OPERATOR QUANTUM ERROR CORRECTION

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This paper is an expanded and more detailed version of the work [1] in which the Operator Quantum Error Correction formalism was introduced. This is a new scheme for the error correction of quantum operations that incorporates the known techniques — i.e. the standard error correction model, the method of decoherence-free subspaces, and the noiseless subsystem method — as special cases, and relies on a generalized mathematical framework for noiseless subsystems that applies to arbitrary quantum operations. We also discuss a number of examples and introduce the notion of "unitarily noiseless subsystems".

1   Introduction

A unified and generalized approach to quantum error correction, called Operator Quantum Error Correction (OQEC), was recently introduced in [1]. This formalism unifies all of the known techniques for the error correction of quantum operations – i.e. the standard model [2, 3, 4, 5], the method of decoherence-free subspaces [6, 7, 8, 9] and the noiseless subsystem method [10, 11, 12] – under a single umbrella. An important new framework introduced as

part of this scheme opens up the possibility of studying noiseless subsystems for arbitrary quantum operations.

This paper is an expanded and more detailed version of the work [1]. We provide complete details for proofs sketched there, and in some cases we present an alternative "operator" approach that leads to new information. Specifically, we show that correction of the general codes introduced in [1] is equivalent to correction of certain operator algebras, and we use this to give a new proof for the main testable conditions in this scheme. In addition, we discuss a number of examples throughout the paper, and introduce the notion of "unitarily noiseless subsystems" as a relaxation of the requirement in the noiseless subsystem formalism for immunity to errors.

2 Preliminaries

2.1 Quantum Operations

Let $\mathcal{H}$ be a (finite-dimensional) Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of operators on $\mathcal{H}$. A *quantum operation* (or *channel*, or *evolution*) on $\mathcal{H}$ is a linear map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is completely positive and preserves traces. Every channel has an "operator-sum representation" of the form $\mathcal{E}(\sigma) = \sum_a E_a \sigma E_a^\dagger$, $\forall \sigma \in \mathcal{B}(\mathcal{H})$, where $\{E_a\} \subseteq \mathcal{B}(\mathcal{H})$ are the Kraus operators (or errors) associated with $\mathcal{E}$. As a convenience we shall write $\mathcal{E} = \{E_a\}$ when the E_a determine $\mathcal{E}$ in this way.

The choice of operators that yield this form is not unique, but if $\mathcal{E} = \{E_a\} = \{F_b\}$ (without loss of generality assume the cardinalities of the sets are the same), then there is a unitary matrix $U = (u_{ab})$ such that $E_a = \sum_b u_{ab} F_b \ \forall a$. The map $\mathcal{E}$ is said to be *unital* or *bistochastic* if $\mathcal{E}(\mathbb{1}) = \sum_a E_a E_a^\dagger = \mathbb{1}$. Trace preservation of $\mathcal{E}$ can be phrased in terms of the error operators via the equation $\sum_a E_a^\dagger E_a = \mathbb{1}$, which is equivalent to the dual map for $\mathcal{E}$ being unital.

2.2 Standard Model for Quantum Error Correction

The "Standard Model" for the error correction of quantum operations [2, 3, 4, 5] consists of triples $(\mathcal{R}, \mathcal{E}, \mathcal{C})$ where $\mathcal{C}$ is a subspace, a *quantum code*, of a Hilbert space $\mathcal{H}$ associated with a given quantum system. The error $\mathcal{E}$ and recovery $\mathcal{R}$ are quantum operations on $\mathcal{B}(\mathcal{H})$ such that $\mathcal{R}$ undoes the effects of $\mathcal{E}$ on $\mathcal{C}$ in the following sense:

$$(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \forall \sigma = P_\mathcal{C} \sigma P_\mathcal{C}, \tag{1}$$

where $P_\mathcal{C}$ is the projection of $\mathcal{H}$ onto $\mathcal{C}$.

When there exists such an $\mathcal{R}$ for a given pair $\mathcal{E}, \mathcal{C}$, the subspace $\mathcal{C}$ is said to be *correctable for $\mathcal{E}$*. The existence of a recovery operation $\mathcal{R}$ of $\mathcal{E} = \{E_a\}$ on $\mathcal{C}$ may be cleanly phrased in terms of the $\{E_a\}$ as follows [4, 5]:

$$P_\mathcal{C} E_a^\dagger E_b P_\mathcal{C} = \lambda_{ab} P_\mathcal{C} \quad \forall a, b \tag{2}$$

for some matrix $\Lambda = (\lambda_{ab})$. It is easy to see that this condition is independent of the operator-sum representation for $\mathcal{E}$.

2.3 Noiseless Subsystems and Decoherence-Free Subspaces

Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{H}$. Let $\mathcal{A}$ be the C*-algebra generated by the E_a, so $\mathcal{A} = \mathrm{Alg}\,\{E_a, E_a^\dagger\}$. This is the set of polynomials in the E_a and $E_a^\dagger$. As a †-algebra (i.e.,

a finite-dimensional C*-algebra [13, 14, 15]), $\mathcal{A}$ has a unique decomposition up to unitary equivalence of the form

$$\mathcal{A} \cong \bigoplus_{J} \left(\mathcal{M}_{m_J} \otimes \mathbb{1}_{n_J} \right), \tag{3}$$

where $\mathcal{M}_{m_J}$ is the full matrix algebra $\mathcal{B}(\mathbb{C}^{m_J})$ represented with respect to a given orthonormal basis and $\mathbb{1}_{n_J}$ is the identity on $\mathbb{C}^{n_J}$. This means there is an orthonormal basis such that the matrix representations of operators in $\mathcal{A}$ with respect to this basis have the form in Eq. (3). Typically $\mathcal{A}$ is called the *interaction algebra* associated with the operation $\mathcal{E}$.

The standard "noiseless subsystem" method of quantum error correction [10, 11, 12] makes use of the operator algebra structure of the *noise commutant* associated with $\mathcal{E}$;

$$\mathcal{A}' = \left\{ \sigma \in \mathcal{B}(\mathcal{H}) : E\sigma = \sigma E \ \forall E \in \{E_a, E_a^\dagger\} \right\}.$$

Observe that when $\mathcal{E}$ is unital, all the states encoded in $\mathcal{A}'$ are immune to the errors of $\mathcal{E}$. Thus, this is in effect a method of passive error correction. The structure of $\mathcal{A}$ given in Eq. (3) implies that the noise commutant is unitarily equivalent to

$$\mathcal{A}' \cong \bigoplus_{J} \left(\mathbb{1}_{m_J} \otimes \mathcal{M}_{n_J} \right). \tag{4}$$

It is obvious from Eqs. (3,4) that elements of $\mathcal{A}'$ are immune to the errors of $\mathcal{A}$ when $\mathcal{E}$ is unital. In [16] the converse of this statement was proved. Specifically, when $\mathcal{E}$ is unital the noise commutant coincides with the fixed point set for $\mathcal{E}$; i.e.,

$$\mathcal{A}' = \mathrm{Fix}(\mathcal{E}) = \left\{ \sigma \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\sigma) = \sum_a E_a \sigma E_a^\dagger = \sigma \right\}. \tag{5}$$

This is precisely the reason that $\mathcal{A}'$ may be used to produce noiseless subsystems for unital $\mathcal{E}$. We note that the noiseless subsystem method may be regarded as containing the method of *decoherence-free subspaces* [6, 7, 8, 9] as a special case, in the sense that this method makes use of the summands $\mathbb{1}_{m_J} \otimes \mathcal{M}_{n_J}$ where $m_J = 1$, inside the noise commutant $\mathcal{A}'$ for encoding information.

While many physical noise models satisfy the unital constraint, the generic quantum operation is non-unital. Below we show how shifting the focus from $\mathcal{A}'$ to $\mathrm{Fix}(\mathcal{E})$ (and related sets) quite naturally leads to the notion of noiseless subsystems that applies to arbitrary quantum operations.

