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QUANTUM MEASUREMENTS AND ENTROPIC BOUNDS ON INFORMATION TRANSMISSION

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While a positive operator valued measure gives the probabilities in a quantum measurement, an instrument gives both the probabilities and the a posteriori states. By interpreting the instrument as a quantum channel and by using the monotonicity theorem for relative entropies many bounds on the classical information extracted in a quantum measurement are obtained in a unified manner. In particular, it is shown that such bounds can all be stated as inequalities between mutual entropies. This approach based on channels gives rise to a unified picture of known and new bounds on the classical information (the bounds by Holevo, by Shumacher, Westmoreland and Wootters, by Hall, by Scutaru, a new upper bound and a new lower one). Some examples clarify the mutual relationships among the various bounds.

Keywords: Instrument, Channel, Quantum information, Entropy, Mutual entropy, Holevo bound
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1 Introduction

A problem which appears in the field of quantum communication and in quantum statistics is the following: a collection of statistical operators, with some a priori probabilities, describes the possible states of a quantum system and an observer wants to decide by means of a quantum measurement in which of these states the system is. The quantity of information extracted by the measurement is the classical mutual information I_c of the input/output joint distribution; interesting upper and lower bounds for I_c, due to the quantum nature of the measurement, are given in the literature [1, 2, 3, 4, 5, 6, 7].

Usually the measurement is described by a *generalized observable* or *positive operator valued* (POV) *measure* which allows to obtain the probabilities for the outcomes of the measurement. However, with respect to a POV measure, a more detailed level of description of the quantum measurement is represented by a different mathematical object, the *instrument*

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[8, 9, 10]: given a state (the preparation) as input, it gives as output not only the probabilities of the outcomes but also the state after the measurement, conditioned on the observed outcome (the a posteriori state). We can think the instrument to be a channel: from a quantum state (the pre-measurement state) to a quantum/classical state (a posteriori state plus probabilities). The mathematical formalization of the idea that an instrument *is* a channel is central in our paper and allows for a unified approach to various bounds for I_c and for related quantities [11, 12].

To maintain things at a sufficiently simple mathematical level, we shall develop and present all the results in the case of a finite-dimensional Hilbert space, a finite alphabet and an instrument with finite outcomes.

In Section 2 we introduce the notion of instrument and we show how to associate a channel to it; some inequalities on various relative entropies are deduced from Ulhmann monotonicity theorem. From such inequalities we obtain in Section 3 some bounds on the quantity of information I_c which can be extracted by using an instrument as decoding apparatus; more precisely, we obtain the bound of Holevo [1] (31), a slight generalization of the bound of Shumacher, Westmoreland, Wootters (SWW) [4] (39) and the new inequalities (35), (47). From the SWW bound we obtain in a straight way also a result by Groenewold, Lindblad, Ozawa [13, 14, 15] on the positivity of the *quantum information gain* given by an instrument. We also show how such bounds can be stated as inequalities between mutual entropies (the relative entropy of a bipartite state with respect to its marginals). In Section 4 we generalize a transformation due to Hall [5], we introduce a new instrument and we obtain another set of bounds on I_c: Hall bound (72), a strengthening of it (76), Scutaru bound [3] (79) and the new inequality (82). All the bounds of Sections 3 and 4 concern a fixed instrument and the associated POV measure; we can say that they quantify the performances of the measurement procedure with respect to the initial ensemble. In Section 5 we give a summary and some examples of the various bounds.

2 Instruments and channels

Let $\mathcal{H} = \mathbb{C}^d$ be the Hilbert space associated with the quantum system QS; we denote by M_d the algebra of the complex $(d\times d)$-matrices and by $\mathcal{S}_d \subset M_d$ the set of statistical operators on $\mathbb{C}^d$.

2.1 Instruments, probabilities and a posteriori states

We consider a measurement on QS represented by a completely positive instrument $\mathcal{I}$ with finitely many outcomes; let us denote by Ω the finite set of possible outcomes (the *value space*). Then, the instrument $\mathcal{I}$ has the structure

$$\mathcal{I}(F)[\rho] = \sum_{\omega\in F} \mathcal{O}(\omega)[\rho], \qquad \forall F \subset \Omega, \quad \forall \rho \in M_d, \tag{1a}$$

$$\mathcal{O}(\omega)[\rho] = \sum_{k\in K} V_k^\omega \rho V_k^{\omega\dagger}, \tag{1b}$$

$$\sum_{\omega\in\Omega} E_{\mathcal{I}}(\omega) = \mathbb{1}, \qquad E_{\mathcal{I}}(\omega) = \sum_{k\in K} V_k^{\omega\dagger} V_k^\omega, \tag{1c}$$

where $V_k^\omega \in M_d$, K is a suitable finite set and $\mathbb{1}$ is the unit element of M_d. Note that $E_{\mathcal{I}}$ is a POV measure, the POV measure associated with $\mathcal{I}$; $\mathcal{O}(\omega)$ is an *operation* [16]. If the pre-measurement state is $\rho \in \mathcal{S}_d$, the probability of the result $\{\omega \in F\}$, $F \subset \Omega$, is

$$P_\rho(F) = \sum_{\omega \in F} p_\rho(\omega) = \mathrm{Tr}\{\mathcal{I}(F)[\rho]\}, \qquad p_\rho(\omega) = \mathrm{Tr}\{E_{\mathcal{I}}(\omega)\rho\} = \mathrm{Tr}\{\mathcal{O}(\omega)[\rho]\}, \tag{2}$$

and the post-measurement state, conditioned on this result, is $\big(\mathrm{Tr}\{\mathcal{I}(F)[\rho]\}\big)^{-1}\mathcal{I}(F)[\rho]$. When F shrinks to a single point, the conditional post-measurement state reduces to what is called the *a posteriori state* [17]

$$\pi_\rho^{\mathcal{I}}(\omega) = \frac{\mathcal{O}(\omega)[\rho]}{p_\rho(\omega)}, \qquad \text{if } \ p_\rho(\omega) > 0\,; \tag{3}$$

this definition has to be completed by defining arbitrarily $\pi_\rho^{\mathcal{I}}(\omega)$ for the points ω for which $p_\rho(\omega) = 0$. The a posteriori state is the state to be attributed to the quantum system QS after the measurement when we know that the result of the measurement has been exactly $\{\omega\}$. On the opposite side, we have the unconditional post-measurement state or *a priori state*

$$\mathcal{I}(\Omega)[\rho] = \sum_{\omega \in \Omega} \mathcal{O}(\omega)[\rho]\,; \tag{4}$$

it is the state to be attributed to the system after the measurement, when the result is not known.

2.2 States, entropies, channels

2.2.1 Algebras and states

To formalize the idea that an instrument is a channel, we need to introduce the spaces $\mathcal{C}(\Omega; M_d)$ of the functions from Ω into M_d and $\mathcal{C}(\Omega) \equiv \mathcal{C}(\Omega; \mathbb{C})$, which are finite C^*-algebras, as M_d; note that $\mathcal{C}(\Omega; M_d) \simeq \mathcal{C}(\Omega) \otimes M_d$. A state on a finite C^*-algebra is a normalized, positive linear functional on the algebra and in our cases we have:

- A state ρ on M_d is identified with a statistical operator, i.e. $\rho \in \mathcal{S}_d$, and ρ applied to an element a of M_d is given by $\langle \rho, a \rangle = \mathrm{Tr}\{\rho a\}$; this is the usual quantum setup.
- A state p on $\mathcal{C}(\Omega)$ is a discrete probability density on Ω and $\langle p, a \rangle = \sum_{\omega \in \Omega} p(\omega) a(\omega)$; this is the classical setup.
- A state Σ on $\mathcal{C}(\Omega; M_d)$ is itself an element of $\mathcal{C}(\Omega; M_d)$ such that $\Sigma(\omega) \geq 0$ and $\sum_{\omega \in \Omega} \mathrm{Tr}\{\Sigma(\omega)\} = 1$; the action of the state Σ on an element $a \in \mathcal{C}(\Omega; M_d)$ is given by $\langle \Sigma, a \rangle = \sum_{\omega \in \Omega} \mathrm{Tr}\{\Sigma(\omega) a(\omega)\}$. Note the quantum/classical hybrid character of this case.

2.2.2 Entropies and relative entropies

Entropies and relative entropies can be defined in very general situations [18], but here we are interested only in the finite case, where the definitions become simpler. In the book by Ohya and Petz [18], the whole Part I is dedicated to the finite-dimensional case, while the rest of the book treats the general case. A finite C^*-algebra $\mathcal{C}$ can always be seen as a subalgebra of block-diagonal matrices in a big matrix algebra M_N and the definition of entropy for states on

$\mathcal{C}$ is derived from the von Neumann definition for states on M_N; the same type of definition applies to the relative entropy ([18], Part I). In some sense this is the general formulation of the trick of embedding classical probabilities into quantum states, a trick by which many results in quantum information theory have been proved. Entropies and relative entropies are non negative; the relative entropy can be infinite. In the case of our three C^*-algebras we have:

- For $\rho_1, \rho_2 \in \mathcal{S}_d$, the entropy is

$$S(\rho_i) = -\operatorname{Tr}\{\rho_i \log \rho_i\} =: S_{\mathrm{q}}(\rho_i) \tag{5a}$$

(the von Neumann entropy), and the relative entropy of ρ_1 with respect to ρ_2 is

$$S(\rho_1 \| \rho_2) = \operatorname{Tr}\{\rho_1(\log \rho_1 - \log \rho_2)\} =: S_{\mathrm{q}}(\rho_1 \| \rho_2)\,. \tag{5b}$$

- In the classical case, for two states p_1, p_2 on $\mathcal{C}(\Omega)$, the entropy is

$$S(p_i) = -\sum_{\omega\in\Omega} p_i(\omega) \log p_i(\omega) =: S_{\mathrm{c}}(p_i) \tag{6a}$$

(the Shannon information), and the relative entropy is

$$S(p_1 \| p_2) = \sum_{\omega\in\Omega} p_1(\omega) \log \frac{p_1(\omega)}{p_2(\omega)} =: S_{\mathrm{c}}(p_1 \| p_2) \tag{6b}$$

(the Kullback-Leibler informational divergence).

- For two states Σ_1, Σ_2 on $\mathcal{C}(\Omega; M_d)$ we have

$$S(\Sigma_i) = -\sum_{\omega\in\Omega} \operatorname{Tr}\{\Sigma_i(\omega) \log \Sigma_i(\omega)\} = S_{\mathrm{c}}(p_i) + \sum_{\omega\in\Omega} p_i(\omega) S_{\mathrm{q}}\big(\sigma_i(\omega)\big), \tag{7a}$$

$$\begin{aligned} S(\Sigma_1 \| \Sigma_2) &= \sum_{\omega\in\Omega} \operatorname{Tr}\big\{\Sigma_1(\omega)\big(\log \Sigma_1(\omega) - \log \Sigma_2(\omega)\big)\big\} \\ &= S_{\mathrm{c}}(p_1 \| p_2) + \sum_{\omega\in\Omega} p_1(\omega) S_{\mathrm{q}}\big(\sigma_1(\omega) \big\| \sigma_2(\omega)\big), \end{aligned} \tag{7b}$$

$$p_i(\omega) := \operatorname{Tr}\{\Sigma_i(\omega)\}, \qquad \sigma_i(\omega) := \frac{\Sigma_i(\omega)}{p_i(\omega)}\,. \tag{8}$$

In both equations (7a) and (7b) the first step is by definition and the second one by simple computations; in (8), when $p_i(\omega) = 0$, $\sigma_i(\omega)$ is defined arbitrarily.

In the previous formulas we have used the subscripts “c” for “classical” and “q” for “quantum” to underline the cases in which the entropy and the relative entropy are of pure classical character or of pure quantum one.

2.2.3 Mutual entropy and χ-quantities.

In classical information theory a key concept is that of mutual information which is the relative entropy of a joint distribution p_{XY} with respect to the product of its marginals p_X, p_Y:

$$S_{\rm c}(p_{XY}\|p_X\otimes p_Y):=\sum_{x,y}p_{XY}(x,y)\log\frac{p_{XY}(x,y)}{p_X(x)p_Y(y)}$$
$$\equiv\sum_x p_X(x)\,S_{\rm c}(p_{Y|X}(\bullet|x)\|p_Y)\equiv\sum_y p_Y(y)\,S_{\rm c}(p_{X|Y}(\bullet|y)\|p_X)\,,\qquad(9)$$

$$p_X(x):=\sum_y p_{XY}(x,y),\qquad p_Y(y):=\sum_x p_{XY}(x,y),\qquad(10a)$$

$$p_{Y|X}(y|x):=\frac{p_{XY}(x,y)}{p_X(x)}\,,\qquad p_{X|Y}(y|x):=\frac{p_{XY}(x,y)}{p_Y(y)}\,.\qquad(10b)$$

The idea of mutual information can be generalized to all the situations when one has states on a tensor product of algebras. Let $\mathcal{C}_i$, $i=1,2$ be two finite C^*-algebras; let Π_{12} be a state on $\mathcal{C}_1\otimes\mathcal{C}_2$; its *marginals* Π_i are its restrictions to the two factors in the tensor product: $\Pi_i:=\Pi_{12}\big|_{\mathcal{C}_i}$. Then, the *mutual information* or the *mutual entropy* of the joint state Π_{12} is its relative entropy with respect to the tensor product of its marginals: $S(\Pi_{12}\|\Pi_1\otimes\Pi_2)$.

