\rangle, |\Phi\rangle)$ as a pair of incomparable states [16, 33]. One of the peculiar feature of such incomparable pairs is that we are unable to say that which state has a greater amount of entanglement content than the other. Also for 2×2 systems there are no pair of pure entangled states which are incomparable to each other. For our purpose, we now explicitly mention the criterion of incomparability for a pair of pure entangled states $|\Psi\rangle, |\Phi\rangle$ of $m \times n$ system where $\min\{m, n\} = 3$. Suppose the Schmidt vectors corresponding to the two states are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively, where $a_1 > a_2 > a_3$, $b_1 > b_2 > b_3$, $a_1 + a_2 + a_3 = 1 = b_1 + b_2 + b_3$. In this case the condition for the pair of states $|\Psi\rangle, |\Phi\rangle$ to be incomparable to each other, can be written in the simplified form that

$$\begin{array}{lll} \texttt{either,} & a_1 > b_1 \ \texttt{and} \ a_3 > b_3 \\ \texttt{or,} & a_1 < b_1 \ \texttt{and} \ a_3 < b_3 \end{array} \tag{3}$$

must hold simultaneously.

4 Incomparability as a Detector for Anti-Unitary Operators

The general class of anti-unitary operations can be defined in the form, $\Gamma = CU$; where C is the conjugation operation and U be the most general type of unitary operation on a qubit, in the form

$$U = \begin{pmatrix} \cos\theta & e^{i\alpha}\sin\theta \\ -e^{i\beta}\sin\theta & e^{i(\alpha+\beta)}\cos\theta \end{pmatrix}$$

Let us consider three qubit states with the spin-directions along x, y, z as,

$$|0_x\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \ , \ \ |0_y\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \ , |0_z\rangle = |0\rangle$$

The action of the operator Γ on these three states can be described as,

$$\begin{array}{l} \Gamma|0_x\rangle = (\frac{\cos\theta + e^{-i\alpha}\sin\theta}{\sqrt{2}})|0\rangle + e^{-i\beta}(\frac{e^{-i\alpha}\cos\theta - \sin\theta}{\sqrt{2}})|1\rangle, \\ \Gamma|0_y\rangle = (\frac{\cos\theta - ie^{-i\alpha}\sin\theta}{\sqrt{2}})|0\rangle - e^{-i\beta}(\frac{ie^{-i\alpha}\cos\theta + \sin\theta}{\sqrt{2}})|1\rangle, \\ \Gamma|0_z\rangle = \cos\theta|0\rangle - e^{-i\beta}\sin\theta|1\rangle \end{array} \tag{4}$$

To prove that this operation Γ is nonphysical and its existence leads to an impossibility, we choose a particular pure bipartite state $|\chi^i\rangle_{AB}$ shared between two spatially separated

parties Alice and Bob in the form,

$$
\begin{aligned}
|\chi^i\rangle_{AB} \quad = \quad & \tfrac{1}{\sqrt{3}}\{|0\rangle_A|0_z\rangle_B|0_z\rangle_B + |1\rangle_A|0_x\rangle_B|0_y\rangle_B \\
& + |2\rangle_A|0_y\rangle_B|0_x\rangle_B\}
\end{aligned}
\tag{5}
$$

The impossibility we want to show here is that by the action of Γ locally we are able to convert a pair of incomparable states deterministically. Now to show incomparability between a pair of pure bipartite states, the minimum Schmidt rank we require is three. So the joint state we consider above is a 3×4 state where Alice has a qutrit and Bob has two qubits. The initial reduced density matrix of Alice's side is then,

$$
\begin{aligned}
\rho_A^i \quad = \quad & \tfrac{1}{3}\{P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + \tfrac{1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0| \\
& + |0\rangle\langle 2| + |2\rangle\langle 0| + |1\rangle\langle 2| + |2\rangle\langle 1|)\}
\end{aligned}
\tag{6}
$$

The Schmidt vector corresponding to the initial state $|\chi^i\rangle_{AB}$ is $(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6})$. Assuming that Bob operates Γ on one of the two qubits, say the last one in his subsystem, the joint state shared between Alice and Bob will transform to

$$
\begin{aligned}
|\chi^f\rangle_{AB} \quad = \quad & \tfrac{1}{\sqrt{3}}\{|0\rangle_A|0_z\rangle_B\Gamma(|0_z\rangle_B) + |1\rangle_A|0_x\rangle_B\Gamma(|0_y\rangle_B) \\
& + |2\rangle_A|0_y\rangle_B\Gamma(|0_x\rangle_B)\}
\end{aligned}
\tag{7}
$$