3   Noiseless Subsystems For Arbitrary Quantum Operations

In this section we describe a generalized mathematical framework for noiseless subsystems that applies to arbitrary (not necessarily unital) quantum operations and serves as a building block for the OQEC scheme presented below. Note that a subsystem that is noiseless for a certain map will also be noiseless for any other map whose Kraus operators are linear combinations of the Kraus operators of the original map. Hence, for the purpose of noiseless encoding, any map whose Kraus operators span is closed under conjugation is equivalent to a unital map. The mathematical framework utilized in [10, 11, 12] produces noiseless subsystems for precisely these kinds of operations, and so may effectively be regarded as restricted to unital

channels. That being said, it is desirable to find a means by which noiseless subsystems can be discovered without relying on the unital nature of an operation, or the structure of its noise commutant. The main result of this section (Theorem 1) shows explicitly how this may be accomplished.

Note that the structure of the algebra $\mathcal{A}$ given in Eq. (3) induces a natural decomposition of the Hilbert space

$$\mathcal{H} = \bigoplus_J \mathcal{H}_J^A \otimes \mathcal{H}_J^B, \tag{6}$$

where the "noisy subsystems" $\mathcal{H}_J^A$ have dimension m_J and the "noiseless subsystems" $\mathcal{H}_J^B$ have dimension n_J. For brevity, we focus on the case where information is encoded in a single noiseless sector of $\mathcal{B}(\mathcal{H})$, and hence

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K} \tag{7}$$

with $\dim(\mathcal{H}^A) = m$, $\dim(\mathcal{H}^B) = n$ and $\dim \mathcal{K} = \dim \mathcal{H} - mn$. We shall write σ^A for operators in $\mathcal{B}(\mathcal{H}^A)$ and σ^B for operators in $\mathcal{B}(\mathcal{H}^B)$. Thus the restriction of the noise commutant $\mathcal{A}'$ to $\mathcal{H}^A \otimes \mathcal{H}^B$ consists of the operators of the form $\sigma = \mathbb{1}^A \otimes \sigma^B$ where $\mathbb{1}^A$ is the identity element of $\mathcal{B}(\mathcal{H}^A)$.

For notational purposes, assume that ordered orthonormal bases have been chosen for $\mathcal{H}^A = \mathrm{span}\{|\alpha_i\rangle\}_{i=1}^m$ and $\mathcal{H}^B = \mathrm{span}\{|\beta_k\rangle\}_{k=1}^n$ that yield the matrix representation of the corresponding subalgebra of $\mathcal{A}'$ as $\mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B) \cong \mathbb{1}_m \otimes \mathcal{M}_n$. We let

$$P_{kl} \equiv |\alpha_k\rangle\langle\alpha_l| \otimes \mathbb{1}^B \quad \forall 1 \leq k, l \leq m \tag{8}$$

denote the corresponding family of "matrix units" in $\mathcal{A}$ associated with this decomposition. The following identities are readily verified and are the defining properties for a family of matrix units:

$$
\begin{aligned}
P_{kl} &= P_{kk} P_{kl} P_{ll} \quad \forall 1 \leq k, l \leq m \\
P_{kl}^\dagger &= P_{lk} \qquad \forall 1 \leq k, l \leq m \\
P_{kl} P_{l'k'} &= \begin{cases} P_{kk'} & \text{if } l = l' \\ 0 & \text{if } l \neq l' \end{cases}.
\end{aligned}
$$

Define the projection $P_{\mathfrak{A}} \equiv P_{11} + \ldots + P_{mm}$, so that $P_{\mathfrak{A}}\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, $P_{\mathfrak{A}}^\perp = \mathbb{1} - P_{\mathfrak{A}}$ and $P_{\mathfrak{A}}^\perp \mathcal{H} = \mathcal{K}$. Further define a superoperator $\mathcal{P}_{\mathfrak{A}}$ by the action $\mathcal{P}_{\mathfrak{A}}(\cdot) = P_{\mathfrak{A}}(\cdot)P_{\mathfrak{A}}$. The following result is readily proved.

Lemma 1 *The map* $\Gamma : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *given by* $\Gamma = \{P_{kl}\}$ *satisfies the following:*

$$\Gamma(\sigma) = \sum_{k,l} P_{kl} \sigma P_{kl}^\dagger = \mathbb{1}^A \otimes (\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}})(\sigma) \in \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B), \tag{9}$$

for all operators $\sigma \in \mathcal{B}(\mathcal{H})$, *so in particular* $\Gamma(\sigma^A \otimes \sigma^B) \propto \mathbb{1}^A \otimes \sigma^B$ *for all* σ^A *and* σ^B.

Note. While we have stated this result as part of a discussion on a subalgebra of a noise commutant, it is valid for any $\dagger$-algebra $\mathfrak{B} \cong \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$ with matrix units $\{P_{kl}\}$ generating the algebra $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$.

We now turn to the generalized noiseless subsystems method. In this framework, the quantum information is encoded in σ^B; i.e., the state of the noiseless subsystem. But it is not necessary for the noisy subsystem to remain in the maximally mixed state $\mathbb{1}^A$ under $\mathcal{E}$, as is the case for noiseless subsystems of unital channels, it could in principle get mapped to any other state.

In order to formalize this idea, define for a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ the set of operators

$$\mathfrak{A} = \{\sigma \in \mathcal{B}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \text{ and } \sigma^B\}. \tag{10}$$

Notice that this set has the structure of a semigroup and includes operator algebras such as $\mathfrak{A}_0 \equiv \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$ and $|\alpha_k\rangle\langle\alpha_k| \otimes \mathcal{B}(\mathcal{H}^B)$. We note that in the formulation below, the operation $\mathcal{E}$ maps the set of operators on the subspace $P_{\mathfrak{A}}\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ to itself.

Lemma 2 *Given a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ and a quantum operation $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$, the following four conditions are equivalent, and are the defining properties of the noiseless subsystem B:*

(1) $\forall\sigma^A \ \forall\sigma^B, \ \exists\tau^A \ : \ \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$

(2) $\forall\sigma^B, \ \exists\tau^A \ : \ \mathcal{E}(\mathbb{1}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$

(3) $\forall\sigma \in \mathfrak{A} \ : \ \big(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{E}\big)(\sigma) = \mathrm{Tr}_A(\sigma).$

Proof. The implications *1.* $\Rightarrow$ *2.* and *1.* $\Rightarrow$ *3.* are trivial. To prove *2.* $\Rightarrow$ *1.*, first let $|\psi\rangle \in \mathcal{H}^B$ and put $P = |\psi\rangle\langle\psi|$. Suppose that $\{|\alpha_k\rangle\}$ is an orthonormal basis for $\mathcal{H}^A$. Then $\sum_{k=1}^m |\alpha_k\rangle\langle\alpha_k| = \mathbb{1}^A$ and by *2.* and the positivity of $\mathcal{E}$ we have for all k,

$$\begin{aligned} 0 \leq \mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes P) &\leq \mathcal{E}(\mathbb{1}^A \otimes P) \\ &= \tau^A \otimes P \\ &= (\mathbb{1}^A \otimes P)(\tau^A \otimes P)(\mathbb{1}^A \otimes P). \end{aligned}$$

It follows that there are positive operators $\sigma_{\psi,k} \in \mathcal{B}(\mathcal{H}^A)$ such that $\mathcal{E}(|\alpha_k\rangle\langle\alpha_k| \otimes P) = \sigma_{\psi,k} \otimes P$ for all k. A standard linearity argument may be used to show that the operators $\sigma_{\psi,k}$ do not depend on $|\psi\rangle$. Condition *1.* now follows from the linearity of $\mathcal{E}$.