For instance, in the case $\mathcal{C}_1=\mathcal{C}(\Omega)$, $\mathcal{C}_2=M_d$, a state Σ on $\mathcal{C}_1\otimes\mathcal{C}_2\simeq\mathcal{C}(\Omega;M_d)$ has marginals p and $\overline{\sigma}:=\sum_\omega\Sigma(\omega)=\sum_\omega p(\omega)\sigma(\omega)$, where $p(\omega)$ and $\sigma(\omega)$ are defined as in Eq. (8). Then, by Eq. (7b) the mutual entropy of Σ is

$$S(\Sigma\|p\otimes\overline{\sigma})=\sum_{\omega\in\Omega}p(\omega)S_{\rm q}\big(\sigma(\omega)\big\|\overline{\sigma}\big)\equiv S_{\rm q}(\overline{\sigma})-\sum_{\omega\in\Omega}p(\omega)S_{\rm q}\big(\sigma(\omega)\big)\qquad(11)$$

In quantum information theory, a couple $\{p,\sigma\}$ of a probability p (let us say on the set Ω) and a family of statistical operators $\sigma(\omega)$ is known as an *ensemble* and

$$\overline{\sigma}=\sum_\omega p(\omega)\sigma(\omega)\qquad(12)$$

is the *average* state of the ensemble. It is trivial to see that the ensemble $\{p,\sigma\}$ is equivalent to the state $\Sigma=\{p(\omega)\sigma(\omega)\}$ on $\mathcal{C}(\Omega;M_d)$; the mutual entropy of this state is called the *χ-quantity* of the ensemble:

$$\chi\{p,\sigma\}:=\sum_{\omega\in\Omega}p(\omega)S_{\rm q}\big(\sigma(\omega)\big\|\overline{\sigma}\big)=S(\Sigma\|p\otimes\overline{\sigma}).\qquad(13)$$

2.2.4 Channels

A (quantum) *channel* Λ ([18] p. 137), or dynamical map, or stochastic map is a completely positive linear map from a finite C^*-algebra $\mathcal{C}_1$ to another one $\mathcal{C}_2$ (but the definition can be extended easily), which transforms states into states. The composition of channels gives again a channel. Channels are usually introduced to describe noisy quantum evolutions, but we shall see that also an instrument can be identified with a channel.

The fundamental *Uhlmann monotonicity theorem* says that channels decrease the relative entropy ([18], Theor. 1.5 p. 21): let $\Lambda:\mathcal{C}_1\to\mathcal{C}_2$ be a channel between finite C^*-algebras; for any two states Σ,Ψ on $\mathcal{C}_1$, the inequality $S(\Sigma\|\Psi)\geq S(\Lambda[\Sigma]\|\Lambda[\Psi])$ holds.

If we have three algebras $\mathcal{A}, \mathcal{C}_1, \mathcal{C}_2$ and three channels $\Lambda_1 : \mathcal{A} \to \mathcal{C}_1$, $\Lambda_2 : \mathcal{A} \to \mathcal{C}_2$, $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$, such that $\Phi \circ \Lambda_1 = \Lambda_2$, we say that the channel Λ_1 is a *refinement* of Λ_2 or that Λ_2 is a *coarse graining* of Λ_1 ([18] p. 138). In this case, for any two states Σ, Ψ on $\mathcal{A}$, we have $S(\Sigma\|\Psi) \geq S(\Lambda_1[\Sigma]\|\Lambda_1[\Psi]) \geq S(\Lambda_2[\Sigma]\|\Lambda_2[\Psi])$.

2.3 Instruments, channels and inequalities on relative entropies

2.3.1 The instrument as a channel

Let us define the linear map $\Lambda_{\mathcal{I}}$ from M_d into $\mathcal{C}(\Omega; M_d)$ by

$$\tau \mapsto \Lambda_{\mathcal{I}}[\tau] , \qquad \Lambda_{\mathcal{I}}[\tau](\omega) := \mathcal{O}(\omega)[\tau] . \tag{14}$$

If $\rho \in \mathcal{S}_d$, then $\Lambda_{\mathcal{I}}[\rho]$ is a state on $\mathcal{C}(\Omega; M_d)$; moreover, by the structure of $\mathcal{O}(\omega)$, $\Lambda_{\mathcal{I}}$ turns out to be completely positive. Therefore, $\Lambda_{\mathcal{I}}$ is a channel, the channel associated with the instrument $\mathcal{I}$. It is also possible to show that any channel from M_d into $\mathcal{C}(\Omega; M_d)$ is the channel associated to a unique instrument. In the case of general instruments, the instrument/channel correspondence is treated in [12].

By Uhlmann monotonicity theorem, we have for any two states ρ and ϕ on M_d

$$S(\rho\|\phi) \geq S(\Lambda_{\mathcal{I}}[\rho]\|\Lambda_{\mathcal{I}}[\phi]) . \tag{15}$$

By Eqs. (7b), (8), (14), (2), (3), inequality (15) becomes

$$S_{\mathrm{q}}(\rho\|\phi) \geq S_{\mathrm{c}}(p_\rho\|p_\phi) + \sum_{\omega\in\Omega} p_\rho(\omega) S_{\mathrm{q}}\big(\pi_\rho^{\mathcal{I}}(\omega)\big\|\pi_\phi^{\mathcal{I}}(\omega)\big). \tag{16}$$

This is a fundamental inequality. A possible interpretation is that the "quantum information" $S_{\mathrm{q}}(\rho\|\phi)$ contained in the couple of quantum states ρ and ϕ is not less than the sum of the classical information $S_{\mathrm{c}}(p_\rho\|p_\phi)$ extracted by the measurement and of the mean "quantum information" $\sum_{\omega\in\Omega} p_\rho(\omega) S_{\mathrm{q}}\big(\pi_\rho^{\mathcal{I}}(\omega)\big\|\pi_\phi^{\mathcal{I}}(\omega)\big)$ left in the a posteriori states.

The POV measure as a channel. In [18], pp. 137-138, another channel is introduced, which involves only the POV measure, by

$$\Lambda_E[\tau](\omega) := \mathrm{Tr}\{E_{\mathcal{I}}(\omega)\tau\} , \qquad \tau \in M_d ; \tag{17}$$

it is easy to check all the properties which define a channel $\Lambda_E : M_d \to \mathcal{C}(\Omega)$. Uhlmann monotonicity theorem applied to this case gives the inequality ([18], pp. 9, 151)

$$S_{\mathrm{q}}(\rho\|\phi) \geq S_{\mathrm{c}}(p_\rho\|p_\phi) , \tag{18}$$

which is weaker than (16). This is due to the fact that inequality (16) has been obtained by using a refinement $\Lambda_{\mathcal{I}}$ of the Ohya-Petz channel Λ_E. Indeed, let us introduce the map $\Phi_{\mathrm{c}} : \mathcal{C}(\Omega; M_d) \to \mathcal{C}(\Omega)$, $\Phi_{\mathrm{c}}[\Sigma](\omega) = \mathrm{Tr}\{\Sigma(\omega)\}$; in some sense, Φ_{c} extracts the classical part of the state Σ. Then, it is easy to check that Φ_{c} is a channel and that $\Lambda_E = \Phi_{\mathrm{c}} \circ \Lambda_{\mathcal{I}}$.

2.3.2 The channel $\mathcal{I}(\Omega)$.

Another inequality is obtained by introducing the channel Φ_{q}, which extracts the quantum part of a state Σ on $\mathcal{C}(\Omega; M_d)$:

$$\Phi_{\mathrm{q}}[\Sigma] := \sum_{\omega\in\Omega} \Sigma(\omega)\,. \tag{19}$$

By Eqs. (19), (14), (4), we get

$$\Phi_{\mathrm{q}} \circ \Lambda_{\mathcal{I}} = \mathcal{I}(\Omega)\,; \tag{20}$$

$\mathcal{I}(\Omega)$ is a channel from M_d into itself, which is a coarse graining of $\Lambda_{\mathcal{I}}$. This gives the inequality

$$S(\Lambda_{\mathcal{I}}[\rho]\|\Lambda_{\mathcal{I}}[\phi]) \geq S(\mathcal{I}(\Omega)[\rho]\|\mathcal{I}(\Omega)[\phi]) \tag{21}$$

or

$$S_{\mathrm{c}}(p_\rho\|p_\phi) + \sum_{\omega\in\Omega} p_\rho(\omega) S_{\mathrm{q}}\big(\pi_\rho^{\mathcal{I}}(\omega)\big\|\pi_\phi^{\mathcal{I}}(\omega)\big) \geq S_{\mathrm{q}}(\mathcal{I}(\Omega)[\rho]\|\mathcal{I}(\Omega)[\phi]). \tag{22}$$

2.3.3 A transpose of the channel Λ_E.

In [18] pp. 141–143 the transpose of a channel with respect to a fixed state is defined; such a definition is particularly simple in the case of the channel Λ_E and allows to introduce a new channel which produces new inequalities of interest in quantum information. Let us fix a quantum state $\phi \in \mathcal{S}_d$, with $p_\phi(\omega) > 0$, $\forall \omega \in \Omega$; according to [18] the ϕ-transpose of Λ_E is a channel $\Lambda_E^\phi : \mathcal{C}(\Omega) \to M_d$, given by

$$\Lambda_E^\phi[f] = \sum_{\omega\in\Omega} \frac{f(\omega)}{p_\phi(\omega)}\, \phi^{1/2} E_{\mathcal{I}}(\omega) \phi^{1/2}\,. \tag{23}$$

As it is easy to check, this channel is such that

$$\Lambda_E^\phi \circ \Lambda_E[\phi] = \Lambda_E^\phi[p_\phi] = \phi\,. \tag{24}$$

Then, the monotonicity theorem gives

$$S(p_1\|p_2) \geq S(\Lambda_E^\phi[p_1]\|\Lambda_E^\phi[p_2])\,; \tag{25}$$

by taking $p_1 = p_\rho$, $p_2 = p_\phi$, it becomes

$$S_{\mathrm{c}}(p_\rho\|p_\phi) \geq S_{\mathrm{q}}(\Lambda_E^\phi[p_\rho]\|\phi). \tag{26}$$

3 Holevo bound and related inequalities

In quantum communication theory often the following scenario is considered: messages are transmitted by encoding the letters in some quantum states, which are possibly corrupted by a quantum noisy channel; at the end of the channel the receiver attempts to decode the message by performing measurements on the quantum system. So, one has an alphabet A (we take it finite) and the letters $\alpha \in A$ are transmitted with some a priori probabilities $p_{\mathrm{i}}(\alpha)$;

$p_{\rm i}$ is a discrete probability density on A. Each letter α is encoded in a quantum state and we denote by $\rho_{\rm i}(\alpha)$ the state associated to the letter α as it arrives to the receiver, after the passage through the transmission channel. We call these states the letter states and we denote by $\{p_{\rm i}, \rho_{\rm i}\}$ the *ensemble* of the states. We have introduced the subscript "i" for "initial" and we shall use "f" for final.

Let us use the instrument $\mathcal{I}$, given in Section 2.1, as decoding apparatus. The conditional probability of the outcome ω, given the input letter α, is

$$p_{\rm f|i}(\omega|\alpha) = {\rm Tr}\{\mathcal{O}(\omega)[\rho_{\rm i}(\alpha)]\} \equiv {\rm Tr}\{E_{\mathcal{I}}(\omega)\rho_{\rm i}(\alpha)\}\,; \tag{27a}$$

then, the joint probability of input and output, the conditional probability of the input given the output and the marginal probability of the output are given by

$$p_{\rm if}(\alpha,\omega) = p_{\rm f|i}(\omega|\alpha)p_{\rm i}(\alpha)\,, \qquad p_{\rm i|f}(\alpha|\omega) = \frac{p_{\rm f|i}(\omega|\alpha)p_{\rm i}(\alpha)}{p_{\rm f}(\omega)}\,, \tag{27b}$$

$$p_{\rm f}(\omega) = \sum_\alpha p_{\rm if}(\alpha,\omega) = \sum_\alpha p_{\rm i}(\alpha)\ {\rm Tr}\{\mathcal{O}(\omega)[\rho_{\rm i}(\alpha)]\} = {\rm Tr}\{\mathcal{O}(\omega)[\eta_{\rm i}]\}\,, \tag{27c}$$

where $\eta_{\rm i}$ is the average state of the initial ensemble, or *initial a priori state*:

$$\eta_{\rm i} := \sum_{\alpha\in A} p_{\rm i}(\alpha)\,\rho_{\rm i}(\alpha). \tag{28}$$

Note that $p_{\rm i|f}(\alpha|\omega)$ is well defined only when $p_{\rm f}(\omega) > 0$, but it can be arbitrarily completed when $p_{\rm f}(\omega) = 0$.

The mean information $I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\}$ on the transmitted letter which can be extracted in this way is the input/output classical mutual information, cf. (9):

$$I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} := S_{\rm c}(p_{\rm if}\|p_{\rm i}\otimes p_{\rm f}) = \sum_\alpha p_{\rm i}(\alpha)\, S_{\rm c}(p_{\rm f|i}(\bullet|\alpha)\|p_{\rm f})\,. \tag{29}$$

3.1 Holevo upper bound and the "transpose channel" lower bound

3.1.1 Holevo bound

Let us introduce *Holevo χ-quantity*, i.e. the χ-quantity of the initial ensemble (cf. Eqs. (11)–(13))

$$\chi\{p_{\rm i}, \rho_{\rm i}\} := \sum_{\alpha\in A} p_{\rm i}(\alpha)\, S_{\rm q}(\rho_{\rm i}(\alpha)\|\eta_{\rm i}) = S_{\rm q}(\eta_{\rm i}) - \sum_{\alpha\in A} p_{\rm i}(\alpha) S_{\rm q}(\rho_{\rm i}(\alpha))\,. \tag{30}$$

By applying the inequality (18) to the states $\rho_{\rm i}(\alpha)$ and $\eta_{\rm i}$ and then by multiplying by $p_{\rm i}(\alpha)$ and summing on α, one gets Holevo inequality [1]

$$I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} \leq \chi\{p_{\rm i}, \rho_{\rm i}\}\,. \tag{31}$$

In the case of a general Hilbert space, general POV measure, general alphabet, this inequality has been proved, just by using the channel Λ_E, by Yuen and Ozawa in [19].