Tracing out Bob's subsystem we again consider the reduced density matrix of Alice's subsystem. The final reduced density matrix is

$$
\begin{aligned}
\rho_A^f \quad = \quad & \tfrac{1}{3}\{P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + \tfrac{1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0| \\
& + |0\rangle\langle 2| + |2\rangle\langle 0| - i|1\rangle\langle 2| + i|2\rangle\langle 1|)\}
\end{aligned}
\tag{8}
$$

The Schmidt vector corresponding to the final state $|\chi^f\rangle_{AB}$ is $(\tfrac{1}{3} + \tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{1}{3} - \tfrac{1}{2\sqrt{3}})$. Interestingly, the Schmidt vector of the final state does not contain the arbitrary parameters of the anti-unitary operator Γ. It is now easy to check that the final and initial Schmidt vectors are incomparable as, $\tfrac{2}{3} > \tfrac{1}{3} + \tfrac{1}{2\sqrt{3}} > \tfrac{1}{3} > \tfrac{1}{6} > \tfrac{1}{3} - \tfrac{1}{2\sqrt{3}}$. Thus we have, $|\chi^i\rangle \nrightarrow |\chi^f\rangle$ so that the transformation of the pure bipartite state $|\chi^i\rangle$ to $|\chi^f\rangle$ by LOCC with certainty is not possible following Nielsen's criteria. Though by applying the anti-unitary operator Γ on Bob's local system the transformation $|\chi^i\rangle \rightarrow |\chi^f\rangle$ is performed exactly. This impossibility emerges out of the impossible operation Γ which we have assumed to be exist and apply it to generate the impossible transformation. Thus we have observed the nonphysical nature of any anti-unitary operator Γ through our detection process. As a particular case one may verify the non-existence of exact universal flipper by our method (choose, $\theta = \pi/2, \alpha = 0, \beta = 0$).

If instead of operating $\Gamma = CU$ we will operate only U, *i.e.*, the general unitary operator, the initial and final density matrices of one side will be seen to be identical, implying that there is not even a violation of No-Signalling principle. This is true as we only operate the unitary operator on any qubit not restricting on any particular choices, such as they will act isotropically for all the qubits, etc. Thus it can not even used to send a signal here.

5   Inner Product Preserving Operations

In this section we relate the impossibility of some inner product preserving operations defined only on the minimum number of qubits $|0_x\rangle, |0_y\rangle, |0_z\rangle$. Here we consider the existence of the operation defined on these three qubits in the following manner,

$$|0_z\rangle \longrightarrow (\alpha|0_z\rangle + \beta|1_z\rangle),$$
$$|0_x\rangle \longrightarrow (\alpha|0_x\rangle + \beta|1_x\rangle),$$
$$|0_y\rangle \longrightarrow (\alpha|0_y\rangle + \beta|1_y\rangle),$$

(9)

where $|\alpha|^2 + |\beta|^2 = 1$.

This operation exactly transforms the input qubit into an arbitrary superposition on the input qubit with its orthogonal one. To verify the possibility or impossibility of existence of this operation we consider a pure bipartite state shared between Alice and Bob:

$$|\Pi^i\rangle_{AB} = \frac{1}{\sqrt{3}}\{|0\rangle_A(|0_z\rangle|0_z\rangle)_B + |1\rangle_A(|0_x\rangle|0_x\rangle)_B$$
$$+ |2\rangle_A(|0_y\rangle|0_y\rangle)_B\}$$

(10)

Reduced density matrix of Alice's side will be of the form,

$$\rho_A^i = \frac{1}{3}\{P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + \frac{1}{2}(|0\rangle\langle1| + |1\rangle\langle0|$$
$$+ |0\rangle\langle2| + |2\rangle\langle0| - i|1\rangle\langle2| + i|2\rangle\langle1|)\}$$

(11)