To prove *3.* $\Rightarrow$ *2.*, first note that since $\mathcal{E}$ and Tr_A are positive and trace preserving, *3.* implies that $\big(\mathcal{P}_{\mathfrak{A}} \circ \mathcal{E}\big)(\sigma) = \mathcal{E}(\sigma)$ for all $\sigma \in \mathfrak{A}$. Now fix $|\psi\rangle \in \mathcal{H}^B$ and put $\sigma = \mathbb{1}^A \otimes P$ where $P = |\psi\rangle\langle\psi|$. Then by *3.* we have

$$\mathrm{Tr}_A\big((\mathbb{1}^A \otimes P)\,\mathcal{E}(\sigma)\,(\mathbb{1}^A \otimes P)\big) = \mathrm{Tr}_A(\sigma).$$

It follows again from the trace preservation and positivity of Tr_A and $\mathcal{E}$ that $\sigma\mathcal{E}(\sigma)\sigma = \mathcal{E}(\sigma)$, and hence there is a τ^A such that $\mathcal{E}(\sigma) = \tau^A \otimes P$. The above argument may now be used to show that τ^A is independent of $|\psi\rangle$, and the rest follows from the linearity of $\mathcal{E}$. $\square$

Definition 1 *The subsystem B is said to be noiseless for $\mathcal{E}$ when it satisfies one — and hence all — of the conditions in Lemma 2.*

We next give necessary and sufficient conditions for a subsystem to be noiseless for a map $\mathcal{E} = \{E_a\}$.

Theorem 1 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as in Eq. (10). Then the following three conditions are equivalent:*

(1) *The B-sector of $\mathfrak{A}$ encodes a noiseless subsystem for $\mathcal{E}$ (decoherence-free subspace in the case m=1), as in Definition 1.*

(2) *The subspace $P_{\mathfrak{A}}\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ is invariant for the operators E_a and the restrictions $E_a|_{P_{\mathfrak{A}}\mathcal{H}}$ belong to the algebra $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$.*

(3) *The following two conditions hold for any choice of matrix units $\{P_{kl} : 1 \leq k, l \leq m\}$ for $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$ as in Eq. (8):*

$$P_{kk}E_aP_{ll} = \lambda_{akl}P_{kl} \quad \forall\, a, k, l \tag{11}$$

for some set of scalars (λ_{akl}) and

$$E_aP_{\mathfrak{A}} = P_{\mathfrak{A}}E_aP_{\mathfrak{A}} \quad \forall\, a. \tag{12}$$

Proof. Since the matrix units $\{P_{kl}\}$ generate $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$ as an algebra, it follows that *3.* is a restatement of *2.* To prove the necessity of Eqs. (11,12) for *1.*, let $\Gamma : \mathcal{B}(\mathcal{H}) \to \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$ be defined by the matrix units for $\mathfrak{A}$ as above and note that Lemma 1 and Lemma 2 imply

$$\big(\Gamma \circ \mathcal{E} \circ \Gamma\big)(\sigma) \propto \Gamma(\sigma) \quad \text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}). \tag{13}$$

As in the proof of Lemma 2, the proportionality factor cannot depend on σ, so the sets of operators $\{P_{ki}E_aP_{jl}\}$ and $\{\lambda P_{k'l'}\}$ define the same map for some scalar λ. We may thus find a set of scalars $\mu_{kiajl,k'l'}$ such that

$$P_{ki}E_aP_{jl} = \sum_{k'l'} \mu_{kiajl,k'l'}P_{k'l'}. \tag{14}$$

Multiplying both sides of this equality on the right by P_l and on the left by P_k, we see that $\mu_{kiajl,k'l'} = 0$ when $k \neq k'$ or $l \neq l'$. This implies Eq. (11) with $\lambda_{akl} = \mu_{kkall,kl}$.

For the second condition, as a consequence of Lemma 2 we have $P_{\mathfrak{A}}^{\perp}\mathcal{E}(P_{\mathfrak{A}}(\sigma))P_{\mathfrak{A}}^{\perp} = 0$ for all $\sigma \in \mathcal{B}(\mathcal{H})$. Equation (12) follows from this observation via consideration of the operator-sum representation (see § 2.1) for $\mathcal{E}$.

To prove sufficiency of Eqs. (11), (12) for *1.*, we use the identity $P_{\mathfrak{A}} = \sum_{k=1}^m P_k$ to establish for all $\sigma = P_{\mathfrak{A}}\sigma \in \mathfrak{A}$,

$$
\begin{aligned}
\mathcal{E}(\sigma) &= (P_{\mathfrak{A}} + P_{\mathfrak{A}}^{\perp})\sum_a E_a\sigma E_a^{\dagger}(P_{\mathfrak{A}} + P_{\mathfrak{A}}^{\perp}) \\
&= \sum_a P_{\mathfrak{A}}E_a\sigma E_a^{\dagger}P_{\mathfrak{A}} \\
&= \sum_{a,k,k'} P_{kk}E_a\sigma E_a^{\dagger}P_{k'k'}.
\end{aligned}
$$

Combining this with the identity

$$\sigma^A \otimes \sigma^B = P_{\mathfrak{A}}(\sigma^A \otimes \sigma^B)P_{\mathfrak{A}} = \sum_{l,l'} P_{ll}(\sigma^A \otimes \sigma^B)P_{l'l'}$$

implies for all $\sigma = \sigma^A \otimes \sigma^B \in \mathfrak{A}$,

$$
\begin{aligned}
\mathcal{E}(\sigma^A \otimes \sigma^B) &= \sum_{a,k,k',l,l'} P_{kk} E_a P_{ll} (\sigma^A \otimes \sigma^B) P_{l'l'} E_a^\dagger P_{k'k'} \\
&= \sum_{a,k,k',l,l'} \lambda_{akl} \bar{\lambda}_{ak'l'} P_{kl} (\sigma^A \otimes \sigma^B) P_{l'k'}.
\end{aligned}
$$

The proof now follows from the fact that the matrix units P_{kl} act trivially on the $\mathcal{B}(\mathcal{H}^B)$ sector. $\square$

Remark. In the case that the semigroup $\mathfrak{A}$ is determined by a matrix block inside the noise commutant $\mathcal{A}'$ for a unital channel $\mathcal{E} = \{E_a\}$, and hence arises through the algebraic approach as in the discussion at the start of this section, the conditions Eqs. (11,12) follow from the structure of $\mathcal{A} = \mathrm{Alg}\{E_a, E_a^\dagger\}$ determined by the matrix units P_{kl}. However, Eqs. (11,12) do not necessarily imply that the noiseless subsystem B is obtained via the noise commutant for $\mathcal{E}$. See [17] for further discussions on this point.

We now discuss a pair of non-unital examples of channels with noiseless subsystems.

Example. As a simple illustration of a noiseless subsystem in a non-unital case, consider the quantum channel $\mathcal{E} : \mathcal{M}_4 \to \mathcal{M}_4$ with errors $\mathcal{E} = \{E_1, E_2\}$ obtained as follows. Fix γ, $0 \le \gamma \le 1$, and with respect to the basis $\{|0\rangle, |1\rangle\}$ let

$$
F_0 = \begin{pmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ \sqrt{1-\gamma} & 0 \end{pmatrix}.
$$

Then define $E_i = F_i \otimes \mathbb{1}_2$, for $i = 0, 1$. That $\sum_i E_i^\dagger E_i = \mathbb{1}_4$ follows from $\sum_i F_i^\dagger F_i = \mathbb{1}_2$, which can be verified straightforwardly.

Decompose $\mathbb{C}^4 = \mathcal{H}^A \otimes \mathcal{H}^B$ with respect to the standard basis, so that $\mathcal{H}^A = \mathcal{H}^B = \mathbb{C}^2$. Then for all $\sigma = \sigma^A \otimes \sigma^B$, we have

$$
\mathcal{E}(\sigma) = \sum_{i=0}^1 E_i (\sigma^A \otimes \sigma^B) E_i^\dagger = \Big(\sum_{i=0}^1 F_i \sigma^A F_i^\dagger \Big) \otimes \sigma^B.
$$

The operator τ^A from Lemma 2 is given by $\tau^A = \sum_i F_i \sigma^A F_i^\dagger$ in this case. It follows that B encodes a noiseless subsystem for $\mathcal{E}$. Also observe that, as opposed to the completely error-free evolution that characterizes the unital case, in this case we have $\mathcal{E}(\mathbb{1}^A \otimes \sigma^B) \ne \mathbb{1}^A \otimes \sigma^B$.