3.1.2 The lower bound

The monotonicity theorem applied to the channel $\Lambda_E^{\eta_{\rm i}}$, the $\eta_{\rm i}$-transpose of Λ_E, gives a new lower bound for $I_{\rm c}$. Firstly, from (23) one has

$$\Lambda_E^{\eta_{\rm i}}[f] = \sum_\omega f(\omega)\sigma(\omega), \tag{32}$$

where we have introduced the family of statistical operators

$$\sigma(\omega) := \frac{1}{p_{\rm f}(\omega)}\, \eta_{\rm i}^{1/2} E_{\mathcal{I}}(\omega) \eta_{\rm i}^{1/2} \,. \tag{33}$$

The probability $p_{\rm f}(\omega)$ could vanish for some ω's, but in this case the positivity implies that also $\eta_{\rm i}^{1/2} E_{\mathcal{I}}(\omega)\eta_{\rm i}^{1/2}$ vanishes and the definition above can be completed arbitrarily for such ω's. Note that the ensemble $\{p_{\rm f}, \sigma\}$ has average

$$\sum_\omega p_{\rm f}(\omega)\sigma(\omega) = \eta_{\rm i} \,. \tag{34}$$

Then, by applying the inequality (26) to the states $\rho_{\rm i}(\alpha)$ and $\eta_{\rm i}$, by multiplying by $p_{\rm i}(\alpha)$ and summing on α, one gets

$$I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} \geq \chi\{p_{\rm i}, \xi\}, \tag{35}$$

where

$$\xi(\alpha) := \sum_\omega p_{\rm f|i}(\omega|\alpha)\sigma(\omega) \,. \tag{36}$$

The ensemble $\{p_{\rm i}, \xi\}$ has average

$$\sum_\alpha p_{\rm i}(\alpha)\xi(\alpha) = \eta_{\rm i} \,. \tag{37}$$

It is possible to show that, according to the definition of transpose given in Ref. [18], the $p_{\rm f}$-transpose of $\Lambda_E^{\eta_{\rm i}}$ would be Λ_E. Therefore, there is a sort of duality between the channels Λ_E and $\Lambda_E^{\eta_{\rm i}}$ and, so, between Holevo bound (31) and the bound (35).

3.2 The bound of Schumacher, Westmoreland, Wootters

Let us consider now the a posteriori states

$$\rho_{\rm f}^\alpha(\omega) := \pi^{\mathcal{I}}_{\rho_{\rm i}(\alpha)}(\omega) = \frac{\mathcal{O}(\omega)[\rho_{\rm i}(\alpha)]}{p_{\rm f|i}(\omega|\alpha)}\,, \qquad \rho_{\rm f}(\omega) := \pi^{\mathcal{I}}_{\eta_{\rm i}}(\omega) = \frac{\mathcal{O}(\omega)[\eta_{\rm i}]}{p_{\rm f}(\omega)}\,. \tag{38}$$

By applying the inequality (16) to the states $\rho_{\rm i}(\alpha)$ and $\eta_{\rm i}$ and then by multiplying by $p_{\rm i}(\alpha)$ and summing on α, one gets

$$\chi\{p_{\rm i}, \rho_{\rm i}\} \geq I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} + \sum_\omega p_{\rm f}(\omega)\, \chi\{p_{\rm i|f}(\bullet|\omega), \rho_{\rm f}^\bullet(\omega)\}. \tag{39}$$

The average state of the ensemble $\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega), \rho_{\mathrm{f}}^{\bullet}(\omega)\}$ is

$$\sum_{\alpha} p_{\mathrm{i}|\mathrm{f}}(\alpha|\omega)\, \rho_{\mathrm{f}}^{\alpha}(\omega) = \rho_{\mathrm{f}}(\omega). \tag{40}$$

Note that

$$\sum_{\omega} p_{\mathrm{f}}(\omega)\, \chi\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega), \rho_{\mathrm{f}}^{\bullet}(\omega)\} \equiv \sum_{\omega} p_{\mathrm{f}}(\omega)\, S_{\mathrm{q}}\big(\rho_{\mathrm{f}}(\omega)\big) - \sum_{\alpha,\omega} p_{\mathrm{if}}(\alpha,\omega)\, S_{\mathrm{q}}\big(\rho_{\mathrm{f}}^{\alpha}(\omega)\big) \tag{41}$$

is the mean χ-quantity left in the a posteriori states by the instrument. Inequality (39) gives an upper bound on $I_{\mathrm{c}}\{p_{\mathrm{i}}, \rho_{\mathrm{i}}; E_{\mathcal{I}}\}$ stronger than (31); indeed, the extra term vanishes when $\rho_{\mathrm{f}}^{\alpha}(\omega)$ is almost surely independent from α, as in the case of a von Neumann complete measurement, but for a generic instrument it is positive.

The original SWW bound [4] is inequality (39) in the case of an instrument with no sum on k in the definition (1b) of the operations $\mathcal{O}(\omega)$. Eq. (39) is a slight generalization to the case of (1b) with sums and was already proven in [11]; a different proof, more similar to the SWW original one, was given after in [20]. Inequality (39) has been generalized to the infinite and continuous case in [12].

Roughly, Eq. (39) says that the quantum information contained in the initial ensemble $\{p_{\mathrm{i}}, \rho_{\mathrm{i}}\}$ is greater than the classical information extracted in the measurement plus the mean quantum information left in the a posteriori states. Inequality (39) can be seen also as giving some kind of information/disturbance trade-off, a subject to which the paper [7], which contains a somewhat related inequality, is devoted.

Let us introduce the *a priori final states*

$$\eta_{\mathrm{f}}^{\alpha} := \mathcal{I}(\Omega)[\rho_{\mathrm{i}}(\alpha)] = \sum_{\omega} \mathcal{O}(\omega)[\rho_{\mathrm{i}}(\alpha)] = \sum_{\omega} p_{\mathrm{f}|\mathrm{i}}(\omega|\alpha)\, \rho_{\mathrm{f}}^{\alpha}(\omega), \tag{42a}$$

$$\eta_{\mathrm{f}} := \mathcal{I}(\Omega)[\eta_{\mathrm{i}}] = \sum_{\omega} \mathcal{O}(\omega)[\eta_{\mathrm{i}}] = \sum_{\omega} p_{\mathrm{f}}(\omega)\, \rho_{\mathrm{f}}(\omega) = \sum_{\alpha,\omega} p_{\mathrm{if}}(\alpha,\omega)\, \rho_{\mathrm{f}}^{\alpha}(\omega) = \sum_{\alpha} p_{\mathrm{i}}(\alpha)\, \eta_{\mathrm{f}}^{\alpha}\,. \tag{42b}$$

By using the expression of a χ-quantity in terms of entropies (11)–(13), one can check that the following identity holds

$$\chi\{p_{\mathrm{if}}, \rho_{\mathrm{f}}^{\bullet}\} = \chi\{p_{\mathrm{f}}, \rho_{\mathrm{f}}\} + \sum_{\omega} p_{\mathrm{f}}(\omega)\, \chi\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega), \rho_{\mathrm{f}}^{\bullet}(\omega)\}. \tag{43}$$

Both the new ensembles $\{p_{\mathrm{if}}, \rho_{\mathrm{f}}^{\bullet}\}$ and $\{p_{\mathrm{f}}, \rho_{\mathrm{f}}\}$ have η_{f} as average state. By using this identity, inequality (39) can be rewritten in the slightly more symmetric equivalent form

$$I_{\mathrm{c}}\{p_{\mathrm{i}}, \rho_{\mathrm{i}}; E_{\mathcal{I}}\} \leq \chi\{p_{\mathrm{i}}, \rho_{\mathrm{i}}\} + \chi\{p_{\mathrm{f}}, \rho_{\mathrm{f}}\} - \chi\{p_{\mathrm{if}}, \rho_{\mathrm{f}}^{\bullet}\}. \tag{44}$$

3.3 The generalized Groenewold-Lindblad inequality

Given an instrument $\mathcal{I}$ and a statistical operator η, an interesting quantity, which can be called the *quantum information gain*, is

$$I_{\mathrm{q}}(\eta;\mathcal{I}) = S_{\mathrm{q}}(\eta) - \sum_{\omega} S_{\mathrm{q}}\big(\pi_{\eta}^{\mathcal{I}}(\omega)\big)\, p_{\eta}(\omega)\,; \tag{45}$$

this is nothing but the entropy of the pre-measurement state minus the mean entropy of the a posteriori states.

By using the expression of a χ-quantity in terms of entropies and mean entropies, as in Eq. (30), one can see that inequality (39) is equivalent to

$$I_{\mathrm{q}}(\eta_{\mathrm{i}};\mathcal{I}) - \sum_{\alpha} p_{\mathrm{i}}(\alpha)\, I_{\mathrm{q}}(\rho_{\mathrm{i}}(\alpha);\mathcal{I}) \geq I_{\mathrm{c}}\{p_{\mathrm{i}},\rho_{\mathrm{i}};E_{\mathcal{I}}\} \geq 0\,. \tag{46}$$

Note that, once the instrument is fixed, $I_{\mathrm{q}}(\eta_{\mathrm{i}};\mathcal{I})$ depends only on η_{i}, while both $I_{\mathrm{c}}\{p_{\mathrm{i}},\rho_{\mathrm{i}};E_{\mathcal{I}}\}$ and $\sum_{\alpha} p_{\mathrm{i}}(\alpha)\, I_{\mathrm{q}}(\rho_{\mathrm{i}}(\alpha);\mathcal{I})$ depend on the demixture $\{p_{\mathrm{i}},\rho_{\mathrm{i}}\}$ of η_{i}.

An interesting question is when the quantum information gain is positive. Groenewold has conjectured [13] and Lindblad [14] has proved that the quantum information gain is non negative for an instrument of the von Neumann-Lüders type. The general case has been settled down by Ozawa, who has introduced the a posteriori states for general instruments in [17] and in [15] has proved a general result on instruments preserving pure states, which here we state only in the finite dimensional and discrete case.

Theorem 1 *For an instrument $\mathcal{I}$ as in Eq.* (1)*, the two following statements are equivalent:*

(a) *the instrument $\mathcal{I}$ sends any pure input state into almost surely pure a posteriori states;*

(b) $I_{\mathrm{q}}(\eta;\mathcal{I}) \geq 0$*, for all statistical operators η.*

Now the proof is an easy application of inequality (46); this proof works also in the general case [12].

Proof. To prove that (b) implies (a) is trivial; it is enough to put a pure state η into the definition, which gives

$$0 \leq I_{\mathrm{q}}(\eta;\mathcal{I}) = -\sum_{\omega} S_{\mathrm{q}}\big(\pi_{\eta}^{\mathcal{I}}(\omega)\big)\, p_{\eta}(\omega)\,.$$

This implies that the a posteriori states $\pi_{\eta}^{\mathcal{I}}(\omega)$ are p_{η}-almost surely pure, because the von Neumann entropy vanishes only on the pure states.

To show that (a) implies (b), the non trivial part in Ozawa's proof, let η_{i} be a generic state and $\{p_{\mathrm{i}},\rho_{\mathrm{i}}\}$ be a demixture of it into pure states; then, by (a) $I_{\mathrm{q}}(\rho_{\mathrm{i}}(\alpha);\mathcal{I}) = 0$ and (46) reduces to $I_{\mathrm{q}}(\eta_{\mathrm{i}};\mathcal{I}) \geq I_{\mathrm{c}}\{p_{\mathrm{i}},\rho_{\mathrm{i}};E_{\mathcal{I}}\} \geq 0$, which is (b). □

A sufficient condition for $\mathcal{I}$ being a pure state preserving instrument is to take $|K| = 1$ in (1b), but this is not necessary. The complete characterization of the structure of a pure state preserving instrument has been given in [22].

Inequality (46) is also interesting in itself, because it gives a link between the quantum information gain in the case of a pre-measurement state η_{i} and the mean quantum information gain in the case of a demixture of η_{i}, a link which holds true for any kind of instrument. The amount of quantum information has been studied and its meaning discussed also in [21, 20], where also the connections with inequality (39) and with pure state preserving instruments have been pointed out.

3.4 Post-measurement χ-quantities

By applying the inequality (22) to the states $\rho_\mathrm{i}(\alpha)$ and η_i and then by multiplying by $p_\mathrm{i}(\alpha)$ and summing on α, one gets

$$I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\}+\sum_\omega p_\mathrm{f}(\omega)\,\chi\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega),\rho_\mathrm{f}^\bullet(\omega)\}\geq\chi\{p_\mathrm{i},\eta_\mathrm{f}^\bullet\}. \tag{47}$$

By Eqs. (42) the average state of the ensemble $\{p_\mathrm{i},\eta_\mathrm{f}^\bullet\}$ is η_f.