The Schmidt vector corresponding to the initial joint state $|\chi^f\rangle_{AB}$ is $(\frac{1}{3} + \frac{1}{2\sqrt{3}}, \frac{1}{3}, \frac{1}{3} - \frac{1}{2\sqrt{3}})$. If Bob has a machine which operates on the three input qubits $|0_x\rangle, |0_y\rangle, |0_z\rangle$ as defined in equation (9) and he operates that machine on his local system (say, on the last qubit). Then the joint state between Alice and Bob will evolve as,

$$|\Pi^f\rangle_{AB} = \frac{1}{\sqrt{3}}\{|0\rangle_A|0_z\rangle_B(\alpha|0_z\rangle + \beta|1_z\rangle)_B + |1\rangle_A|0_x\rangle_B$$
$$(\alpha|0_x\rangle + \beta|1_x\rangle)_B + |2\rangle_A|0_y\rangle_B(\alpha|0_y\rangle + \beta|1_y\rangle)_B\}$$

(12)

Final reduced density matrix of Alice's side will be of the form,

$$\rho_A^f = \frac{1}{3}\{P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + p(|0\rangle\langle1| + |1\rangle\langle0|)$$
$$+ q|0\rangle\langle2| + \overline{q}|2\rangle\langle0| + r|1\rangle\langle2| + \overline{r}|2\rangle\langle1|)\}$$

(13)

where $p = \frac{1}{2}\{|\alpha|^2 - |\beta|^2 + \alpha\,\overline{\beta} + \beta\,\overline{\alpha}\}$, $q = \frac{1}{2}\{|\alpha|^2 + i|\beta|^2 + \alpha\,\overline{\beta} - i\beta\,\overline{\alpha}\}$ and $r = \frac{1}{2}\{\alpha\,\overline{\beta} + \beta\,\overline{\alpha} - i\}$.

The eigenvalue equation turns out to be,

$$x^3 - (p\overline{p} + q\overline{q} + r\overline{r})x + pr\overline{q} + \overline{pr}q = 0,$$

where we denote $1 - 3\lambda = x$.

To compare the initial and final state we have to check whether the initial and final eigenvalues will satisfy either of the relations of equation (3). We rewrite, the above eigenvalue equation as

$$x^3 - 3Ax + B = 0$$

(14)

with, $A = \frac{1}{3}(p\overline{p} + q\overline{q} + r\overline{r}) \geq 0$ and $B = pr\overline{q} + \overline{pr}q$. The eigenvalues can then be written as $\{\lambda_1 \equiv \frac{1}{3}[1 - 2\sqrt{A}\cos(\frac{2\pi}{3} + \theta)], \lambda_2 \equiv \frac{1}{3}[1 - 2\sqrt{A}\cos\theta], \lambda_3 \equiv \frac{1}{3}[1 - 2\sqrt{A}\cos(\frac{2\pi}{3} - \theta)]\}$ where $\cos3\theta = \frac{-B}{2\sqrt{A^3}}$. We discuss the matter case by case (for details, see Appendix A).

Case-1 : For $B < 0$, we see an incomparability between the initial and final joint states if $A = \frac{1}{4}$. In case $A < \frac{1}{4}$ we observe that either there is an incomparability between the initial and final states or the entanglement content of the final state is larger than that of the initial states. Lastly if $A > \frac{1}{4}$ we also see a case of incomparability if the condition

$2\sqrt{A}\cos(\frac{2\pi}{3} + \theta) > -\frac{\sqrt{3}}{2}$ holds. Numerical searches support that for real values of (α, β) incomparability is seen almost everywhere in this region.

Case-2 : For $B = 0$, we found that there do not arise a case of incomparability. It is also seen that there is always an increase of entanglement by LOCC if $A < \frac{1}{4}$, which is the only possibility for real values of α, β.

Case-3 : For $B > 0$ we also get a similar result like Case-1. Only the condition for incomparability in case $A > \frac{1}{4}$ is changed to the form that $2\sqrt{A}\cos\varphi < \frac{\sqrt{3}}{2}$ where $\varphi = \min\{\theta, (\frac{2\pi}{3} - \theta)\} \in (\frac{\pi}{6}, \frac{\pi}{3})$. It must be noted that for real values of α, β this subcase do not arise at all.

In particular if we check the values of α, β be such that they represents the operations flipping(i.e., $\alpha = 0$) and Hadamard(i.e., $\alpha = \beta = \frac{1}{\sqrt{2}}$) respectively, we find from the above that in both the cases the initial and final states are incomparable.