Example. We next present a non-unital channel with a pair of noiseless subsystems; one that is supported by the noise commutant, and one that is not. We shall explicitly indicate Eqs. (11,12) in this case. Let $\mathcal{E} = \{E_0, E_1\}$ be the channel on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ with Kraus operators defined with respect to the computational basis by

$$
\begin{aligned}
E_0 &= \alpha \big(|00\rangle\langle 00| + |11\rangle\langle 11| \big) + |01\rangle\langle 01| + |10\rangle\langle 10|, \\
E_1 &= \beta \big(|00\rangle\langle 00| + |10\rangle\langle 00| + |01\rangle\langle 11| + |11\rangle\langle 11| \big),
\end{aligned}
$$

where $0 < q < 1$ is fixed, and $\alpha = \sqrt{1-2q}$ and $\beta = \sqrt{q}$. (Notice that $\mathcal{E}$ is non-unital; $\mathcal{E}(\mathbb{1}) \ne \mathbb{1}$.)

Let $\mathcal{H}^{B_1} = \mathrm{span}\{|01\rangle, |10\rangle\}$ and $\mathcal{H}^{A_1} = \mathbb{C}$, so that $\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1} = \mathcal{H}^{B_1}$. We may regard $|0_L\rangle = |01\rangle$ and $|1_L\rangle = |10\rangle$ as logical zero and logical one states in this case. Let $Q = |01\rangle\langle 01| + |10\rangle\langle 10|$. Then

$$
E_0 Q = Q = Q E_0 = Q E_0 Q
$$

$$E_1 Q = 0 = Q E_1 Q.$$

Thus, Eqs. (11,12) are satisfied and it follows from Theorem 1 that B_1 is a noiseless subsystem (a subspace in this case) for $\mathcal{E}$. To see this explicitly, let $\sigma \in \mathcal{B}(\mathcal{H}^{B_1})$ be arbitrary, and so

$$\sigma = a|01\rangle\langle 01| + b|01\rangle\langle 10| + c|10\rangle\langle 01| + d|10\rangle\langle 10|,$$

for some $a, b, c, d \in \mathbb{C}$. Then

$$\mathcal{E}(\sigma) = E_0 \sigma E_0^\dagger + E_1 \sigma E_1^\dagger = \sigma,$$

and the conditions of Lemma 2 are satisfied for all $\sigma \in \mathcal{B}(\mathcal{H}^{B_1}) = \mathcal{B}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1})$. Observe that a typical operator $\sigma \in \mathcal{B}(\mathcal{H}^{B_1})$ satisfies $E_1 \sigma = 0 \neq \sigma E_1$, and hence this noiseless subsystem is not supported by the noise commutant for $\mathcal{E}$.

There is another noiseless subsystem for $\mathcal{E}$ which is supported by the noise commutant. Decompose $\mathbb{C}^4 = \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_2}$ into the product of a pair of single qubit systems $\mathcal{H}^{A_2} = \text{span}\{|\alpha_1\rangle, |\alpha_2\rangle\} = \mathbb{C}^2$ and $\mathcal{H}^{B_2} = \text{span}\{|\beta_1\rangle, |\beta_2\rangle\} = \mathbb{C}^2$ such that

$$
\begin{aligned}
|\alpha_1\rangle \otimes |\beta_1\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|\alpha_1\rangle \otimes |\beta_2\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|\alpha_2\rangle \otimes |\beta_1\rangle &= \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
|\alpha_2\rangle \otimes |\beta_2\rangle &= \frac{|10\rangle - |01\rangle}{\sqrt{2}}.
\end{aligned}
$$

As noted below, $|0_L\rangle = |\beta_1\rangle$ and $|1_L\rangle = |\beta_2\rangle$ are logical zero and logical one states that remain immune to the errors of $\mathcal{E}$. For $1 \leq k, l \leq 2$, let

$$
\begin{aligned}
P_{kl} &= |\alpha_k\rangle\langle\alpha_l| \otimes \mathbb{1}^{B_2} \\
&= |\alpha_k\rangle\langle\alpha_l| \otimes (|\beta_1\rangle\langle\beta_1| + |\beta_2\rangle\langle\beta_2|) \\
&= (|\alpha_k\rangle \otimes |\beta_1\rangle)(\langle\alpha_l| \otimes \langle\beta_1|) + (|\alpha_k\rangle \otimes |\beta_2\rangle)(\langle\alpha_l| \otimes \langle\beta_2|)
\end{aligned}
$$

be the matrix units associated with this decomposition, and notice that these operators are given by

$$
\begin{aligned}
P_{11} &= |00\rangle\langle 00| + |11\rangle\langle 11| \quad &P_{12} &= |00\rangle\langle 10| + |11\rangle\langle 01| \\
P_{21} &= |10\rangle\langle 00| + |01\rangle\langle 11| \quad &P_{22} &= |10\rangle\langle 10| + |01\rangle\langle 01|.
\end{aligned}
$$

We calculate to find:

$$
\begin{aligned}
P_{11} E_0 P_{11} &= \alpha P_{11} \quad &P_{11} E_0 P_{22} &= 0 P_{12} \\
P_{22} E_0 P_{11} &= 0 P_{21} \quad &P_{22} E_0 P_{22} &= P_{22} \\
P_{11} E_1 P_{11} &= \beta P_{11} \quad &P_{11} E_1 P_{22} &= 0 P_{12} \\
P_{22} E_1 P_{11} &= 0 P_{21} \quad &P_{22} E_1 P_{22} &= 0 P_{22}.
\end{aligned}
$$

Thus, Eqs. (11,12) are satisfied and it follows from Theorem 1 that B_2 is a noiseless subsystem for $\mathcal{E}$. As an illustration of the conditions from Lemma 2 in this case, one can check that

$$\mathcal{E}(\mathbb{1}_2 \otimes \sigma) = \begin{pmatrix} 1-q & q \\ q & 1+q \end{pmatrix} \otimes \sigma \qquad \forall \sigma \in \mathcal{B}(\mathcal{H}^{B_2}),$$

where the tensor decomposition $\mathbb{C}^4 = \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_2}$ is given above.

4   Operator Quantum Error Correction

The unified scheme for quantum error correction consists of a triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ where again $\mathcal{R}$ and $\mathcal{E}$ are quantum operations on some $\mathcal{B}(\mathcal{H})$, but now $\mathfrak{A}$ is a semigroup in $\mathcal{B}(\mathcal{H})$ defined as above with respect to a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$.

Definition 2 *Given such a triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ we say the B-sector of $\mathfrak{A}$ is correctable for $\mathcal{E}$ if*

$$\left(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{R} \circ \mathcal{E} \right)(\sigma) = \mathrm{Tr}_A(\sigma) \quad \textit{for all} \quad \sigma \in \mathfrak{A}. \tag{15}$$

In other words, $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ is a correctable triple if the $\mathcal{H}^B$ sector of the semigroup $\mathfrak{A}$ encodes a noiseless subsystem for the error map $\mathcal{R} \circ \mathcal{E}$. Thus, substituting $\mathcal{E}$ by $\mathcal{R} \circ \mathcal{E}$ in Lemma 2 offers alternative equivalent definitions of a correctable triple. Since correctable codes consist of operator semigroups and algebras, we refer to this scheme as *Operator Quantum Error Correction* (OQEC). Observe that the standard model for error correction is given by the particular case in the OQEC model that occurs when $m = \dim \mathcal{H}^A = 1$. Lemma 2 shows that the decoherence-free subspace and noiseless subsystem methods are captured in this model when $\mathcal{R} = \mathrm{id}$ is the identity channel and, respectively, $m = 1$ and $m \geq 1$.

While we focus on the general setting of operator semigroups $\mathfrak{A}$ as correctable codes, it is important to note that correctability of a given $\mathfrak{A}$ is equivalent to the precise correction of the †-algebra

$$\mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$$

in the following sense. (Note the difference between $\mathfrak{A}_0$ just defined and $\mathfrak{A} = \{\sigma = \sigma^A \otimes \sigma^B : \sigma^{A,B} \in \mathcal{B}(H^{A,B})\}$; in the former case the A sector is restricted to the maximally mixed state while in the latter it is not.)

