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DETERMINISTIC LOCAL CONVERSION OF INCOMPARABLE STATES BY COLLECTIVE LOCC

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Incomparability of pure bipartite entangled states under deterministic LOCC is a very strange phenomena. We find two possible ways of getting our desired pure entangled state which is incomparable with the given input state, by collective LOCC with certainty. The first one is by providing some pure entanglement through the lower dimensional maximally-entangled states or using further less amount of entanglement and the next one is by collective operation on two pairs which are individually incomparable. It is quite surprising that we are able to achieve maximally entangled states of any Schmidt rank from a finite number of 2×2 pure entangled states only by deterministic LOCC. We provide general theory for the case of 3×3 system of incomparable states by the above processes where incomparability seems to be the most hardest one.

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1 Introduction

The necessity of quantum entanglement on different quantum information and communication tasks [1, 2] naturally raises the question of manipulating entanglement shared between different parties. It is very often useful when shared parties are able to manipulate entanglement by local operation and classical communication(LOCC). Within this restricted scenario, Bennet et al. [3] provided the asymptotic conversion of entanglement by LOCC for bipartite systems. Things are qualitatively different if we restrict ourselves to finite regime, i.e., only finite number of copies are available, even for pure bipartite entangled states. Nielsen [4] provided a necessary and sufficient condition for the local conversion of pure bipartite entangled states for single copy case with certainty. Vidal extended this to the case of probabilistic local conversion (SLOCC) of two pure bipartite entangled states [5]. Further, Morikoshi [6] investigated the recovery of entanglement loss in the process of local conversion and several other groups studied the possibility and impossibility of entanglement manipulation in different context [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In many practical situation it is often necessary to use a particular entangled state for a specific task. Therefore the possibility of extending the set of states which produces the target

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state of our interest, beyond those satisfying Nielsen's criterion, is of importance. Jonathan and Plenio [8] investigated the case where in place of single copy transformation we also allow collective operation by assigning some entanglement. The phenomena is known as catalytic conversion. Similar to the situation of partial recovery, Bandyopadhyay et.al. [7, 11] and Feng et.al. [9] studied the possibility of local conversion of incomparable states (those not satisfying Nielsen's criteria) by collective operation where some entanglement may be recovered. The class of incomparable states may be further reduced by the process of multiple copy transformation [12]. Still there remain large class of incomparable states where deterministic conversion by LOCC are not possible. Recently, Ishizaka [17] showed that using ppt-bound entanglement local conversion of any two states (no matter what the Schmidt rank of the states are) is always possible, at least with some probability. So the problem of finding target state of our interest now remains for the case of deterministic local conversion. In this paper our aim is to resolve the incomparability of two pure bipartite states having the same Schmidt rank, by deterministic LOCC. We first show that, using the entanglement of some lower dimensional pure state, it is always possible to break the incomparability in any dimension. This method is found to be successful in providing maximally entangled states of any dimension from next lower dimensional maximally entangled states with the help of suitable non-maximally pure entangled states. Actually it is possible to achieve any maximally entangled states by a finite number of 2×2 pure entangled states only. We also observed that a special kind of mutual catalysis which we rather call mutual co-operation is very much useful in converting the pairs of incomparable states. We discuss exhaustively all our analysis for 3×3 systems in which incomparability seems to be the hardest one.