Thus we get almost in all cases some kind of violation of physical laws implying that the kind of inner product preserving operations defined on only three states is nonphysical in nature and we observe for a large class of such inner-product-preserving operation incomparability senses.

To conclude this work proves a close relation between anti-unitary operators and the existence of incomparable states. Incomparability shows it is also able to detect nonphysical operations like Hadamard and some other inner-product preserving operations. This work also shows an interplay between LOCC, nonphysical operations and the entanglement behavior of quantum systems.

Acknowledgements

References

1. A. K. Ekert (1991), *Quantum Cryptography Based on Bell's Theorem*, Phys. Rev. Lett., **67**, 661.
2. C. H. Bennett and S. J. Wiesner (1992), *Communication via one- and two-particle operation on Einstein-Podolsky-Rosen states*, Phys. Rev. Lett., **69**, 2881.
3. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters (1993), *Teleporting an Unknown Quantum State via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett., **70**, 1895.
4. W. K. Wootters and W. H. Zurek (1982), *A single quantum can not be cloned*, Nature, **299**, 802.
5. D. Dieks (1982), *Communication by EPR devices*, Phys. Lett. A, **92**, 271.
6. H. P. Yuen (1986), *Amplification of quantum states and noiseless photon amplification*, Phys. Lett. A, **113**, 405.
7. A. K. Pati and S. L. Braunstein (2000), *Impossibility of deleting an unknown quantum state*, Nature, **404**, 164.
8. W. H. Zurek (2000), *Quantum cloning : Schrodinger's sheep*, Nature, **404**, 40.
9. N. Gisin and S. Popescu (1999), *Spin Flips and Quantum Information for Antiparallel Spins*, Phys. Rev. Lett., **83**, 432-435.
10. V. Buzek, M. Hillery and R. F. Werner (1999), *Optimal manipulations with qubits: Universal-NOT gate*, Phys. Rev. A, **60**, R2626-R2629.

11. S. Massar (2000), *Collective versus local measurements on two parallel or antiparallel spins*, Phys. Rev. A, **62**, 040101(R).

12. A. K. Pati (2002), *General impossible operation in quantum information*, Phys. Rev. A, **66**, 062319.

13. S. Ghosh, A. Roy and U. Sen (2000), *Antiparallel spin does not always contain more information*, Phys. Rev. A, **63**, 014301.

14. A. K. Pati (2000), *Minimum classical bit for remote preparation and measurement of a qubit*, Phys. Rev. A, **63**, 014302.

15. K. Kraus (1983), *States, Effects, and Operations*, Lecture Notes in Physics, Vol.190, Springer, (Berlin).

16. M. A. Nielsen (1999), *Conditions for a class of entanglement transformation*, Phys. Rev. Lett., **83**, 436.

17. M. A. Nielsen and I. L. Chuang (2000), *Quantum Computation and Quantum Information*, Cambridge University Press, (Cambridge).

18. N. Gisin (1998), *Quantum cloning without signalling*, Phys. Lett. A, **242**, 1-3.

19. L. Hardy and D. Song (1999), *No signalling and probabilistic quantum claning*, Phys. Lett. A, **259**, 331-333.

20. A. K. Pati and S. L. Braunstein (2003), *Quantum deleting and signalling*, Phys. Lett. A, **315**, 208-212.

21. A. K. Pati (2000), *Probabilistic exact cloning and probabilistic no-signalling*, Phys. Lett. A, **270**, 103.

22. I. Chattopadhyay, S. K. Choudhary, G. Kar, S. Kunkri and D. Sarkar (2006), *No-Flipping as a consequence of No-Signalling and Non-increase of Entanglement under LOCC*, Phys. Lett. A, **351**, 384-387.

23. M. Horodecki, R. Horodecki, A. Sen(De) and U. Sen, *No-Deleting and no-cloning principles as consequences of conservation of quantum information*, arXiv: quant-ph/0306044.

24. C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters (1999), *Quantum nonlocality without entanglement*, Phys. Rev. A, **59**, 1070-1091.

25. C. Simon, V. Buzek and N. Gisin (2001), *No-Signaling Condition and Quantum Dynamics*, Phys. Rev. Lett., **87**, 170405.

26. I. Chattopadhyay and D. Sarkar (2006), *Impossibility of exact flipping of three arbitrary quantum states Via Incomparability*, Phys. Rev. A, **73**, 044303.