Theorem 2 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as in Eq. (10). Then the B-sector of $\mathfrak{A}$ is correctable for $\mathcal{E}$ if and only if there is a quantum operation $\mathcal{R}$ on $\mathcal{B}(\mathcal{H})$ such that*

$$(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \forall \sigma \in \mathfrak{A}_0. \tag{16}$$

Proof. If Eq. (16) holds, then condition *2.* of Lemma 2 holds for $\mathcal{R} \circ \mathcal{E}$ with $\tau^A = \mathbb{1}^A$ and hence the B-sector of $\mathfrak{A}$ is correctable for $\mathcal{E}$. For the converse, suppose that condition *2.* of Lemma 2 holds for $\mathcal{R} \circ \mathcal{E}$. Note that the map $\Gamma' = \{\frac{1}{\sqrt{m}} P_{kl}\}$ is trace preserving on $\mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$. Thus by Lemma 1 we have for all σ^B,

$$(\Gamma' \circ \mathcal{R} \circ \mathcal{E})(\mathbb{1}^A \otimes \sigma^B) = \Gamma'(\tau^A \otimes \sigma^B) \propto \mathbb{1}^A \otimes \sigma^B. \tag{17}$$

By trace preservation the proportionality factor must be one, and hence Eq. (16) is satisfied for $(\Gamma' \circ \mathcal{R}) \circ \mathcal{E}$. The map Γ' may be extended to a quantum operation on $\mathcal{B}(\mathcal{H})$ by including the projection $P_{\mathfrak{A}}^{\perp}$ onto $\mathcal{K}$ as a Kraus operator. As this does not effect the calculation Eq. (17), the result follows. $\square$

We next derive a testable condition that characterizes correctable codes for a given channel $\mathcal{E}$ in terms of its error operators and generalizes Eq. (2) for the standard model. We first glean some interesting peripheral information.

Lemma 3 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let P be a projection on $\mathcal{H}$. If $\mathcal{E}(P) = P$, then the range space $\mathcal{C}$ for P is invariant for every E_a; that is,*

$$E_a P = P E_a P \quad \forall a.$$

Proof. Let $|\psi\rangle$ belong to $\mathcal{C} = P\mathcal{H}$. Then by hypothesis and the positivity of $\mathcal{E}$, for each a we have

$$E_a|\psi\rangle\langle\psi|E_a^\dagger \le \sum_b E_b|\psi\rangle\langle\psi|E_b^\dagger = \mathcal{E}(|\psi\rangle\langle\psi|) \le \mathcal{E}(P) = P.$$

Thus $P^\perp(E_a|\psi\rangle\langle\psi|E_a^\dagger)P^\perp \le P^\perp P P^\perp = 0$ and so $P^\perp E_a|\psi\rangle = 0$. As both $|\psi\rangle$ and a were arbitrary the result follows. $\square$

An adjustment of this proof shows that more is true when $\mathcal{E}$ is contractive ($\mathcal{E}(\mathbb{1}) \le \mathbb{1}$). Specifically, $\mathcal{E}(P) \le P$ if and only if $E_a P = P E_a P$ for all a in this event. In the special case of unital operations one can further obtain the following [16].

Proposition 1. If $\mathcal{E} = \{E_a\}$ is a unital quantum operation and P is a projector, then $\mathcal{E}(P) = P$ if and only if the range space for P reduces each E_a; that is, $P E_a = E_a P$ for all a.

We now prove necessary and sufficient conditions for a semigroup $\mathfrak{A}$ to be correctable for a given error model. Sufficiency was first proven in [18]. We assume that matrix units $\{P_{kl}\}$ inside $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$ have been identified as above.

Theorem 3 *Let $\mathcal{E} = \{E_a\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as in Eq. (10). Then the B-sector of $\mathfrak{A}$ is correctable for $\mathcal{E}$ if and only if for any choice of matrix units $\{P_{kl}\}$ for $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$ as in Eq. (8), there are scalars $\Lambda = (\lambda_{abkl})$ such that*

$$P_{kk} E_a^\dagger E_b P_{ll} = \lambda_{abkl} P_{kl} \quad \forall a, b, k, l. \tag{18}$$

Proof. To prove necessity, by Theorem 2 we can assume there is a quantum operation $\mathcal{R}$ on $\mathcal{B}(\mathcal{H})$ such that $\mathcal{R} \circ \mathcal{E}$ acts as the identity channel on $\mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B) \subseteq \mathcal{B}(\mathcal{H})$. For brevity, we shall first suppose that $\mathcal{R} = \mathrm{id}$ is the identity channel.

Let $\mathcal{C} = P_{\mathfrak{A}}\mathcal{H}$ be the range of the projection $P_{\mathfrak{A}} = P_{11} + \ldots + P_{mm}$. Then since $P_{\mathfrak{A}} \in \mathfrak{A}_0$ we have $\mathcal{E}(P_{\mathfrak{A}}) = P_{\mathfrak{A}}$ and so Lemma 3 gives us $P_{\mathfrak{A}} E_a|_{\mathcal{C}} = E_a|_{\mathcal{C}}$ for all a.

With $\mathcal{B}(\mathcal{C})$ naturally regarded as imbedded inside $\mathcal{B}(\mathcal{H})$, define a completely positive map $\mathcal{E}_{\mathcal{C}} : \mathcal{B}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$ via

$$\sigma \mapsto \mathcal{E}_{\mathcal{C}}(\sigma) = P_{\mathfrak{A}}\mathcal{E}(\sigma)|_{\mathcal{C}} = P_{\mathfrak{A}}\mathcal{E}(P_{\mathfrak{A}}\sigma P_{\mathfrak{A}})|_{\mathcal{C}}$$

for all $\sigma \in \mathcal{B}(\mathcal{C})$. Then we have

$$\sum_a (P_{\mathfrak{A}}E_a|_{\mathcal{C}})^\dagger (P_{\mathfrak{A}}E_a|_{\mathcal{C}}) = \sum_a P_{\mathfrak{A}} E_a^\dagger E_a|_{\mathcal{C}} = P_{\mathfrak{A}}\mathbb{1}_{\mathcal{H}}|_{\mathcal{C}} = \mathbb{1}_{\mathcal{C}},$$

and so $\mathcal{E}_{\mathcal{C}}$ defines a quantum operation on $\mathcal{B}(\mathcal{C})$. Moreover, $\mathcal{E}_{\mathcal{C}}$ is unital as

$$\mathcal{E}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) = P_{\mathfrak{A}}\mathcal{E}(P_{\mathfrak{A}})|_{\mathcal{C}} = \mathbb{1}_{\mathcal{C}}.$$

Thus by hypothesis and Eq. (5) we have

$$\mathfrak{A}_0|_{\mathcal{C}} \subseteq \mathrm{Fix}(\mathcal{E}_{\mathcal{C}}) = \{P_{\mathfrak{A}}E_a|_{\mathcal{C}}, P_{\mathfrak{A}}E_a^\dagger|_{\mathcal{C}}\}',$$

where the latter commutant is computed inside $\mathcal{B}(\mathcal{C})$. It follows that

$$\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B = (\mathfrak{A}_0|_{\mathcal{C}})' \supseteq \{P_{\mathfrak{A}}E_a|_{\mathcal{C}}, P_{\mathfrak{A}}E_a^\dagger|_{\mathcal{C}}\}'' = \mathrm{C}^*(\{P_{\mathfrak{A}}E_a|_{\mathcal{C}}\}).$$

Since the P_{kl} form a set of matrix units that generate $(P_{\mathfrak{A}}\mathfrak{A}_0|_{\mathcal{C}})' = \mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$ as a vector space, there are scalars $\mu_{akl} \in \mathbb{C}$ such that

$$P_{kk} E_a P_{ll} = P_{kk}(P_{\mathfrak{A}}E_a|_{\mathcal{C}})P_{ll} = \mu_{akl} P_{kl}.$$

We now turn to the general case and suppose $\mathcal{R} = \{R_b\}$. The noise operators for the operation $\mathcal{R} \circ \mathcal{E}$ are $\{R_b E_a\}$ and thus we may find scalars μ_{abkl} such that

$$P_{kk} R_b E_a P_{ll} = \mu_{abkl} P_{kl} \quad \forall a, b, k, l.$$

Consider the products

$$
\begin{aligned}
\left(P_{kk} R_b E_a P_{ll} \right)^\dagger \left(P_{k'k'} R_b E_{a'} P_{l'l'} \right) &= \left(\overline{\mu_{abkl}} P_{lk} \right) \left(\mu_{a'bk'l'} P_{k'l'} \right) \\
&= \begin{cases} (\overline{\mu_{abkl}} \mu_{a'bkl'}) P_{ll'} & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases} .
\end{aligned}
$$