Similarly to (43), also a second identity holds:

$$\chi\{p_\mathrm{f},\rho_\mathrm{f}\}+\sum_\omega p_\mathrm{f}(\omega)\,\chi\big\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega),\rho_\mathrm{f}^\bullet(\omega)\big\}=\chi\{p_\mathrm{i},\eta_\mathrm{f}^\bullet\}+\sum_\alpha p_\mathrm{i}(\alpha)\,\chi\big\{p_{\mathrm{f}|\mathrm{i}}(\bullet|\alpha),\rho_\mathrm{f}^\alpha\big\}. \tag{48}$$

By (42a), the ensemble $\big\{p_{\mathrm{f}|\mathrm{i}}(\bullet|\alpha),\rho_\mathrm{f}^\alpha\big\}$ has average state η_f^α. By this identity, inequality (47) is equivalent to

$$I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\}+\sum_\alpha p_\mathrm{i}(\alpha)\,\chi\big\{p_{\mathrm{f}|\mathrm{i}}(\bullet|\alpha),\rho_\mathrm{f}^\alpha\big\}\geq\chi\{p_\mathrm{f},\rho_\mathrm{f}\}. \tag{49}$$

3.5 Mutual entropy formulation

3.5.1 The initial and the final state

Let us introduce the algebras

$$\mathcal{C}_0:=\mathcal{C}(A),\qquad\mathcal{C}_1:=M_d,\qquad\mathcal{C}_2:=\mathcal{C}(\Omega). \tag{50}$$

As seen in Paragraph 2.2.3, the initial ensemble $\{p_\mathrm{i},\rho_\mathrm{i}\}$ can be seen as a state Σ_i^{01} on $\mathcal{C}_0\otimes\mathcal{C}_1\simeq\mathcal{C}(A;M_d)$. By using a superscript which indicates the algebras on which a state is acting, we can write

$$\Sigma_\mathrm{i}^{01}:=\{p_\mathrm{i}(\alpha)\rho_\mathrm{i}(\alpha)\},\qquad\Sigma_\mathrm{i}^{0}=\{p_\mathrm{i}(\alpha)\},\qquad\Sigma_\mathrm{i}^{1}=\{\eta_\mathrm{i}\}, \tag{51}$$

for the initial state and its marginals. By (13), Holevo χ-quantity (30) coincides with the initial mutual entropy

$$S(\Sigma_\mathrm{i}^{01}\|\Sigma_\mathrm{i}^{0}\otimes\Sigma_\mathrm{i}^{1})=\chi\{p_\mathrm{i},\rho_\mathrm{i}\}. \tag{52}$$

By dilating the channel $\Lambda_\mathcal{I}$ (14) with the identity we obtain the *measurement channel*

$$\Lambda:\mathcal{C}_0\otimes\mathcal{C}_1\to\mathcal{C}_0\otimes\mathcal{C}_1\otimes\mathcal{C}_2\,,\qquad\Lambda:=\mathbb{1}\otimes\Lambda_\mathcal{I}\,. \tag{53}$$

Then, by applying the measurement channel to the initial state we obtain the final state

$$\Sigma_\mathrm{f}^{012}:=\Lambda[\Sigma_\mathrm{i}^{01}]=\{p_\mathrm{i}(\alpha)\Lambda_\mathcal{I}[\rho_\mathrm{i}(\alpha)](\omega)\}=\{p_\mathrm{if}(\alpha,\omega)\rho_\mathrm{f}^\alpha(\omega)\}, \tag{54a}$$

whose marginals are

$$\begin{aligned}\Sigma_\mathrm{f}^{01}&=\{p_\mathrm{i}(\alpha)\eta_\mathrm{f}^\alpha\},\qquad\Sigma_\mathrm{f}^{02}=\{p_\mathrm{if}(\alpha,\omega)\},\qquad\Sigma_\mathrm{f}^{12}=\{p_\mathrm{f}(\omega)\rho_\mathrm{f}(\omega)\},\\ \Sigma_\mathrm{f}^{0}&=\Sigma_\mathrm{i}^{0}=\{p_\mathrm{i}(\alpha)\},\qquad\Sigma_\mathrm{f}^{1}=\{\eta_\mathrm{f}\},\qquad\Sigma_\mathrm{f}^{2}=\{p_\mathrm{f}(\omega)\}.\end{aligned} \tag{54b}$$

Moreover, one gets easily

$$\Lambda[\Sigma_\mathrm{i}^{0}\otimes\Sigma_\mathrm{i}^{1}]=\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{12}\,. \tag{55}$$

3.5.2 Mutual entropies and inequalities

By the definitions of Section 2.2.2 it is easy to compute all the mutual entropies related to the final state. The mutual entropy involving only the classical part of the final state turns out to be the input/output classical mutual information:

$$S(\Sigma_\mathrm{f}^{02}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{2}) = S_\mathrm{c}(p_\mathrm{if}\|p_\mathrm{i}\otimes p_\mathrm{f}) = I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\}. \tag{56}$$

Then, the remaining mutual entropies turn out to be

$$S(\Sigma_\mathrm{f}^{01}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{1}) = \chi\{p_\mathrm{i},\eta_\mathrm{f}^{\bullet}\}, \qquad S(\Sigma_\mathrm{f}^{12}\|\Sigma_\mathrm{f}^{1}\otimes\Sigma_\mathrm{f}^{2}) = \chi\{p_\mathrm{f},\rho_\mathrm{f}\}, \tag{57a}$$

$$S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{02}\otimes\Sigma_\mathrm{f}^{1}) = \chi\{p_\mathrm{if},\rho_\mathrm{f}^{\bullet}\}, \tag{57b}$$

$$S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{12}) = I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} + \sum_\omega p_\mathrm{f}(\omega)\,\chi\{p_{\mathrm{i}|\mathrm{f}}(\bullet|\omega),\rho_\mathrm{f}^{\bullet}(\omega)\}, \tag{57c}$$

$$S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{01}\otimes\Sigma_\mathrm{f}^{2}) = I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} + \sum_\alpha p_\mathrm{i}(\alpha)\,\chi\{p_{\mathrm{f}|\mathrm{i}}(\bullet|\alpha),\rho_\mathrm{f}^{\alpha}\}, \tag{57d}$$

$$S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{2}\otimes\Sigma_\mathrm{f}^{3}) = I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} + \chi\{p_\mathrm{if},\rho_\mathrm{f}^{\bullet}\}. \tag{57e}$$

Note that the expressions of the mutual entropies involve the χ-quantities of all the ensembles entering into play.

Uhlmann monotonicity theorem and Eqs. (54a), (55) give us the inequality

$$S(\Sigma_\mathrm{i}^{01}\|\Sigma_\mathrm{i}^{0}\otimes\Sigma_\mathrm{i}^{1}) \geq S(\Lambda[\Sigma_\mathrm{i}^{01}]\|\Lambda[\Sigma_\mathrm{i}^{0}\otimes\Sigma_\mathrm{i}^{1}]) = S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{12}). \tag{58}$$

By Eqs. (52) and (57c), one has that this inequality is equivalent to the SWW bound (39).

It is trivial to see that the operation of restricting states on a tensor product to one of the factors is a channel; therefore, we have also the inequality

$$S(\Sigma_\mathrm{f}^{012}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{12}) \geq S(\Sigma_\mathrm{f}^{01}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{1}), \tag{59}$$

which, by (57c) and (57a), is equivalent to inequality (47). All the other inequalities which can be obtained are implied by the previous ones trivially or via the identities (43), (48). Among these inequalities there is

$$S(\Sigma_\mathrm{i}^{01}\|\Sigma_\mathrm{i}^{0}\otimes\Sigma_\mathrm{i}^{1}) \geq S(\Sigma_\mathrm{f}^{02}\|\Sigma_\mathrm{f}^{0}\otimes\Sigma_\mathrm{f}^{2}), \tag{60}$$

which, by (13), (56), is equivalent to Holevo bound (31).

To express inequality (35) in terms of mutual entropies let us introduce the new channel

$$\Gamma : \mathcal{C}_0\otimes\mathcal{C}_2 \to \mathcal{C}_0\otimes\mathcal{C}_1 \tag{61a}$$

by

$$\Gamma[f](\alpha) = \sum_\omega f(\alpha,\omega)\sigma(\omega), \qquad \forall f\in\mathcal{C}_0\otimes\mathcal{C}_2\,. \tag{61b}$$

Then, the monotonicity theorem gives

$$S(p_{\mathrm{if}}\|p_{\mathrm{i}}\otimes p_{\mathrm{f}}) \geq S(\Gamma[p_{\mathrm{if}}]\|\Gamma[p_{\mathrm{i}}\otimes p_{\mathrm{f}}]); \tag{62}$$

but one has

$$\Gamma[p_{\mathrm{if}}](\alpha) = \sum_{\omega} p_{\mathrm{if}}(\alpha,\omega)\sigma(\omega) = p_{\mathrm{i}}(\alpha)\xi(\alpha)\,, \tag{63a}$$

$$\Gamma[p_{\mathrm{i}}\otimes p_{\mathrm{f}}](\alpha) = \sum_{\omega} p_{\mathrm{i}}(\alpha)p_{\mathrm{f}}(\omega)\sigma(\omega) = p_{\mathrm{i}}(\alpha)\eta_{\mathrm{i}}\,, \tag{63b}$$

and, so, inequality (62) is equivalent to the bound (35). Note that $\Gamma[p_{\mathrm{i}}\otimes p_{\mathrm{f}}] = \Gamma[p_{\mathrm{if}}]\big|_{\mathcal{C}_0}\otimes \Gamma[p_{\mathrm{if}}]\big|_{\mathcal{C}_1}$ so that both sides of (62) are mutual entropies.

4 Hall bound and generalizations

In [5] Hall exhibits a transformation on the initial ensemble and on the POV measure which leaves invariant I_{c} but not the initial χ-quantity and in this way produces a new upper bound on the classical information. Inspired by Hall transformation, a new instrument can be constructed in such a way that the analogous of inequality (39) produces an upper bound on I_{c} stronger than both Hall and Holevo bounds.

For simplicity in this section we assume that η_{i} is invertible.

4.1 A generalization of Hall transformation

4.1.1 A new instrument $\mathcal{J}$

Let us set

$$M(\alpha) := \sqrt{p_{\mathrm{i}}(\alpha)}\;\rho_{\mathrm{i}}(\alpha)^{1/2}\eta_{\mathrm{i}}^{-1/2}\,, \qquad \mathcal{G}(\alpha)[\tau] := M(\alpha)\tau M(\alpha)^*\,, \quad \forall \tau\in M_d\,; \tag{64a}$$

by Eq. (28) the operators $M(\alpha)$ satisfy the normalization condition

$$\sum_{\alpha} M(\alpha)^* M(\alpha) = \mathbb{1}\,. \tag{64b}$$

Then, the position

$$\mathcal{J}(B) := \sum_{\alpha\in B}\mathcal{G}(\alpha), \qquad B\subset A, \tag{64c}$$

defines an instrument with value space A. The instrument $\mathcal{J}$ has been constructed by using only the old initial ensemble $\{p_{\mathrm{i}},\rho_{\mathrm{i}}\}$. The associated POV measure is

$$E_{\mathcal{J}}(\alpha) = M(\alpha)^* M(\alpha) = p_{\mathrm{i}}(\alpha)\,\eta_{\mathrm{i}}^{-1/2}\rho_{\mathrm{i}}(\alpha)\eta_{\mathrm{i}}^{-1/2}\,. \tag{64d}$$

Now, we can construct the associated channel and a posteriori states, as in Section 2: $\forall \tau\in M_d$, $\forall\rho\in\mathcal{S}_d$, one has

$$\Lambda_{\mathcal{J}}[\tau](\alpha) = \mathcal{G}(\alpha)[\tau] = M(\alpha)\tau M(\alpha)^*, \tag{65}$$

$$\pi_\rho^{\mathcal{J}}(\alpha) = \big(\operatorname{Tr}\{M(\alpha)^* M(\alpha)\rho\}\big)^{-1} M(\alpha)\rho M(\alpha)^*. \tag{66}$$

Let us stress that $\mathcal{J}$ sends pure states into a.s. pure a posteriori states; therefore, by Theorem 1 one has

$$I_{\rm q}\{\rho;\mathcal{J}\} \equiv S_{\rm q}(\rho) - \sum_\alpha \mathrm{Tr}\{E_{\mathcal{J}}(\alpha)\rho\}\, S_{\rm q}\big(\pi_\rho^{\mathcal{J}}(\alpha)\big) \geq 0. \tag{67}$$

4.1.2 A new initial ensemble and the replacements

Now we consider $\{p_{\rm f},\sigma\}$ (33) as initial ensemble for $\mathcal{J}$; recall that its average state is $\eta_{\rm i}$ (28). It is easy to verify that

$$\mathrm{Tr}\{E_{\mathcal{J}}(\alpha)\sigma(\omega)\} = p_{\rm i|f}(\alpha|\omega); \tag{68}$$

together with the substitution of $p_{\rm i}$ with $p_{\rm f}$, this gives that $p_{\rm if}$ is left invariant and that $p_{\rm f}$ is substituted by $p_{\rm i}$. Therefore, we have

$$I_{\rm c}\{p_{\rm f},\sigma;E_{\mathcal{J}}\} = I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\}. \tag{69}$$

Indeed, the POV measure $E_{\mathcal{J}}$ and the states $\sigma(\omega)$ have been constructed by Hall just in order to have this equality.