27. A. Bhar, I. Chattopadhyay and D. Sarkar (2006), *No-Cloning and No-Deleting theorems through the existence of Incomparable states under LOCC*, Quantum Information Proccesing (Accepted), arXiv: quant-ph/0606206.

28. R. Bhatia (1997), *Matrix Algebra*, Springer-Verlag, (New York).

29. M. A. Nielsen and J. Kempe (2001), *Separable States Are More Disordered Globally Than Locally*, Phys. Rev. Lett., **86**, 5184.

30. T. Hiroshima (2003), *Majorization Criterion for Distillability of a Bipartite Quantum States*, Phys. Rev. Lett., **91**, 057902.

31. Y. -J. Han, Y. -S. Zhang and G. -C. Guo (2005), *Compatibility relations between the reduced and global density matrices*, Phys. Rev. A, **71**, 052306.

32. M. A. Nielsen (2002), *An introduction to majorization and its application to quantum mechanics*, Home-page(M. A. Nielsen).

33. I. Chattopadhyay and D. Sarkar (2005), *Deterministic Local Conversion of Incomparable States by Collective LOCC*, Quantum Information and Computation, **5**, 247-257.

Appendix A

Case-1 : $B < 0$ This implies $3\theta \in [0, \frac{\pi}{2}) \bigcup (\frac{3\pi}{2}, 2\pi]$. We analyze this in two section.

If $3\theta \in [0, \frac{\pi}{2})$ we have, $\frac{\sqrt{3}}{2} < \cos\theta \leq 1 \Rightarrow \lambda_2 \in [\frac{1}{3}(1 - 2\sqrt{A}), \frac{1}{3}(1 - \sqrt{3A}))$. Again $0 \leq \theta < \frac{\pi}{6} \Rightarrow -\frac{\sqrt{3}}{2} < \cos(\frac{2\pi}{3} + \theta) \leq -\frac{1}{2} \Rightarrow \lambda_1 \in [\frac{1}{3}(1 + \sqrt{A}), \frac{1}{3}(1 + \sqrt{3A}))$. Finally,

$0 \leq \theta < \frac{\pi}{6} \Rightarrow \cos(\frac{2\pi}{3} - \theta) \in [-\frac{1}{2}, 0) \Rightarrow \lambda_3 \in (\frac{1}{3}, \frac{1}{3}(1 + \sqrt{A})]$.

Otherwise $3\theta \in (\frac{3\pi}{2}, 2\pi]$, i.e., $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3}]$, we have, $\lambda_3 \in [\frac{1}{3}(1 - 2\sqrt{A}), \frac{1}{3}(1 - \sqrt{3A}))$, $\lambda_2 \in [\frac{1}{3}(1 + \sqrt{A}), \frac{1}{3}(1 + \sqrt{3A}))$ and $\lambda_1 \in (\frac{1}{3}, \frac{1}{3}(1 + \sqrt{A})]$.

Thus in both the cases $\lambda^f_{MAX} \in [\frac{1}{3}(1 + \sqrt{A}), \frac{1}{3}(1 + \sqrt{3A}))$ and $\lambda^f_{MIN} \in [\frac{1}{3}(1 - 2\sqrt{A}), \frac{1}{3}(1 - \sqrt{3A}))$.

For $A = \frac{1}{4}$ we observe that $\lambda^f_{MIN} \in [0, \frac{1}{3}(1 - \frac{\sqrt{3}}{2})) < \lambda^i_{MIN}$ and $\lambda^f_{MAX} \in [\frac{1}{2}, \frac{1}{3}(1 + \frac{\sqrt{3}}{2})) < \lambda^i_{MAX}$ which implies that $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ are incomparable.

If $A < \frac{1}{4}$ then $\lambda^f_{MAX} \leq \lambda^i_{MAX}$. So, in case $\lambda^f_{MIN} \leq \lambda^i_{MIN}$ the states $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ are incomparable, otherwise we have $\lambda^f_{MIN} \geq \lambda^i_{MIN}$ then $E(|\Pi^i\rangle_{AB}) < E(|\Pi^f\rangle_{AB})$. For real values of α, β we can express A, B as

$$A = \frac{1}{4} + \frac{1}{6}[2\alpha^2\beta^2 + 3\alpha\beta(\alpha^2 - \beta^2)]$$
$$B = \frac{\beta}{4}(\alpha^2 - \beta^2 + 2\alpha\beta)[\alpha(2\alpha^2 + 1) + \beta(\alpha^2 - \beta^2)] \tag{A.1}$$

Numerical evidences support that for real α, β most of the cases show incomparability between $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$.