Noting that $\mathcal{C}$ is invariant for the noise operators $R_b E_a$ by Lemma 3, for fixed a, a' and l, l' we use $\sum_b R_b^\dagger R_b = \mathbb{1}$ to obtain

$$
\begin{aligned}
\left(\sum_{b,k} \overline{\mu_{abkl}} \mu_{a'bkl'} \right) P_{ll'} &= \sum_{b,k} \left(P_{ll} E_a^\dagger R_b^\dagger P_{kk} \right) \left(P_{kk} R_b E_{a'} P_{l'l'} \right) \\
&= \sum_b P_{ll} E_a^\dagger R_b^\dagger P_{\mathfrak{A}} R_b E_{a'} P_{l'l'} \\
&= P_{ll} E_a^\dagger \left(\sum_b R_b^\dagger R_b \right) E_{a'} P_{l'l'} \\
&= P_{ll} E_a^\dagger E_{a'} P_{l'l'}
\end{aligned}
$$

The proof is completed by setting $\lambda_{aa'll'} = \sum_{b,k} \overline{\mu_{abkl}} \mu_{a'bkl'}$ for all a, a' and l, l'.

For sufficiency, let us assume that Eq. (18) holds. Let $\sigma_k = |\alpha_k\rangle\langle\alpha_k| \in \mathcal{B}(\mathcal{H}^A)$, for $1 \leq k \leq m$, and define a quantum operation $\mathcal{E}_k : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H})$ by $\mathcal{E}_k(\rho^B) \equiv \mathcal{E}(\sigma_k \otimes \rho^B)$. With $P \equiv P_{\mathfrak{A}}$ and $E_{a,k} \equiv E_a P|\alpha_k\rangle$, it follows that $\mathcal{E}_k = \{E_{a,k}\}$. We shall find a quantum operation that globally corrects all of the errors $E_{a,k}$.

To do this, first note that we may define a quantum operation $\mathcal{E}_B : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H})$ with error model

$$\mathcal{E}_B = \left\{ \frac{1}{\sqrt{m}} E_{a,k} : \forall a, \forall 1 \leq k \leq m \right\}.$$

Then Eq. (18) and $P = \sum_k P_{kk}$ give us

$$
\begin{aligned}
\mathbb{1}^B E_{a,k}^\dagger E_{b,l} \mathbb{1}^B &= \mathbb{1}^B \langle\alpha_k| P E_a^\dagger E_b P |\alpha_l\rangle \mathbb{1}^B \\
&= \sum_{k',l'} \mathbb{1}^B \langle\alpha_k| P_{k'k'} E_a^\dagger E_b P_{l'l'} |\alpha_l\rangle \mathbb{1}^B \\
&= \sum_{k',l'} \lambda_{abk'l'} \mathbb{1}^B \langle\alpha_k| P_{k'l'} |\alpha_l\rangle \mathbb{1}^B = \lambda_{abkl} \mathbb{1}^B .
\end{aligned}
$$

In particular, Standard QEC implies the existence of a quantum operation $\mathcal{R} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^B)$ such that $(\mathcal{R} \circ \mathcal{E}_B)(\rho^B) = \rho^B$ for all ρ^B.

This implies that

$$
\begin{aligned}
(\mathcal{R} \circ \mathcal{E})(\mathbb{1}^A \otimes \rho^B) &= \mathcal{R}\left(\sum_k \mathcal{E}_k(\rho^B) \right) \\
&= m \mathcal{R}\left(\sum_{k,a} \frac{1}{m} E_{a,k} \rho^B E_{a,k}^\dagger \right) \\
&= m \mathcal{R} \circ \mathcal{E}_B(\rho^B) = m\rho^B .
\end{aligned}
$$

Hence we may define a channel $I_{\mathfrak{A}} : \mathcal{B}(\mathcal{H}^B) \to \mathcal{B}(\mathcal{H})$ via $I_{\mathfrak{A}}(\rho^B) = \frac{1}{m}(\mathbb{1}^A \otimes \rho^B)$. Thus, on defining $\mathcal{R}' \equiv I_{\mathfrak{A}} \circ \mathcal{R}$, we obtain

$$\left(\mathcal{R}' \circ \mathcal{E}\right)(\mathbb{1}^A \otimes \rho^B) = \mathbb{1}^A \otimes \rho^B \quad \forall \, \rho^B \in \mathcal{B}(\mathcal{H}^B).$$

The result now follows from an application of Theorem 2. □

Remark. The necessity of Eq. (18) for correction was initially established in [1]. Here we have provided a new operator algebra proof based on Eq. (5) and Theorem 2. In the original draft of this paper, we established sufficiency of Eq. (18) up to a set of technical conditions. More recently, sufficiency was established in full generality in [18]. In [18], two proofs of sufficiency were given; the first casts this condition into information theoretic language, and a sketch was given for the second. Here we have presented an operator algebra version (based on Theorem 2) of the proof of sufficiency sketched in [18].

Let us note that Eq. (18) is independent of the choice of basis $\{|\alpha_k\rangle\}$ that define the family P_{kl} and of the operator-sum representation for $\mathcal{E}$. In particular, under the changes $|\alpha'_k\rangle = \sum_l u_{kl}|\alpha_l\rangle$ and $F_a = \sum_b w_{ab}E_b$, the scalars Λ change to $\lambda'_{abkl} = \sum_{a'b'k'l'} \overline{u}_{kk'} u_{l'l} \overline{w}_{aa'} w_{bb'} \lambda_{abkl}$.

Equation (18) generalizes the quantum error correction condition Eq. (2) to the case where information is encoded in operators, not necessarily restricted to act on a fixed code subspace $\mathcal{C}$. However, observe that setting $k = l$ in Eq. (18) gives the standard error correction condition Eq. (2) with $P_{\mathcal{C}} = P_{kk}$. This leads to the following result.

Theorem 4 *If* $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ *is a correctable triple for some semigroup* $\mathfrak{A}$ *defined as in Eq. (10), then* $(\mathcal{P}_k \circ \mathcal{R}, \mathcal{E}, P_{kk}\mathfrak{A}P_{kk})$ *is a correctable triple according to the standard definition Eq. (2), where* P_{kk} *is any minimal reducing projection of* $\mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B)$, *and the map* $\mathcal{P}_k$ *is defined by* $\mathcal{P}_k(\cdot) = \sum_l P_{kl}(\cdot)P^{\dagger}_{kl}$.

Proof. Let $\sigma \in |\alpha_k\rangle\langle\alpha_k| \otimes \mathcal{B}(\mathcal{H}^B)$, so that $\sigma = P_{kk}\sigma P_{kk}$. Let $\mathcal{E} = \{E_a\}$ and $\mathcal{R} = \{R_b\}$. By Theorem 1 there are scalars λ_{abkl} such that $P_{kk}R_bE_aP_{ll} = \lambda_{abkl}P_{kl}$ $\forall \, a, b, k, l$. It follows that

$$
\begin{aligned}
(\mathcal{P}_k \circ \mathcal{R} \circ \mathcal{E})(\sigma) &= \sum_{a,b,l} P_{kl}R_bE_aP_{kk}\sigma P_{kk}E^{\dagger}_a R^{\dagger}_b P_{lk} \\
&= \sum_{a,b,l} (\lambda_{ablk}P_{kk})\sigma(\overline{\lambda_{ablk}}P_{kk}) \\
&= \Big(\sum_{a,b,l} |\lambda_{ablk}|^2\Big)\sigma.
\end{aligned}
$$

Thus $(\mathcal{P}_k \circ \mathcal{R} \circ \mathcal{E})(\sigma) \propto \sigma$ for all $\sigma \in |\alpha_k\rangle\langle\alpha_k| \otimes \mathcal{B}(\mathcal{H}^B)$, the proportionality factor independent of σ. In fact, this factor is one. To see this, fix k and note that Theorem 1 shows that

$$R_bE_aP_{kk} = R_bE_aP_{\mathfrak{A}}P_{kk} = P_{\mathfrak{A}}R_bE_aP_{\mathfrak{A}}P_{kk} = P_{\mathfrak{A}}R_bE_aP_{kk} \quad \forall \, a, b.$$