One can also check that under Hall transformation the states $\sigma(\omega)$ (33) become the states $\rho_{\rm i}(\alpha)$. Summarizing, we have that the following replacements have to be made:

$$\begin{aligned} A \rightleftarrows \Omega\,, \qquad & p_{\rm if} \to p_{\rm if}\,, \qquad p_{\rm i}(\alpha) \rightleftarrows p_{\rm f}(\omega)\,, \\ p_{\rm f|i}(\omega|\alpha) \rightleftarrows p_{\rm i|f}(\alpha|\omega)\,, \qquad & \rho_{\rm i}(\alpha) \rightleftarrows \sigma(\omega)\,, \qquad \eta_{\rm i} \to \eta_{\rm i}\,. \end{aligned} \tag{70a}$$

By Eqs. (33), (64a), (66) we obtain also

$$\rho_{\rm f}^\alpha(\omega) \to \pi^{\mathcal{J}}_{\sigma(\omega)}(\alpha) = \rho_{\rm i}(\alpha)^{1/2}\,\frac{E_{\mathcal{I}}(\omega)}{p_{\rm f|i}(\omega|\alpha)}\,\rho_{\rm i}(\alpha)^{1/2}\,, \qquad \rho_{\rm f}(\omega) \to \pi^{\mathcal{J}}_{\eta_{\rm i}}(\alpha) = \rho_{\rm i}(\alpha); \tag{70b}$$

the first quantity is defined similarly to (33). Moreover,

$$\eta_{\rm f}^\alpha \to \eta_{\mathcal{J}}^\omega := \sum_\alpha p_{\rm i|f}(\alpha|\omega)\pi^{\mathcal{J}}_{\sigma(\omega)}(\alpha) = \sum_\alpha \frac{p_{\rm i}(\alpha)}{p_{\rm f}(\omega)}\,\rho_{\rm i}(\alpha)^{1/2}E_{\mathcal{I}}(\omega)\rho_{\rm i}(\alpha)^{1/2}\,, \tag{70c}$$

$$\eta_{\rm f} \to \sum_\alpha p_{\rm i}(\alpha)\pi^{\mathcal{J}}_{\eta_{\rm i}}(\alpha) = \eta_{\rm i}. \tag{70d}$$

4.2 The new bounds

4.2.1 Hall bound

Let us consider now Holevo bound for the new set up:

$$I_{\rm c}\{p_{\rm f},\sigma;E_{\mathcal{J}}\} \leq \chi\{p_{\rm f},\sigma\}. \tag{71}$$

By (69), (70a) we get

$$I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} \leq \chi\{p_{\rm f},\sigma\} \equiv \sum_\omega p_{\rm f}(\omega)\, S_{\rm q}\big(\sigma(\omega)\big\|\eta_{\rm i}\big), \tag{72}$$

which is Hall bound $\big($Eq. (19) of [5]$\big)$. This bound is discussed also in Refs. [6, 24, 25]; the "continuous" version of it is given in [12].

4.2.2 The new upper bound for I_c

Having defined a new instrument and not only a POV measure, we obtain from (39) the inequality

$$\chi\{p_\mathrm{f},\sigma\} \geq I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} + \sum_\alpha p_\mathrm{i}(\alpha)\,\chi\{p_{\mathrm{f}|\mathrm{i}}(\bullet|\alpha),\pi^\mathcal{J}_{\sigma(\bullet)}(\alpha)\}, \tag{73}$$

which gives a stronger bound than Hall's one (72). In order to render more explicit this bound, it is convenient to start from the equivalent form (46), which now reads

$$I_\mathrm{q}\{\eta_\mathrm{i};\mathcal{J}\} \geq I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} + \sum_\omega p_\mathrm{f}(\omega)\,I_\mathrm{q}\{\sigma(\omega);\mathcal{J}\}. \tag{74}$$

By Eqs. (64d), (67), (70b) we obtain

$$I_\mathrm{q}\{\eta_\mathrm{i};\mathcal{J}\} = \chi\{p_\mathrm{i},\rho_\mathrm{i}\}. \tag{75}$$

Therefore, Eq. (74) gives the new bound

$$I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} \leq \chi\{p_\mathrm{i},\rho_\mathrm{i}\} - \sum_\omega p_\mathrm{f}(\omega)\,I_\mathrm{q}\{\sigma(\omega);\mathcal{J}\}; \tag{76}$$

let us stress that $I_\mathrm{q}\{\sigma(\omega);\mathcal{J}\} \geq 0$ because of Eq. (67). More explicitly, by Eqs. (64d), (33), (67), we have

$$\sum_\omega p_\mathrm{f}(\omega)\,I_\mathrm{q}\{\sigma(\omega);\mathcal{J}\} = \sum_\omega p_\mathrm{f}(\omega)\,S_\mathrm{q}\big(\sigma(\omega)\big) - \sum_{\alpha,\omega} p_\mathrm{if}(\alpha,\omega)\,S_\mathrm{q}\big(\pi^\mathcal{J}_{\sigma(\omega)}(\alpha)\big), \tag{77}$$

where $\sigma(\omega)$ is given by (33) and $\pi^\mathcal{J}_{\sigma(\omega)}(\alpha)$ by (70b). The general version of the bound (76) has been presented in [12].

4.2.3 Scutaru lower bound

By (70a) one gets that the states ξ (36) have to be replaced by

$$\epsilon(\omega) := \sum_\alpha p_{\mathrm{i}|\mathrm{f}}(\alpha|\omega)\rho_\mathrm{i}(\alpha)\,; \tag{78}$$

recalling also that p_i has to be replaced by p_f, one gets that the bound (35) becomes

$$I_\mathrm{c}\{p_\mathrm{i},\rho_\mathrm{i};E_\mathcal{I}\} \geq \chi\{p_\mathrm{f},\epsilon\}\,. \tag{79}$$

Note that

$$\sum_\omega p_\mathrm{f}(\omega)\epsilon(\omega) = \eta_\mathrm{i}\,. \tag{80}$$

This bound was obtained, directly in the "continuous case", by Scutaru in [3]; he used Uhlmann monotonicity theorem and a "classical→quantum" channel Ψ mapping states on $\mathcal{C}(A)$ (discrete probability densities on A) into states on M_d: if h is any discrete probability density on A, then

$$\Psi[h] = \sum_\alpha h(\alpha)\rho_\mathrm{i}(\alpha)\,. \tag{81}$$

This channel is exactly the one we have used; indeed, with the symbols of Paragraph 2.3.3, one can check that $\Psi = \Lambda_{E_{\mathcal{J}}}^{\eta_{\rm i}}$. Therefore, Scutaru channel Ψ is the $\eta_{\rm i}$-transpose of the "quantum→classical" channel associated to the POV measure introduced by Hall and Hall (72) and Scutaru (79) bounds are linked one to the other exctly as Holevo bound (31) is linked to the bound (35).

4.2.4 An upper bound on Holevo χ-quantity

By (69), (70a), (70b), inequality (49) gives

$$I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} + \sum_\omega p_{\rm f}(\omega)\,\chi\{p_{\rm i|f}(\bullet|\omega),\pi^{\mathcal{J}}_{\sigma(\omega)}\} \geq \chi\{p_{\rm i},\rho_{\rm i}\}; \tag{82}$$

the average state of the ensemble $\{p_{\rm i|f}(\bullet|\omega),\pi^{\mathcal{J}}_{\sigma(\omega)}\}$ is $\eta^\omega_{\mathcal{J}}$ defined in (70c). Let us stress that Holevo χ-quantity depends only on the initial ensemble, while the l.h.s. of inequality (82) depends also on the POV measure.

In the Subsection 3.5 all the inequalities of Section 3 have been shown to be inequalities between mutual entropies. As the results of this section have been obtained from those of Section 3 only by changing instrument, also all inequalities of the present section can be obviously stated as inequalities between mutual entropies.

5 Summary of the inequalities and examples

5.1 The main inequalities

The mutual information $I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\}$ is a key object, which quantifies the ability of the POV measure $E_{\mathcal{I}}$ in extracting the information codified in the initial ensemble. Let us summarize all the inequality involving $I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\}$.

In Section 3 we obtained the new lower bound (35), the generalization (39) of the bound by Shumacher, Westmoreland, Wootters and the bound by Holevo (31); we can summarize their definitions and relationships by

$$B_{\rm Hlv} := \chi\{p_{\rm i},\rho_{\rm i}\}, \qquad b_{\rm nlb} := \chi\{p_{\rm i},\xi\}, \tag{83a}$$

$$B_{\rm SWW} := \chi\{p_{\rm i},\rho_{\rm i}\} - \sum_\omega p_{\rm f}(\omega)\,\chi\{p_{\rm i|f}(\bullet|\omega),\rho^\bullet_{\rm f}(\omega)\}, \tag{83b}$$

$$0 \leq b_{\rm nlb} \leq I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} \leq B_{\rm SWW} \leq B_{\rm Hlv}\,. \tag{84}$$

We are using b for a lower bound and B for an upper bound.

In Section 4 we obtained Scutaru bound (79), the new upper bound (76) and Hall bound (72); summarizing we have

$$0 \leq b_{\rm Scu} \leq I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} \leq B_{\rm nub} \leq \begin{cases} B_{\rm Hall} \\ B_{\rm Hlv} \end{cases} \tag{85}$$

$$b_{\rm Scu} := \chi\{p_{\rm f},\epsilon\}, \tag{86a}$$

$$B_{\rm nub} := \chi\{p_{\rm i},\rho_{\rm i}\} - \sum_\omega p_{\rm f}(\omega)\,I_{\rm q}\{\sigma(\omega);\mathcal{J}\}, \tag{86b}$$

$$B_{\rm Hall} := \chi\{p_{\rm f},\sigma\}. \tag{86c}$$

Finally, the inequalities (47) and (82) can be written as

$$I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\}\geq\begin{cases}b_1\\ b_2\end{cases}\tag{87}$$

$$b_1:=\chi\{p_{\rm i},\eta_{\rm f}^\bullet\}-\sum_\omega p_{\rm f}(\omega)\,\chi\{p_{\rm i|f}(\bullet|\omega),\rho_{\rm f}^\bullet(\omega)\},\tag{88a}$$

$$b_2:=\chi\{p_{\rm i},\rho_{\rm i}\}-\sum_\omega p_{\rm f}(\omega)\,\chi\{p_{\rm i|f}(\bullet|\omega),\pi^{\mathcal{J}}_{\sigma(\omega)}\}.\tag{88b}$$

However, b_1 and b_2 are not necessarily non-negative and, therefore, (87) does not give always effective lower bounds on $I_{\rm c}$.

A notion related to that of classical mutual information, but not linked to a specific measurement, is the accessible information of an ensemble [23]: it is the supremum over all the POV measures of the classical mutual information extracted by the quantum measurement

$$I_{\rm acc}\{p_{\rm i},\rho_{\rm i}\}:=\sup_E I_{\rm c}\{p_{\rm i},\rho_{\rm i};E\}.\tag{89}$$

The only bound from above for $I_{\rm acc}\{p_{\rm i},\rho_{\rm i}\}$ is Holevo bound, because only this bound does not depend on the measurement. From below $I_{\rm acc}\{p_{\rm i},\rho_{\rm i}\}$ is bounded by the subentropy introduced in [2] and, trivially, by $I_{\rm c}\{p_{\rm i},\rho_{\rm i};E\}$ computed for any fixed E and by any of its lower bounds.

The subentropy of a density matrix ρ is

$$Q(\rho)=-\sum_k\left(\prod_{\ell:\,\ell\neq k}\frac{\lambda_k}{\lambda_k-\lambda_\ell}\right)\lambda_k\log\lambda_k\,,\tag{90}$$

where the λ_k are the eigenvalues of ρ $\big([2]$, Eq. $(8)\big)$. The bound based on the subentropy $\big([2]$, Eq. $(33)\big)$ is

$$I_{\rm acc}\{p_{\rm i},\rho_{\rm i}\}\geq b_{\rm subent}\equiv Q(\eta_{\rm i})-\sum_\alpha p_{\rm i}(\alpha)Q\big(\rho_{\rm i}(\alpha)\big).\tag{91}$$

5.2 A rank-one POV measure

As a first example, let us consider a measurement described by a POV measure made up of rank-one elements:

$$E_{\mathcal{I}}(\omega)=\mu(\omega)|\psi(\omega)\rangle\langle\psi(\omega)|,\tag{92a}$$

$$\|\psi(\omega)\|=1\,,\qquad\mu(\omega)\geq0\,,\qquad\sum_\omega\mu(\omega)|\psi(\omega)\rangle\langle\psi(\omega)|=\mathbb{1}\,.\tag{92b}$$

This gives

$$p_\rho(\omega)=\mu(\omega)\langle\psi(\omega)|\rho\,\psi(\omega)\rangle,\qquad\rho\in\mathcal{S}_d\,,\tag{92c}$$

$$p_{\rm f|i}(\omega|\alpha)=\mu(\omega)\langle\psi(\omega)|\rho_{\rm i}(\alpha)\,\psi(\omega)\rangle,\qquad p_{\rm f}(\omega)=\mu(\omega)\langle\psi(\omega)|\eta_{\rm i}\,\psi(\omega)\rangle,\tag{93}$$

$$I_{\mathrm{c}}\{p_{\mathrm{i}},\rho_{\mathrm{i}};E_{\mathcal{I}}\}=\sum_{\alpha,\omega}p_{\mathrm{i}}(\alpha)\langle\psi(\omega)|\rho_{\mathrm{i}}(\alpha)\,\psi(\omega)\rangle\mu(\omega)\,\log\frac{\langle\psi(\omega)|\rho_{\mathrm{i}}(\alpha)\,\psi(\omega)\rangle}{\langle\psi(\omega)|\eta_{\mathrm{i}}\,\psi(\omega)\rangle}\,. \tag{94}$$