Lastly if $A \geq \frac{1}{4}$ then $\lambda^i_{MAX} < \lambda^f_{MAX}$. Thus incomparability between $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ will hold if $\lambda^i_{MIN} < \lambda^f_{MIN}$. For this we get the condition that $2\sqrt{A}\cos\phi < \frac{\sqrt{3}}{2}$ where $\phi = \min\{\theta, \frac{2\pi}{3} - \theta\} \in (\frac{\pi}{6}, \frac{\pi}{3})$. For real values of α, β from equation(A.1), we see $A > \frac{1}{4}$ implies $B > 0$. Thus for real α, β this subcase do not arises.

Case-2 : $B = 0$. Here the final eigenvalues are $\{\frac{1}{3}(1 + \sqrt{3A}), \frac{1}{3}, \frac{1}{3}(1 - \sqrt{3A})\}$. Thus, $E(|\Pi^i\rangle_{AB}) \geq E(|\Pi^f\rangle_{AB})$ if $A \geq \frac{1}{4}$. Incomparability between the initial and final joint states $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ will not occur in this case.

Hence for all values of α, β for which $A < \frac{1}{4}$ there is an increase of entanglement by applying the local operation defined in equation(9) in Bob's system. This impossibility indicate the impossibility of the operation defined in (9) for those values of α, β which satisfy $A < \frac{1}{4}$. And for real values of α, β, in all possibilities for $B = 0$ we have $A < \frac{1}{4}$. This case always shows an increase of entanglement.

Case-3 : $B > 0$. Here $3\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \Rightarrow \theta \in (\frac{\pi}{6}, \frac{\pi}{2}) \Rightarrow \cos\theta \in (\frac{\sqrt{3}}{2}, 0) \Rightarrow \lambda_2 \in (\frac{1}{3}(1 - \sqrt{3A}), \frac{1}{3})$. Again $\theta \in (\frac{\pi}{6}, \frac{\pi}{2}) \Rightarrow \cos(\frac{2\pi}{3} + \theta) \in (-1, -\frac{\sqrt{3}}{2}) \Rightarrow \lambda_1 \in (\frac{1}{3}(1 + \sqrt{3A}), \frac{1}{3}(1 + 2\sqrt{A}))$. Lastly, $\theta \in (\frac{\pi}{6}, \frac{\pi}{2}) \Rightarrow \cos(\frac{2\pi}{3} - \theta) \in (0, \frac{\sqrt{3}}{2}) \Rightarrow \lambda_3 \in (\frac{1}{3}(1 - \sqrt{3A}), \frac{1}{3})$.

Hence in this case $\lambda^f_{MAX} \in (\frac{1}{3}(1 + \sqrt{3A}), \frac{1}{3}(1 + 2\sqrt{A}))$ and $\lambda^f_{MIN} \in (\frac{1}{3}(1 - \sqrt{3A}), \frac{1}{3})$. So for $A = \frac{1}{4}$ we have $\lambda^i_{MAX} < \lambda^f_{MAX}$ and $\lambda^i_{MIN} < \lambda^f_{MIN}$ implies that $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ are incomparable.

Again for $A \leq \frac{1}{4}$ we see, $\lambda^f_{MIN} > \lambda^i_{MIN}$. Thus if $\lambda^f_{MAX} > \lambda^i_{MAX}$ then the states $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ are incomparable or if $\lambda^f_{MAX} < \lambda^i_{MAX}$ then $E(|\Pi^i\rangle_{AB}) < E(|\Pi^f\rangle_{AB})$.

Lastly if $A \geq \frac{1}{4}$ then $\lambda^i_{MAX} < \lambda^f_{MAX}$. Incomparability between the initial and final joint states $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$ will hold if $\lambda^i_{MIN} < \lambda^f_{MIN}$. For this we get the condition that $2\sqrt{A}\cos(\frac{2\pi}{3} + \theta) > -\frac{\sqrt{3}}{2}$. From equation (A.1) we find, for real values of α and β, numerical results support that in most of the cases there is an incomparability between $|\Pi^i\rangle_{AB}$, $|\Pi^f\rangle_{AB}$.