Hence, trace preservation of $\mathcal{R} \circ \mathcal{E}$ yields

$$
\begin{aligned}
\Big(\sum_{a,b,l} |\lambda_{ablk}|^2\Big)P_{kk} &= \sum_{a,b,l} (P_{kk}E^{\dagger}_a R^{\dagger}_b P_{ll})(P_{ll}R_bE_aP_{kk}) \\
&= P_{kk}\Big(\sum_{a,b} E^{\dagger}_a R^{\dagger}_b P_{\mathfrak{A}}R_bE_a\Big)P_{kk} \\
&= P_{kk}\Big(\sum_{a,b} E^{\dagger}_a R^{\dagger}_b R_bE_a\Big)P_{kk} = P_{kk}.
\end{aligned}
$$

As k was arbitrary, the result follows. $\square$

Remark. Theorem 4 has important consequences. Given a map $\mathcal{E}$, the existence of a correctable code subspace $\mathcal{C}$ — captured by the standard error correction condition Eq. (2) — is a prerequisite to the existence of any known type of error correction or prevention scheme (including the generalizations introduced here and in [1]). Moreover, Theorem 4 shows how to transform any one of these error correction or prevention techniques into a standard error correction scheme. However, while OQEC does not lead to new families of codes, it does allow for simpler correction procedures. See [19, 20] for further discussions on this point.

Remark. As a special case, Theorem 4 demonstrates that to every noiseless subsystem, there is an associated QEC code obtained by projecting the A-sector to a pure state. This is complementary to Theorem 6 of [10] which demonstrates that every QEC scheme composed of a triple $(\mathcal{R}, \mathcal{E}, \mathcal{C})$ arises as a noiseless subsystem of the map $\mathcal{E} \circ \mathcal{R}$.

We conclude this section by exhibiting the 2-qubit case of a new class of quantum channels, together with correctable subsystems, that is covered by OQEC, but for which the recovery operation does not fit into the Standard QEC protocol.

First, let us recall briefly that the motivating class of channels $\mathcal{E} = \{E_a\}$ which satisfy Eq. (2) occur when the restrictions $E_a|_{P_{\mathcal{C}}\mathcal{H}} = E_a|_{\mathcal{C}}$ of the error operators to $\mathcal{C}$ are scalar multiples of unitary operators U_a such that the subspaces $U_a\mathcal{C}$ are mutually orthogonal. In fact, this case describes any error model that satisfies Eq. (2), up to a linear transformation of the error operators. In this situation the positive scalar matrix Λ is diagonal. A correction operation here may be constructed by an application of the measurement operation determined by the subspaces $U_a\mathcal{C}$, followed by the reversals of the corresponding restricted unitaries $U_a P_{\mathcal{C}}$. Specifically, if P_a is the projection of $\mathcal{H}$ onto $U_a\mathcal{C}$, then $\mathcal{R} = \{U_a^\dagger P_a\}$ satisfies Eq. (1) for $\mathcal{E}$ on $\mathcal{C}$. The following is a generalization of this class of channels to the OQEC setting. For clarity we focus on the 2-qubit case.

Example. Let $\{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\}$ and $\{|a_1\rangle, |b_1\rangle, |a_2\rangle, |b_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^4$. Let P_1 be the projection onto $\mathrm{span}\{|a\rangle, |b\rangle\}$ and P_2 the projection onto $\mathrm{span}\{|a'\rangle, |b'\rangle\}$. Let Q_i, $i = 1, 2$, be the projection onto $\mathrm{span}\{|a_i\rangle, |b_i\rangle\}$. Define operators U_1, U_1', U_2, U_2' on $\mathbb{C}^4$ as follows:

$$\left\{ \begin{array}{rcl} U_1|a\rangle & = & |a_1\rangle \\ U_1|b\rangle & = & |b_1\rangle \\ U_1'|a'\rangle & = & |a_1\rangle \\ U_1'|b'\rangle & = & |b_1\rangle \end{array} \right. \quad \left\{ \begin{array}{rcl} U_2|a\rangle & = & |a_2\rangle \\ U_2|b\rangle & = & |b_2\rangle \\ U_2'|a'\rangle & = & |a_2\rangle \\ U_2'|b'\rangle & = & |b_2\rangle \end{array} \right. ,$$

and put $U_1 P_2 \equiv U_1' P_1 \equiv U_2 P_2 \equiv U_2' P_1 \equiv 0$. Then these operators are "partial isometries" and satisfy $U_1 = U_1 P_1$, $U_1' = U_1' P_2$, $U_2 = U_2 P_1$, $U_2' = U_2' P_2$. The operators $\mathcal{E} = \{E_1, E_2\}$ define a quantum channel where

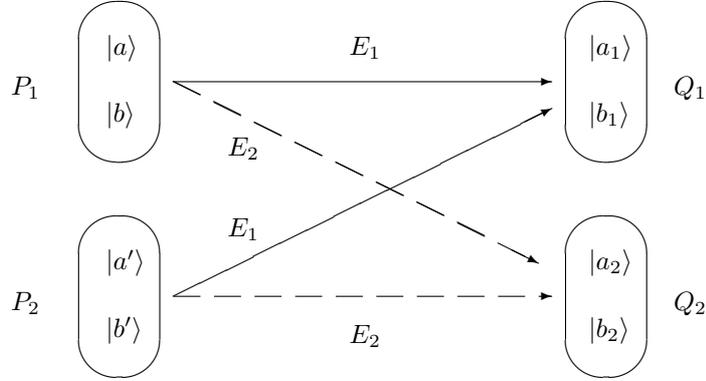$$E_1 = \frac{1}{\sqrt{2}}(U_1 P_1 + U_1' P_2)$$

$$E_2 = \frac{1}{\sqrt{2}}(U_2 P_1 - U_2' P_2).$$

The action of E_1 and E_2 is indicated in Figure 1.

Here the matrix units are given by

$$P_1 = P_{11} = |a\rangle\langle a| + |b\rangle\langle b|$$

Fig. 1.

$$P_2 = P_{22} = |a'\rangle\langle a'| + |b'\rangle\langle b'|$$

$$P_{12} = |a\rangle\langle a'| + |b\rangle\langle b'|$$

$$P_{21} = |a'\rangle\langle a| + |b'\rangle\langle b|.$$

For trace preservation, observe that

$$
\begin{aligned}
E_1^\dagger E_1 &= \frac{1}{2}\big(P_1 U_1^\dagger + P_2 (U_1')^\dagger\big)\big(U_1 P_1 + U_1' P_2\big) \\
&= \frac{1}{2}\big(P_{11} + P_{12} + P_{21} + P_{22}\big).
\end{aligned}
$$

Similarly, we compute

$$E_2^\dagger E_2 = \frac{1}{2}\big(P_{11} - P_{12} - P_{21} + P_{22}\big).$$

Thus we have $E_1^\dagger E_1 + E_2^\dagger E_2 = P_{11} + P_{22} = \mathbb{1}_4$. Equations (18) are computed as follows:

$$P_k E_i^\dagger E_i P_k = \frac{1}{2} P_k \quad \text{for} \quad i, k = 1, 2,$$

$$P_k E_i^\dagger E_j P_l = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad k, l = 1, 2,$$

$$P_1 E_1^\dagger E_1 P_2 = \frac{1}{2} P_{12} = \big(\tfrac{1}{2} P_{21}\big)^\dagger = \big(P_2 E_1^\dagger E_1 P_1\big)^\dagger,$$

$$P_1 E_2^\dagger E_2 P_2 = \frac{-1}{2} P_{12} = \big(\tfrac{-1}{2} P_{21}\big)^\dagger = \big(P_2 E_2^\dagger E_2 P_1\big)^\dagger.$$