By (1c) and the positivity of $\sum_{k\in K}V_k^{\omega\dagger}V_k^{\omega}$ one can prove that for any instrument $\mathcal{I}$ compatible with the POV measure (92) it must be

$$V_k^{\omega}=|\phi_k(\omega)\rangle\langle\psi(\omega)|,\qquad \sum_k\|\phi_k(\omega)\|^2=\mu(\omega). \tag{95}$$

By inserting this into the definition (3) of the a posteriori states, one gets that

$$\pi_\rho^{\mathcal{I}}(\omega)=\frac{1}{\mu(\omega)}\sum_k|\phi_k(\omega)\rangle\langle\phi_k(\omega)|=:\pi(\omega),\qquad\forall\rho\in\mathcal{S}_d; \tag{96}$$

the a posteriori states depend on the instrument, but are independent from the pre-measurement state. Then, we have $\rho_{\mathrm{f}}^{\alpha}(\omega)=\rho_{\mathrm{f}}(\omega)=\pi(\omega)$ and

$$\sum_\omega p_{\mathrm{f}}(\omega)\,\chi\{p_{\mathrm{i|f}}(\bullet|\omega),\rho_{\mathrm{f}}^{\bullet}(\omega)\}=0\,. \tag{97}$$

Moreover, one can check that the states $\sigma(\omega)$ and $\pi^{\mathcal{J}}_{\sigma(\omega)}(\alpha)$ are pure, that implies

$$\sum_\omega p_{\mathrm{f}}(\omega)\,I_{\mathrm{q}}\{\sigma(\omega);\mathcal{J}\}=0\,. \tag{98}$$

The consequence is that the SWW bound (39) and the new upper bound (76) reduce to Holevo bound (31). Moreover, we get $\chi\{p_{\mathrm{f}},\sigma\}=S_{\mathrm{q}}(\eta_{\mathrm{i}})$; so, the original Hall bound (72) is worst than Holevo bound, as already noticed by Hall himself [5]. Summarizing, the four upper bounds are related by

$$B_{\mathrm{SWW}}=B_{\mathrm{nub}}=B_{\mathrm{Hlv}}\equiv S_{\mathrm{q}}(\eta_{\mathrm{i}})-\sum_{\alpha\in A}p_{\mathrm{i}}(\alpha)S_{\mathrm{q}}(\rho_{\mathrm{i}}(\alpha))\le B_{\mathrm{Hall}}\equiv S_{\mathrm{q}}(\eta_{\mathrm{i}}). \tag{99}$$

Let us consider now the lower bounds. The statistical operators ξ and ϵ in the new lower bound (83a) and in Scutaru bound (86a) are now given by

$$\xi(\alpha)=\sum_\omega\mu(\omega)\,\frac{\langle\psi(\omega)|\rho_{\mathrm{i}}(\alpha)\,\psi(\omega)\rangle}{\langle\psi(\omega)|\eta_{\mathrm{i}}\,\psi(\omega)\rangle}\,\eta_{\mathrm{i}}^{1/2}|\psi(\omega)\rangle\langle\psi(\omega)|\eta_{\mathrm{i}}^{1/2}\,, \tag{100a}$$

$$\epsilon(\omega)=\sum_\alpha p_{\mathrm{i}}(\alpha)\,\frac{\langle\psi(\omega)|\rho_{\mathrm{i}}(\alpha)\,\psi(\omega)\rangle}{\langle\psi(\omega)|\eta_{\mathrm{i}}\,\psi(\omega)\rangle}\,\rho_{\mathrm{i}}(\alpha). \tag{100b}$$

By (97), Eq. (88a) gives the effective lower bound

$$b_1=\chi\{p_{\mathrm{i}},\eta_{\mathrm{f}}^{\bullet}\}\ge 0\,; \tag{101}$$

moreover, the states $\eta_{\mathrm{f}}^{\alpha}$ turn out to be given by

$$\eta_{\mathrm{f}}^{\alpha}=\sum_\omega p_{\mathrm{f|i}}(\omega|\alpha)\pi(\omega). \tag{102}$$

Finally, by the fact that the states $\pi^{\mathcal{J}}_{\sigma(\omega)}(\alpha)$ are pure, we get from (88b)

$$b_2 = \chi\{p_{\rm i}, \rho_{\rm i}\} - \sum_\omega p_{\rm f}(\omega)\, S_{\rm q}(\eta^\omega_{\mathcal{J}}), \tag{103}$$

with

$$\eta^\omega_{\mathcal{J}} = \frac{1}{\langle \psi(\omega) | \eta_{\rm i}\, \psi(\omega)\rangle} \sum_\alpha p_{\rm i}(\alpha) \rho_{\rm i}(\alpha)^{1/2} |\psi(\omega)\rangle\langle\psi(\omega)| \rho_{\rm i}(\alpha)^{1/2}\,. \tag{104}$$

5.2.1 A complete von Neumann measurement

An interesting case of rank-one POV measure is certainly that one of a complete von Neumann measurement. Let us consider here only the case of a projection valued measure, which diagonalizes $\eta_{\rm i}$:

$$\Omega = \{1, \ldots, d\}, \qquad \langle \psi(\omega)|\psi(\omega')\rangle = \delta_{\omega\omega'}\,, \qquad \mu(\omega) = 1\,, \tag{105a}$$

$$\eta_{\rm i} = \sum_{\omega=1}^d \lambda_\omega |\psi(\omega)\rangle\langle\psi(\omega)|. \tag{105b}$$

Moreover, we construct the instrument by the usual reduction postulate, so that

$$\pi(\omega) = E_{\mathcal{I}}(\omega) = |\psi(\omega)\rangle\langle\psi(\omega)|. \tag{106}$$

Then, we have

$$p_{\rm f|i}(\omega|\alpha) = \langle \psi(\omega)|\rho_{\rm i}(\alpha)\, \psi(\omega)\rangle, \qquad p_{\rm f}(\omega) = \lambda_\omega, \tag{107}$$

$$I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} = S_{\rm q}(\eta_{\rm i}) - \sum_\alpha p_{\rm i}(\alpha) S_{\rm c}\big(p_{\rm f|i}(\bullet|\alpha)\big). \tag{108}$$

As before, only Holevo bound survives as upper bound.

About the lower bounds, now we have

$$\epsilon(\omega) = \sum_\alpha p_{\rm i}(\alpha)\, \frac{\langle \psi(\omega)|\rho_{\rm i}(\alpha)\, \psi(\omega)\rangle}{\lambda_\omega}\, \rho_{\rm i}(\alpha), \tag{109}$$

$$\eta^\alpha_{\rm f} = \xi(\alpha) = \sum_\omega \langle \psi(\omega)|\rho_{\rm i}(\alpha)\, \psi(\omega)\rangle \pi(\omega). \tag{110}$$

This gives

$$b_1 = b_{\rm nlb} = I_{\rm c}\{p_{\rm i}, \rho_{\rm i}; E_{\mathcal{I}}\} \geq b_{\rm Scu} \equiv S_{\rm q}(\eta_{\rm i}) - \sum_\omega \lambda_\omega\, S_{\rm q}\big(\epsilon(\omega)\big). \tag{111}$$

Finally, the quantity $\eta^\omega_{\mathcal{J}}$ appearing in the expression (103) of b_2 becomes

$$\eta^\omega_{\mathcal{J}} = \sum_\alpha \frac{p_{\rm i}(\alpha)}{\lambda_\omega}\, \rho_{\rm i}(\alpha)^{1/2} |\psi(\omega)\rangle\langle\psi(\omega)| \rho_{\rm i}(\alpha)^{1/2}\,. \tag{112}$$

5.2.2 The case of commuting letter states

Let us consider now the case in which all the $\rho_{\rm i}(\alpha)$ are commuting operators; it is known that this is the only case in which Holevo bound is attained [1, 25].

Let us choose $E_{\mathcal{I}}(\omega) = |\psi(\omega)\rangle\langle\psi(\omega)|$ to be a joint spectral measure of all the operators $\rho_{\rm i}(\alpha)$; because, necessarily, also $\eta_{\rm i}$ is diagonalized by $E_{\mathcal{I}}$, this is a particularization of the case of Subsection 5.2.1. Then, we have

$$\rho_{\rm i}(\alpha) = \sum_\omega \kappa_\omega^\alpha \pi(\omega), \qquad \kappa_\omega^\alpha \geq 0\,, \quad \sum_\omega \kappa_\omega^\alpha = 1\,, \quad \sum_\alpha p_{\rm i}(\alpha)\kappa_\omega^\alpha = \lambda_\omega\,, \tag{113a}$$

$$\eta_{\mathcal{J}}^\omega = \pi(\omega), \qquad S_{\rm q}(\eta_{\mathcal{J}}^\omega) = 0\,, \tag{113b}$$

$$\epsilon(\omega) = \sum_{\omega'} \frac{q_{12}(\omega,\omega')}{\lambda(\omega)}\,\pi(\omega')\,, \qquad q_{12}(\omega,\omega') := \sum_\alpha p_{\rm i}(\alpha)\kappa_\omega^\alpha \kappa_{\omega'}^\alpha\,; \tag{113c}$$

let us note that q_{12} is a joint discrete probability density with marginals $q_1(\omega) = q_2(\omega) = \lambda_\omega$. Then, all the previous equalities/inequalities reduce to

$$B_{\rm Hall} \geq B_{\rm SWW} = B_{\rm nub} = B_{\rm Hlv} = I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} = b_1 = b_2 = b_{\rm nlb} \geq b_{\rm Scu} \equiv S_{\rm c}(q_{12}\|q_1\otimes q_2). \tag{114}$$

5.3 Pure initial states

When all the initial states $\rho_{\rm i}(\alpha)$ are pure, Holevo χ-quantity reduces to the von Neumann entropy: $\chi\{p_{\rm i},\rho_{\rm i}\} = S_{\rm q}(\eta_{\rm i})$. Moreover, from Eqs. (64d), (66) we have that $E_{\mathcal{J}}(\alpha)$ is a rank-one POV measure and that $\mathcal{J}$ purifies any initial state: $\pi_\rho^{\mathcal{J}}(\alpha) = \rho_{\rm i}(\alpha)$, $\forall\rho\in\mathcal{S}_d$. Then, Eqs. (70), (78) give

$$\pi_{\sigma(\omega)}^{\mathcal{J}}(\alpha) = \pi_{\eta_{\rm i}}^{\mathcal{J}}(\alpha) = \rho_{\rm i}(\alpha), \qquad \eta_{\mathcal{J}}^\omega = \epsilon(\omega), \tag{115}$$

which imply also

$$\sum_\alpha p_{\rm i}(\alpha)\,\chi\big\{p_{\rm f|i}(\bullet|\alpha),\pi_{\sigma(\bullet)}^{\mathcal{J}}(\alpha)\big\} = 0\,. \tag{116}$$

Therefore one obtains that inequality (73) reduces to Eq. (72), that Hall bound is better than Holevo bound in this case and that inequality (82) becomes equivalent to Scutaru bound (79):

$$b_2 = b_{\rm Scu} \leq I_{\rm c}\{p_{\rm i},\rho_{\rm i};E_{\mathcal{I}}\} \leq B_{\rm Hall} = B_{\rm nub} \leq B_{\rm Hlv} \equiv S_{\rm q}(\eta_{\rm i}). \tag{117}$$

The instrument $\mathcal{I}$ is pure

When the initial states are pure and, moreover, the instrument $\mathcal{I}$ sends pure states into pure a posteriori states, one has also that the states $\rho_{\rm f}^\alpha(\omega)$ are pure and

$$\sum_\omega p_{\rm f}(\omega)\,\chi\{p_{\rm i|f}(\bullet|\omega),\rho_{\rm f}^\bullet(\omega)\} = \sum_\omega p_{\rm f}(\omega)\,S_{\rm q}\big(\rho_{\rm f}(\omega)\big).$$

Then, the SWW bound (83b) reduces to

$$B_{\rm SWW} = I_{\rm q}(\eta_{\rm i};\mathcal{I}) \equiv S_{\rm q}(\eta_{\rm i}) - \sum_\omega p_{\rm f}(\omega)S_{\rm q}\big(\rho_{\rm f}(\omega)\big)\,. \tag{118}$$

5.4 Examples based on a two-level atom

Here we give two examples based on a two-state system. This case is particularly suited to construct examples which allow for explicit calculations. The eigenvalues of a density matrix $\rho \in \mathcal{S}_2$ are

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4D}\right), \qquad D := \det \rho\,, \qquad 0 \leq D \leq \frac{1}{4}\,. \tag{119a}$$

Then, the von Neumann entropy and the subentropy can be written as

$$S_\mathrm{q}(\rho) = \sqrt{1-4D}\,[1 - \log(2\lambda_+)] - \lambda_- \log D\,, \tag{119b}$$

$$Q(\rho) = S_\mathrm{q}(\rho) - \frac{D}{\sqrt{1-4D}} \log \frac{\lambda_+}{\lambda_-}\,. \tag{119c}$$

5.4.1 Pure initial states and good counting measurement

Let us give now a simple example of the situation of Section 5.3. We consider a two-level atom whose ground and excited states are $|0\rangle = \binom{0}{1}$ and $|1\rangle = \binom{1}{0}$, respectively. After the preparation, the atom is left isolated and, if it is in the excited state, it can emit a photon. For what concerns the measurement, assume that we are able only to count the number (0 or 1) of photons emitted in the time interval $(0,t)$. The instrument is

$$\mathcal{O}_t(0)[\rho] = \mathrm{e}^{-\frac{\Gamma}{2}|1\rangle\langle 1|t}\,\rho\,\mathrm{e}^{-\frac{\Gamma}{2}|1\rangle\langle 1|t}, \tag{120a}$$

$$\mathcal{O}_t(1)[\rho] = \int_0^t \mathrm{d}s\,\Gamma|0\rangle\langle 1|\mathcal{O}_s(0)[\rho]|1\rangle\langle 0| = \left(1 - \mathrm{e}^{-\Gamma t}\right)|0\rangle\langle 1|\rho|1\rangle\langle 0|, \tag{120b}$$

where Γ is the decay rate. The associated POV measure is

$$E_t(0) = \mathrm{e}^{-\Gamma t}|1\rangle\langle 1| + |0\rangle\langle 0|, \qquad E_t(1) = \left(1 - \mathrm{e}^{-\Gamma t}\right)|1\rangle\langle 1|. \tag{121}$$

In this example, due to the presence of the time t, we shall use the subscript "t" instead of "f" for the final quantities; we shall also write the various bounds as functions of $\Gamma t =: x$.