Define

$$V_{11} = U_1 P_1, \quad V_{12} = U_1' P_2, \quad V_{21} = U_2 P_1, \quad V_{22} = U_2' P_2$$

and observe that

$$V_{11} V_{11}^\dagger = U_1 P_1 U_1^\dagger = Q_1 = U_1' P_2 (U_1')^\dagger = V_{12} V_{12}^\dagger$$

$$V_{21} V_{21}^\dagger = U_2 P_1 U_2^\dagger = Q_2 = U_2' P_2 (U_2')^\dagger = V_{22} V_{22}^\dagger.$$

Then a calculation shows that the channel

$$\mathcal{R} = \left\{ \frac{1}{\sqrt{2}} V_{jk}^{\dagger} Q_j : 1 \leq j, k \leq 2 \right\}$$

corrects for all errors induced by $\mathcal{E}$ on $\mathfrak{A}_0 \cong \mathbb{1}_2 \otimes \mathcal{M}_2$. Specifically, $(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma$ for all $\sigma \in \mathcal{B}(\mathbb{C}^4)$ which have a matrix representation of the form $\sigma = \left(\begin{smallmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{smallmatrix} \right)$, $\sigma_1 \in \mathcal{M}_2$, with respect to the ordered basis $\{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\}$ for $\mathbb{C}^4$. That is, $(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma$ for all $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in \mathbb{C}$ and all

$$\begin{aligned} \sigma &= \alpha_{11} (|a\rangle\langle a| + |a'\rangle\langle a'|) + \alpha_{12} (|a\rangle\langle b| + |a'\rangle\langle b'|) \\ &\quad + \alpha_{21} (|b\rangle\langle a| + |b'\rangle\langle a'|) + \alpha_{22} (|b\rangle\langle b| + |b'\rangle\langle b'|). \end{aligned}$$

Thus $\mathcal{R}$ corrects all $\sigma = \mathbb{1}_2 \otimes \sigma_1$ that are "equally balanced" with respect to the standard bases for the ranges of P_1 and P_2. Further, by Theorem 2 we know $\mathcal{R}$ corrects the associated semigroup $\mathfrak{A}$ in the sense of Definition 2.

Remark. We note that recent work [19] presents physically motivated examples in which correction of subsystems is accomplished within the OQEC framework. Furthermore, a general class of recovery procedures based on the stabilizer formalism was recently presented in [20]. In particular, this work builds on OQEC to demonstrate how certain stabilizer codes can be simplified by incorporating gauge qubits. These have the effect of reducing the number of syndrome measurements required to correct the error map and extend the class of physical realizations of the logical operations on the encoded data.

5   Unitarily Noiseless Subsystems

In this section we discuss error triples $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ such that the restriction of $\mathcal{R}$ to $\mathcal{E}(\mathfrak{A})$ is a unitary operation. Consideration of this case leads to a generalization of the noiseless subsystem protocol that falls under the OQEC umbrella. Let us first consider a direct generalization of the fixed point set algebraic approach as in Eq. (5). Here we have the equation

$$\mathcal{E}(\sigma) = U \sigma U^{\dagger} \quad \forall \sigma \in \mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B), \tag{19}$$

for some unitary operator U. When $\mathfrak{A}_0$ satisfies Eq. (19) for a unitary U we shall say that $\mathfrak{A}_0$ is a *unitarily noiseless subsystem* (UNS) for $\mathcal{E}$. Of course, a subsystem $\mathfrak{A}_0$ that satisfies Eq. (19) is not noiseless, but it may be easily corrected by applying the reversal operation $U^{\dagger}(\cdot)U$. As we indicate below, this can lead to new non-trivial correctable subsystems not obtained under the noiseless subsystem regime. If $\mathcal{E}$ is a unital operation, it is possible to explicitly compute all UNS's for $\mathcal{E}$.

Theorem 5 *If $\mathcal{E} = \{E_a\}$ is a unital quantum operation on $\mathcal{B}(\mathcal{H})$ and U is a unitary on $\mathcal{H}$, then the corresponding unitarily noiseless subsystem $\mathfrak{A}_0$ is equal to the commutant of the operators $\{U^{\dagger} E_a\}$;*

$$\mathfrak{A}_0 = \left\{ \sigma \in \mathcal{B}(\mathcal{H}) : \mathcal{E}(\sigma) = \sum_a E_a \sigma E_a^{\dagger} = U \sigma U^{\dagger} \right\} = \left\{ U^{\dagger} E_a \right\}'.$$

Proof. The set of σ that satisfy Eq. (19) is equal to the set of σ that satisfy $U^{\dagger} \mathcal{E}(\sigma) U = \sigma$. Thus, here we are considering the fixed point set for the unital operation $U^{\dagger} \mathcal{E}(\cdot) U$, which has noise operators $\{U^{\dagger} E_a\}$. The result now follows from Eq. (5). $\square$

Let us consider a simple example of how this scheme can be used to identify new correctable codes for a given channel.

Example. Let $Z_1 = Z \otimes \mathbb{1}_2$ and $Z_2 = \mathbb{1}_2 \otimes Z$ with the Pauli matrix $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Then, with respect to the standard orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for $\mathbb{C}^4$, we have

$$\{Z_1, Z_2\}' = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \right\},$$

Hence there are no non-trivial noiseless subsystems for the corresponding channel $\mathcal{E} = \{Z_1, Z_2\}$. However, if we let $U \in \mathcal{B}(\mathbb{C}^4)$ be the unitary

$$U|ij\rangle = \left\{ \begin{array}{ll} |ij\rangle & \text{if } i \neq 1 \text{ or } j \neq 1 \\ -|11\rangle & \text{if } i = 1 \text{ and } j = 1 \end{array} \right. ,$$

then we compute

$$\{U^\dagger Z_1, U^\dagger Z_2\}' = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ e & 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{C} \right\}.$$

In particular, the $\dagger$-algebra $\mathfrak{A}_0 = \{U^\dagger Z_i\}'$ is unitarily equivalent to $\mathfrak{A}_0 \cong \mathcal{M}_2 \oplus \mathbb{C} \oplus \mathbb{C}$. Thus, a single qubit code subspace may be corrected. Specifically, all operators $\sigma \in \mathfrak{A}_0$ may be corrected by applying $U^\dagger(\cdot)U$ since they satisfy $\mathcal{E}(\sigma) = U\sigma U^\dagger$.

In a similar manner we can extend this discussion to the case of noiseless subsystems for arbitrary quantum operations. The analogue of Eq. (19) in this case is

$$\forall \sigma^A \, \forall \sigma^B, \; \exists \tau^A \; : \; \mathcal{E}(\sigma^A \otimes \sigma^B) = U(\tau^A \otimes \sigma^B)U^\dagger, \tag{20}$$

where U is a fixed unitary on $\mathcal{H}$. In effect, this is the special case of the OQEC formulation Eq. (15) where the recovery $\mathcal{R}$ is unitary. In this context the conditions of Lemma 2 yield the following.

Theorem 6 *Given a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, a map $\mathcal{E}$ on $\mathcal{B}(\mathcal{H})$ and a unitary U on $\mathcal{H}$, the following three conditions are equivalent:*

1. *Eq. (20) is satisfied.*

2. *$\forall \sigma^B, \; \exists \tau^A \; : \; \mathcal{E}(\mathbb{1}^A \otimes \sigma^B) = U(\tau^A \otimes \sigma^B)U^\dagger$.*

3. *$\forall \sigma \in \mathfrak{A} \; : \; \big(\text{Tr}_A \circ \mathcal{P}_\mathfrak{A} \circ \mathcal{U}^{-1} \circ \mathcal{E} \big)(\sigma) = \text{Tr}_A(\sigma).$*

where $\mathcal{U}^{-1}(\cdot) = U^\dagger(\cdot)U$.

6   Conclusion

We have presented a detailed analysis of the OQEC formalism for error correction in quantum computing. This approach provides a unified framework for investigations into both active and passive error correction techniques. Fundamentally, we have generalized the setting for

correction from states to operators. The condition from standard quantum error correction was shown to be necessary for any of these schemes to be feasible. Included in this formalism is a scheme for identifying noiseless subsystems that applies to arbitrary (not necessarily unital) quantum operations. We also introduced the notion of unitarily noiseless subsystems as a natural relaxation of the noiseless subsystem condition.

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