Assume that we are able to prepare the atom in the ground state $|0\rangle$ and, by a suitable pulse, in the state $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$; so, our initial states are

$$\rho_\mathrm{i}(0) = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_\mathrm{i}(1) = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{122}$$

Moreover, let us assume that the a priori probabilities are equal:

$$p_\mathrm{i}(0) = p_\mathrm{i}(1) = \frac{1}{2}\,. \tag{123}$$

Then, the initial average state is

$$\eta_\mathrm{i} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}. \tag{124}$$

The various probability can be easily computed; we give the results in Appendix A. Then, the explicit expression of the classical mutual information turns out to be

$$I_\mathrm{c}(\Gamma t) := I_\mathrm{c}\{p_\mathrm{i}, \rho_\mathrm{i}; E_t\} = \frac{3}{2} + \frac{1+\mathrm{e}^{-\Gamma t}}{4}\log\left(1 + \mathrm{e}^{-\Gamma t}\right) - \frac{3+\mathrm{e}^{-\Gamma t}}{4}\log\left(3 + \mathrm{e}^{-\Gamma t}\right); \tag{125}$$

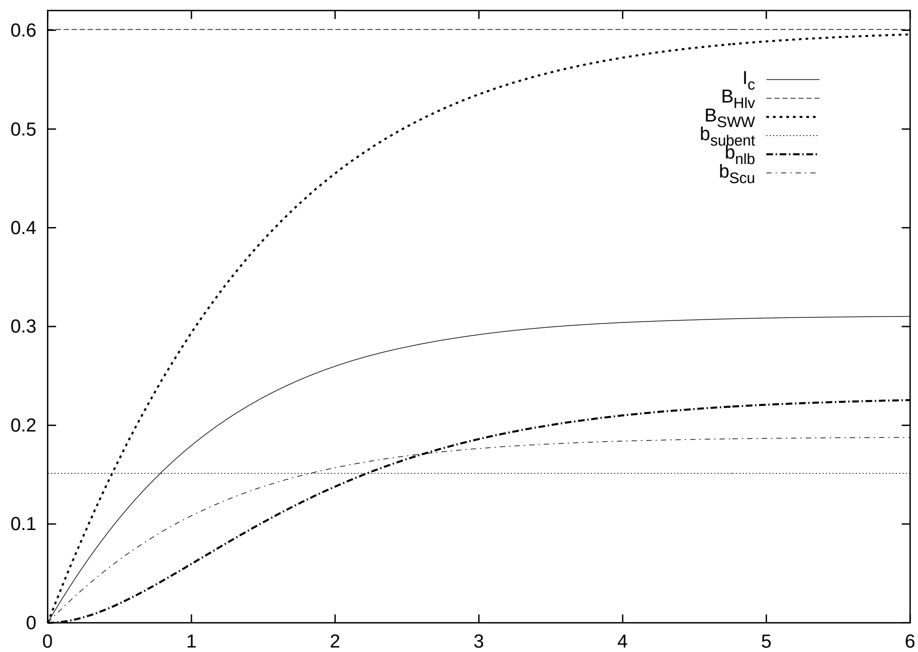

Fig. 1. The classical mutual information and the various bounds as functions of $x = \Gamma t$: the example of Section 5.4.1. In this case $B_{\rm nub}(x) = B_{\rm Hall}(x) = B_{\rm SWW}(x)$, $b_2(x) = b_{\rm Scu}(x)$, $b_1(x) < 0$.

its maximum value is for large times:

$$\sup_{x>0} I_{\rm c}(x) = \lim_{x\to+\infty} I_{\rm c}(x) = \frac{3\,(2-\log 3)}{4} \simeq 0.311278\,. \tag{126}$$

Let us consider now the various bounds; all the determinants needed in the formulas are given in Appendix A. First of all we have Holevo bound and the subentropy bound

$$B_{\rm Hlv} = S_{\rm q}(\eta_{\rm i}) \simeq 0.600876\,, \qquad b_{\rm subent} = Q(\eta_{\rm i}) = S_{\rm q}(\eta_{\rm i}) - \frac{\log\big(\sqrt{2}+1\big)}{2\sqrt{2}} \simeq 0.151314\,. \tag{127}$$

The computations of the determinants give that also the SWW bound (118) reduces to Hall one; we get

$$B_{\rm nub}(\Gamma t) = B_{\rm Hall}(\Gamma t) = B_{\rm SWW}(\Gamma t) = S_{\rm q}(\eta_{\rm i}) - \frac{3+{\rm e}^{-\Gamma t}}{4}\, S_{\rm q}\big(\rho_t(0)\big). \tag{128}$$

Finally we have

$$b_{\rm nlb}(\Gamma t) = S_{\rm q}(\eta_{\rm i}) - \frac{1}{2}\, S_{\rm q}\big(\xi_t(0)\big) - \frac{1}{2}\, S_{\rm q}\big(\xi_t(1)\big), \tag{129}$$

$$b_2(\Gamma t) = b_{\rm Scu}(\Gamma t) = S_{\rm q}(\eta_{\rm i}) - \frac{3+{\rm e}^{-\Gamma t}}{4}\, S_{\rm q}\big(\epsilon_t(0)\big). \tag{130}$$

By numerical computations one can check that $b_1(\Gamma t) < 0$. In Figure 1 the various bounds are plotted as functions of the length of the time interval $x = \Gamma t$.

5.4.2 Mixed initial states and imperfect measurement

In the previous example many bounds turned out to be the same; to have a more generic situation, we modify that example by rendering not pure one of the initial states and by adding some more imperfection in the instrument.

We consider again a two-level atom, but now, when we try to count the number (0 or 1) of photons emitted in the time interval $(0,t)$ a spurious count can be registered with a small probability, due to some imperfection in the instrumentation. Let us say that now the instrument is

$$\mathcal{O}_t(1)[\rho] = \left(1 - \mathrm{e}^{-\Gamma t}\right)\left(\frac{49}{50}\,|0\rangle\langle 1|\rho|1\rangle\langle 0| + \frac{1}{50}\,\rho\right), \tag{131a}$$

$$\mathcal{O}_t(0)[\rho] = \frac{49}{50}\,\mathrm{e}^{-\frac{\Gamma}{2}|1\rangle\langle 1|t}\,\rho\,\mathrm{e}^{-\frac{\Gamma}{2}|1\rangle\langle 1|t} + \frac{\mathrm{e}^{-\Gamma t}}{50}\,\rho\,, \tag{131b}$$

where Γ is the decay rate. The associated POV measure is

$$E_t(1) = \left(1 - \mathrm{e}^{-\Gamma t}\right)\left(|1\rangle\langle 1| + \frac{1}{50}\,|0\rangle\langle 0|\right), \qquad E_t(0) = \mathrm{e}^{-\Gamma t}|1\rangle\langle 1| + \frac{49 + \mathrm{e}^{-\Gamma t}}{50}\,|0\rangle\langle 0|. \tag{132}$$

We are able to prepare the atom in the ground state $|0\rangle$. We would also prepare the state $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ by a suitable pulse, but some imperfection again allows us only to obtain a mixture of this state with the ground state. So, let us say that our initial states are

$$\rho_{\mathrm{i}}(0) = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{133a}$$

$$\rho_{\mathrm{i}}(1) = \frac{9}{10}\,\frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right) + \frac{1}{10}|0\rangle\langle 0| = \begin{pmatrix} 9/20 & 9/20 \\ 9/20 & 11/20 \end{pmatrix}. \tag{133b}$$

Moreover, let us assume that the a priori probabilities are

$$p_{\mathrm{i}}(0) = \frac{4}{9}\,, \qquad p_{\mathrm{i}}(1) = \frac{5}{9}\,. \tag{134}$$

Then, the initial average state is the same as in the previous section:

$$\eta_{\mathrm{i}} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}. \tag{135}$$

The various probabilities can be easily computed and are written down in Appendix B. Then, the classical mutual information becomes

$$\begin{aligned} I_{\mathrm{c}}(\Gamma t) := I_{\mathrm{c}}\{p_{\mathrm{i}}, \rho_{\mathrm{i}}; E_t\} &= \frac{1 - \mathrm{e}^{-\Gamma t}}{25}\left(\frac{2}{9}\,\log\frac{4}{53} + \frac{461}{72}\,\log\frac{461}{265}\right) \\ &+ \frac{98 + 2\mathrm{e}^{-\Gamma t}}{225}\,\log\frac{4\left(49 + \mathrm{e}^{-\Gamma t}\right)}{147 + 53\mathrm{e}^{-\Gamma t}} + \frac{539 + 461\mathrm{e}^{-\Gamma t}}{1800}\,\log\frac{539 + 461\mathrm{e}^{-\Gamma t}}{5\left(147 + 53\mathrm{e}^{-\Gamma t}\right)} \\ &\overset{t\to\infty}{\simeq} 0.21822\,. \end{aligned} \tag{136}$$

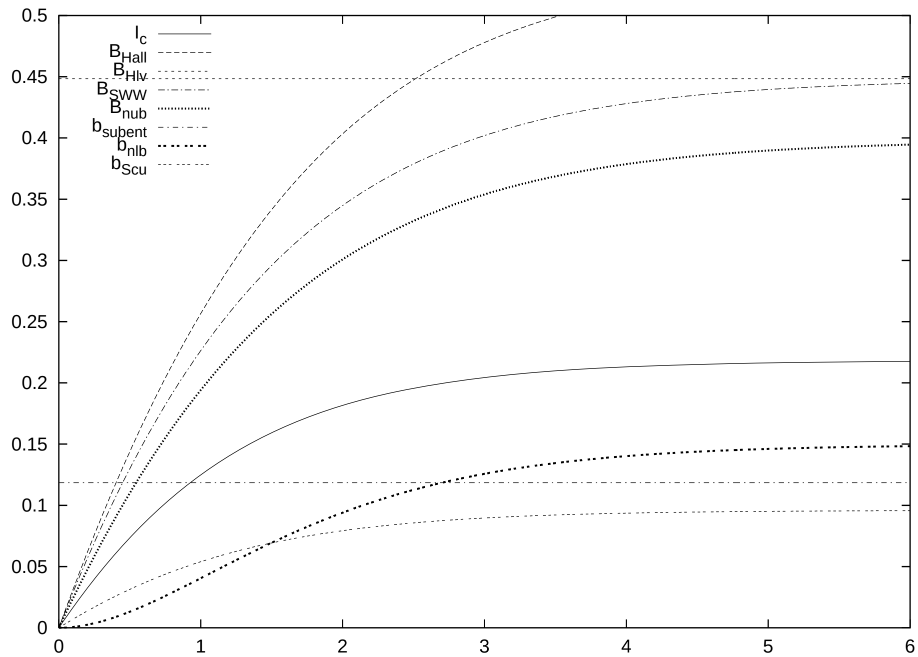

Fig. 2. The classical mutual information and the various bounds as functions of $x = \Gamma t$: the example of Section 5.4.2. In this case $b_1(x) < 0$, $b_2(x) \leq 0$.

To calculate the various bounds, we need many determinants, again given in Appendix B. Then, we have the various bounds: Holevo bound

$$B_{\rm Hlv} := \chi\{p_{\rm i}, \rho_{\rm i}\} = S_{\rm q}(\eta_{\rm i}) - \frac{5}{9}\, S_{\rm q}\big(\rho_{\rm i}(1)\big) \simeq 0.448368\,, \tag{137}$$

Hall bound

$$B_{\rm Hall}(\Gamma t) := \chi\{p_t, \sigma_t\} = S_{\rm q}(\eta_{\rm i}) - p_t(0) S_{\rm q}\big(\sigma_t(0)\big) - p_t(1) S_{\rm q}\big(\sigma_t(1)\big), \tag{138}$$

the new lower bound

$$b_{\rm nlb}(\Gamma t) := \chi\{p_{\rm i}, \xi_t\} = S_{\rm q}(\eta_{\rm i}) - \frac{4}{9}\, S_{\rm q}\big(\xi_t(0)\big) - \frac{5}{9}\, S_{\rm q}\big(\xi_t(1)\big), \tag{139}$$

SWW bound

$$\begin{aligned} B_{\rm SWW}(\Gamma t) &:= \chi\{p_{\rm i}, \rho_{\rm i}\} - \sum_\omega p_t(\omega)\, \chi\{p_{{\rm i}|t}(\bullet|\omega), \rho_t^\bullet(\omega)\} \\ &= B_{\rm Hlv} - \sum_\omega \big[p_t(\omega) S_{\rm q}\big(\rho_t(\omega)\big) - p_{{\rm i}t}(1,\omega) S_{\rm q}\big(\rho_t^1(\omega)\big)\big]\,, \end{aligned} \tag{140}$$

the new upper bound

$$B_{\mathrm{nub}}(\Gamma t) := \chi\{p_{\mathrm{i}}, \rho_{\mathrm{i}}\} - \sum_{\omega} p_t(\omega) I_{\mathrm{q}}\{\sigma_t(\omega); \mathcal{J}\} \\ = B_{\mathrm{Hall}}(\Gamma t) - \frac{5}{9}\, S_{\mathrm{q}}\big(\rho_{\mathrm{i}}(1)\big) + \sum_{\omega} p_{\mathrm{i}t}(1,\omega) S_{\mathrm{q}}\big(\pi^{\mathcal{J}}_{\sigma_t(\omega)}(1)\big), \tag{141}$$

Scutaru bound

$$b_{\mathrm{Scu}}(\Gamma t) := \chi\{p_t, \epsilon_t\} = S_{\mathrm{q}}(\eta_{\mathrm{i}}) - p_t(0) S_{\mathrm{q}}\big(\epsilon_t(0)\big) - p_t(1) S_{\mathrm{q}}\big(\epsilon_t(1)\big), \tag{142}$$

the subentropy lower bound for the accessible information

$$b_{\mathrm{subent}} = B_{\mathrm{Hlv}} - d(0.125) + \frac{5}{9}\, d(0.045) \simeq 0.118467\,, \tag{143a}$$

$$d(x) := \frac{x}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{1-\sqrt{1-4x}}\,. \tag{143b}$$

By numerical computations one can check that $b_1(\Gamma t) < 0$ and $b_2(\Gamma t) \leq 0$. In Figure 2 the various bounds are plotted as functions of the length of the time interval $x = \Gamma t$.

5.4.3 A special feature of the two ensembles

In Section 5.2.1 we have considered a POV measure made up of the eigenprojections of the initial average state η_{i} and in Section 5.2.2 we have recalled that this choice saturates Holevo inequality in the case of commuting letter states. However, when the letter states do not commute, not only the eigenprojections of η_{i} do not give necessarily the best measurement, but they can even be the worst choice, as shown by the case of the ensembles of Sections 5.4.1 and 5.4.2.

The average state η_{i} is the same in both cases, see Eqs. (124) and (135). Its eigenprojections are $P_\pm = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} \mp 1 & \pm 1 \\ \pm 1 & \sqrt{2} \pm 1 \end{pmatrix}$, for which we get $\mathrm{Tr}\{P_\pm \rho\} = \frac{2 \pm \sqrt{2}}{4}$ for any density matrix of the form $\rho = \begin{pmatrix} a & a \\ a & 1-a \end{pmatrix}$. But this is the form of all the letter states of Sections 5.4.1 and 5.4.2; therefore, in both cases, $p_{\mathrm{f}|\mathrm{i}}(\pm|\alpha) = p_{\mathrm{f}}(\pm)$ and, so, $I_{\mathrm{c}} = 0$.

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Appendix A. Two-level system, first example

The various probabilities needed in the example are

$$p_{t|\mathrm{i}}(0|0) = 1\,, \quad p_{t|\mathrm{i}}(1|0) = 0\,, \qquad p_{t|\mathrm{i}}(0|1) = \frac{1+\mathrm{e}^{-x}}{2}\,, \quad p_{t|\mathrm{i}}(1|1) = \frac{1-\mathrm{e}^{-x}}{2}\,, \tag{A.1a}$$

$$p_t(0) = \frac{3+\mathrm{e}^{-x}}{4}\,, \qquad p_t(1) = \frac{1-\mathrm{e}^{-x}}{4}\,, \tag{A.1b}$$

$$p_{\mathrm{i}t}(0,0) = \frac{1}{2}\,, \quad p_{\mathrm{i}t}(1,0) = \frac{1+\mathrm{e}^{-x}}{4}\,, \quad p_{\mathrm{i}t}(0,1) = 0\,, \quad p_{\mathrm{i}t}(1,1) = \frac{1-\mathrm{e}^{-x}}{4}\,, \tag{A.1c}$$

$$p_{\mathrm{i}|t}(0|1) = 0\,, \quad p_{\mathrm{i}|t}(1|1) = 1\,, \qquad p_{\mathrm{i}|t}(0|0) = \frac{2}{3+\mathrm{e}^{-x}}\,, \quad p_{\mathrm{i}|t}(1|0) = \frac{1+\mathrm{e}^{-x}}{3+\mathrm{e}^{-x}}\,. \tag{A.1d}$$

For what concerns the determinants involved in the upper bounds, we have

$$\det \eta_{\mathrm{i}} = \frac{1}{8}\,, \qquad \det \rho_{\mathrm{i}}(\alpha) = 0\,. \tag{A.2}$$

Then, Eq. (33) gives $\det \sigma_t(\omega) = \dfrac{\det \eta_{\mathrm{i}}\ \det E_t(\omega)}{p_t(\omega)^2}$ and we get

$$\det \sigma_t(0) = \frac{2\,\mathrm{e}^{-x}}{\left(3+\mathrm{e}^{-x}\right)^2}\,, \qquad \det \sigma_t(1) = 0\,. \tag{A.3}$$

By direct computations, we obtain

$$\det \rho_t^{\alpha}(\omega) = \det \rho_t(1) = 0\,, \qquad \det \rho_t(0) = \frac{2\,\mathrm{e}^{-x}}{\left(3+\mathrm{e}^{-x}\right)^2}\,, \tag{A.4}$$

$$\det \eta_t^0 = 0\,, \qquad \det \eta_t^1 = \frac{\mathrm{e}^{-x}}{4}\left(1-\mathrm{e}^{-x}\right), \qquad \det \eta_t = \frac{\mathrm{e}^{-x}}{16}\left(3-\mathrm{e}^{-x}\right). \tag{A.5}$$

Finally, we get

$$\xi_t(0) = \sigma_t(0), \qquad \xi_t(1) = \frac{2}{3+\mathrm{e}^{-x}}\,\eta_{\mathrm{i}}^{1/2}\left[\left(3-\mathrm{e}^{-x}\right)|1\rangle\langle 1| + \left(1+\mathrm{e}^{-x}\right)|0\rangle\langle 0|\right]\eta_{\mathrm{i}}^{1/2}\,, \tag{A.6a}$$

$$\det \xi_t(0) = \frac{2\,\mathrm{e}^{-x}}{\left(3+\mathrm{e}^{-x}\right)^2}\,, \qquad \det \xi_t(1) = \frac{\left(3-\mathrm{e}^{-x}\right)\left(1+\mathrm{e}^{-x}\right)}{2\left(3+\mathrm{e}^{-\Gamma t}\right)^2}\,, \tag{A.6b}$$

$$\epsilon_t(1) = \rho_{\mathrm{i}}(1), \qquad \epsilon_t(0) = \frac{1}{2\left(3+\mathrm{e}^{-x}\right)}\begin{pmatrix} 1+\mathrm{e}^{-x} & 1+\mathrm{e}^{-x} \\ 1+\mathrm{e}^{-x} & 5+\mathrm{e}^{-x} \end{pmatrix}, \tag{A.7a}$$

$$\det \epsilon_t(0) = \frac{1+\mathrm{e}^{-x}}{\left(3+\mathrm{e}^{-x}\right)^2}\,, \qquad \det \epsilon_t(1) = 0\,. \tag{A.7b}$$

Appendix B. Two-level system, second example

First of all, the various probabilities are

$$p_{t|\mathrm{i}}(0|0) = \frac{49 + \mathrm{e}^{-x}}{50}\,, \qquad p_{t|\mathrm{i}}(1|0) = \frac{1 - \mathrm{e}^{-x}}{50}\,, \tag{B.1a}$$

$$p_{t|\mathrm{i}}(0|1) = \frac{539 + 461\,\mathrm{e}^{-x}}{1000}\,, \qquad p_{t|\mathrm{i}}(1|1) = \frac{461\,(1 - \mathrm{e}^{-x})}{1000}\,, \tag{B.1b}$$

$$p_t(0) = \frac{147 + 53\,\mathrm{e}^{-x}}{200}\,, \qquad p_t(1) = \frac{53\,(1 - \mathrm{e}^{-x})}{200}\,, \tag{B.1c}$$

$$p_{\mathrm{i}t}(0,0) = \frac{2\,(49 + \mathrm{e}^{-x})}{225}\,, \qquad p_{\mathrm{i}t}(1,0) = \frac{539 + 461\,\mathrm{e}^{-x}}{1800}\,, \tag{B.1d}$$

$$p_{\mathrm{i}t}(0,1) = \frac{2\,(1 - \mathrm{e}^{-x})}{225}\,, \qquad p_{\mathrm{i}t}(1,1) = \frac{461\,(1 - \mathrm{e}^{-x})}{1800}\,, \tag{B.1e}$$

$$p_{\mathrm{i}|t}(0|1) = \frac{16}{477}\,, \qquad p_{\mathrm{i}|t}(0|0) = \frac{16\,(49 + \mathrm{e}^{-x})}{9\,(147 + 53\,\mathrm{e}^{-x})}\,, \tag{B.1f}$$

$$p_{\mathrm{i}|t}(1|1) = \frac{461}{477}\,, \qquad p_{\mathrm{i}|t}(1|0) = \frac{539 + 461\,\mathrm{e}^{-x}}{9\,(147 + 53\,\mathrm{e}^{-x})}\,. \tag{B.1g}$$

Then, Eqs. (33), (36), (70b) give

$$\det \sigma_t(\omega) = \frac{\det \eta_{\mathrm{i}}\ \det E_t(\omega)}{p_t(\omega)^2}\,, \qquad \det \pi^{\mathcal{J}}_{\sigma_t(\omega)}(\alpha) = \frac{\det \rho_{\mathrm{i}}(\alpha)\ \det E_t(\omega)}{p_{t|\mathrm{i}}(\omega|\alpha)^2}\,, \tag{B.2}$$

$$\det \xi_t(\alpha) = \det \eta_{\mathrm{i}}\ \det \Big[\sum_{\omega} \frac{p_{t|\mathrm{i}}(\omega|\alpha)}{p_t(\omega)}\, E_t(\omega)\Big]. \tag{B.3}$$

The final result of the computations of the determinants are

$$\det \eta_{\mathrm{i}} = \frac{1}{8}\,, \qquad \det \rho_{\mathrm{i}}(1) = \frac{9}{200}\,, \qquad \det \rho_{\mathrm{i}}(0) = \det \eta_t^0 = 0\,, \tag{B.4}$$

$$\det \eta_t^1 = \frac{9\left[(1 + 49\,\mathrm{e}^{-x})\,(1991 - 441\,\mathrm{e}^{-x}) - 9\left(1 + 49\,\mathrm{e}^{-x/2}\right)^2\right]}{10^6}\,, \tag{B.5}$$

$$\det \eta_t = \frac{(1 + 49\,\mathrm{e}^{-x})\,(199 - 49\,\mathrm{e}^{-x}) - \left(1 + 49\,\mathrm{e}^{-x/2}\right)^2}{4 \times 10^4}\,, \tag{B.6}$$

$$\det \sigma_t(0) = \frac{100\,\mathrm{e}^{-x}\,(49 + \mathrm{e}^{-x})}{(147 + 53\,\mathrm{e}^{-x})^2}\,, \qquad \det \sigma_t(1) = \left(\frac{10}{53}\right)^2\,, \tag{B.7}$$

$$\det \xi_t(0) = \frac{4\,(1274 + 51\,\mathrm{e}^{-x})\,(147 + 2503\,\mathrm{e}^{-x})}{[53\,(147 + 53\,\mathrm{e}^{-x})]^2}\,, \tag{B.8}$$

$$\det \xi_t(1) = \frac{\left(67767 - 14767\,\mathrm{e}^{-x}\right)\left(29351 + 23649\,\mathrm{e}^{-x}\right)}{2\left[530\left(147 + 53\,\mathrm{e}^{-x}\right)\right]^2}\,, \tag{B.9}$$

$$\det \rho_t^0(\omega) = 0, \qquad \det \rho_t^1(1) = \frac{9 \times 443}{(461)^2}\,, \qquad \det \rho_t(1) = \frac{51}{(53)^2}\,, \tag{B.10}$$

$$\det \rho_t^1(0) = \frac{9\,\mathrm{e}^{-x}}{\left(539 + 461\,\mathrm{e}^{-x}\right)^2}\,\left(5341 - 882\,\mathrm{e}^{-x/2} + 541\,\mathrm{e}^{-x}\right), \tag{B.11}$$

$$\det \rho_t(0) = \frac{\mathrm{e}^{-x}\left(4949 + 149\,\mathrm{e}^{-x} - 98\,\mathrm{e}^{-x/2}\right)}{\left(147 + 53\,\mathrm{e}^{-x}\right)^2}\,, \tag{B.12}$$

$$\det \pi^{\mathcal{J}}_{\sigma_t(\omega)}(0) = 0, \qquad \det \pi^{\mathcal{J}}_{\sigma_t(1)}(1) = \left(\frac{30}{461}\right)^2, \tag{B.13}$$

$$\det \pi^{\mathcal{J}}_{\sigma_t(0)}(1) = \frac{900\,\mathrm{e}^{-x}\left(49 + \mathrm{e}^{-x}\right)}{\left(539 + 461\,\mathrm{e}^{-x}\right)^2}\,, \tag{B.14}$$

$$\det \epsilon_t(0) = \frac{\left(539 + 461\,\mathrm{e}^{-x}\right)\left(931 + 69\,\mathrm{e}^{-x}\right)}{200\left(147 + 53\,\mathrm{e}^{-x}\right)^2}\,, \qquad \det \epsilon_t(1) = \frac{69 \times 461}{200 \times 53^2}\,. \tag{B.15}$$