2 Background

Let's take a quick overview of the topic of state transformation by LOCC. We first mention Nielsen's criterion [4] for deterministic local conversion of $|\psi\rangle$ to $|\phi\rangle$, both are pure bipartite states shared between two parties, say, Alice and Bob. Suppose up to local unitary equivalence we write $|\psi\rangle$, $|\phi\rangle$ in the Schmidt basis $\{|i_A\rangle, |i_B\rangle\}$ with decreasing order of Schmidt coefficients as : $|\psi\rangle = \sum_{i=1}^{d} \sqrt{\alpha_i}|i_A i_B\rangle$, $|\phi\rangle = \sum_{i=1}^{d} \sqrt{\beta_i}|i_A i_B\rangle$, where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$, for $i = 1, 2, \cdots, d-1$, $\sum_{i=1}^{d} \alpha_i = 1 = \sum_{i=1}^{d} \beta_i$ and denote $\lambda_\psi \equiv (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $\lambda_\phi \equiv (\beta_1, \beta_2, \cdots, \beta_d)$. Then Nielsen's criterion tells us that $|\psi\rangle \rightarrow |\phi\rangle$ is possible with certainty under LOCC if and only if λ_ψ is majorized by λ_ϕ (denoted by $\lambda_\psi \prec \lambda_\phi$); i.e., $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i$ for each $k = 1, 2, \cdots, d$. One consequence of Nielsen's result is; if $|\psi\rangle \rightarrow |\phi\rangle$ under LOCC with certainty, then $E(|\psi\rangle) \geq E(|\phi\rangle)$ [where $E(\cdot)$ denote the von-Neumann entropy of the reduced density operator of any subsystem and known as the entropy of entanglement]. If the above criterion does not hold, then it is usually denoted by $|\psi\rangle \not\rightarrow |\phi\rangle$. Though it may happen that $|\phi\rangle \rightarrow |\psi\rangle$ under LOCC. Now if it happens that $|\psi\rangle \not\rightarrow |\phi\rangle$ and $|\phi\rangle \not\rightarrow |\psi\rangle$ then we denote it as $|\psi\rangle \not\leftrightarrow |\phi\rangle$ and call $|\psi\rangle, |\phi\rangle$ as a pair of incomparable states. For 2×2 states it is always be the case that either $|\psi\rangle \rightarrow |\phi\rangle$ or $|\phi\rangle \rightarrow |\psi\rangle$. Therefore, we look beyond the 2×2 system of states. Now it is natural to ask that for such incomparable pairs, is it not possible to convert $|\psi\rangle$ to $|\phi\rangle$ by means of LOCC? If we require $|\phi\rangle$ and we have a finite but sufficient number of copies of $|\psi\rangle$ then Vidal's theorem[5] provide us the way of converting probabilistically $|\psi\rangle$ to $|\phi\rangle$ by LOCC. This is of course of no use for deterministic conversion. For the purpose of transferring $|\psi\rangle$ to $|\phi\rangle$ where $|\psi\rangle \not\rightarrow |\phi\rangle$ Jonathan and Plenio [8] found that sometimes collective operation may be useful to convert deterministically. They showed that if we assist the conversion by another pure entangled state $|\chi\rangle$, say catalyst, then the conversion $|\psi\rangle \otimes |\chi\rangle \longrightarrow |\phi\rangle \otimes |\chi\rangle$ may be possible by collective LOCC deterministically. It is interesting in this method that we recover all the entanglement used in the process. But what type of pairs are really catalyzable? It is really hard to categorize. Jonathan and Plenio [8] showed that if the conversion $|\psi\rangle \to |\phi\rangle$ is possible by a catalytic state then $\alpha_1 \leq \beta_1$ and $\alpha_d \geq \beta_d$ hold simultaneously. Also if $|\psi\rangle \to |\phi\rangle$ is possible by catalysis, then $E(|\psi\rangle) \geq E(|\phi\rangle)$. For 3×3 system of states, violation of Nielsen's criteria always implies violation of the necessary condition for catalysis. The existence of catalytic state is first seen for 4×4 incomparable pairs and only in this level a necessary and sufficient condition for the existence of 2×2 system of catalytic state is found until now [19, 18]. Therefore except some numerical evidence it is really hard to find catalyzable pairs. Investigation in this direction is going on by several groups [14]. Now another interesting result is provided by Feng et.al., and other groups [9, 11] which is known as mutual catalysis. The basic objective in this process is: given two pairs of incomparable states, say, $|\psi_1\rangle \not\to |\phi_1\rangle, |\psi_2\rangle \not\to |\phi_2\rangle$, whether $|\psi_1\rangle \otimes |\psi_2\rangle \longrightarrow |\phi_1\rangle \otimes |\phi_2\rangle$ is possible under LOCC with certainty or not. Emphasis is given on the special kind of mutual catalysis (in [11] it is defined as super catalysis), where in the conversion we recover not only the entanglement assisted in the process but more than that, i.e., for some incomparable pair $|\psi\rangle \not\to |\phi\rangle$ there exists $(|\chi\rangle, |\eta\rangle)$ with $E(|\eta\rangle) \geq E(|\chi\rangle)$ and $|\eta\rangle \to |\chi\rangle$ such that by collective local operation $|\psi\rangle \otimes |\chi\rangle \longrightarrow |\phi\rangle \otimes |\eta\rangle$ is possible deterministically. It is interesting that the necessary condition for the existence of such special kind of mutual catalytic pair $(|\chi\rangle, |\eta\rangle)$, is the same as that for catalyst. Hence this type of mutual catalysis is not possible for 3×3 system of incomparability. It is shown, not analytically, but by some numerical examples, that there are systems for which catalyst does not exit but mutual catalysis works. Trivially it is always possible that $|\psi\rangle \otimes |\phi\rangle \longrightarrow |\phi\rangle \otimes |\psi\rangle$ under LOCC with certainty. So existence of mutual catalytic state is always possible. But it is not of use, as our target state $|\phi\rangle$ is not in our hand and in the process of trivial mutual catalysis we have to use it. Next to that, it is found by Bandyopadhyay, et.al. [12], sometimes if we increase the number of copies of the states, then deterministic conversion of incomparable states under LOCC may be possible; i.e., $|\psi\rangle \not\to |\phi\rangle$ but $|\psi\rangle^{\otimes k} \to |\phi\rangle^{\otimes k}$ is possible for some integer k. This phenomena is called multi-copy transformation. A sufficient condition for an incomparable pair to remain incomparable even if we increase the number of copies as large as possible is that, either $\alpha_1 < \beta_1$ and $\alpha_d < \beta_d$, or, $\alpha_1 > \beta_1$ and $\alpha_d > \beta_d$ must hold simultaneously [12]. We call them as strongly incomparable [12]. All pure incomparable states in 3×3 are strongly incomparable. So all the process of catalysis, mutual catalysis with some recovery and increasing number of copies will fail for all 3×3 pure incomparable pair of states and also in all pure strongly incomparable bipartite classes. Therefore one may ask, is it not possible to get the target state under LOCC for such incomparable states?

Here comes the question of using entanglement to reach the target state. By the use of entanglement we mean to forget about recovering the entanglement used in the process, but to concentrate on converting the input state to the desired one. In this paper we discover two paths from $|\psi\rangle$ to $|\phi\rangle$ by collective LOCC deterministically, discussed in two different sections. In the first part we show that for any incomparable pair $|\psi\rangle \not\leftrightarrow |\phi\rangle$, it is always possible to locally transform an incomparable pair of states with certainty, if we provide some amount of pure entanglement. We shall show that by collective LOCC, on the joint system of $|\psi\rangle$ and the next lower dimensional maximally entangled state $|\Psi^{d-1}_{max}\rangle$, we are always able to get the target state $|\phi\rangle$ along with a product state $|P\rangle$. This indicates that in any finite dimension, an incomparable pair can be made to transform if we have some supply of pure entanglement, at least the next lower rank maximally entangled states. Then it is only the matter that beside of using $|\Psi^{d-1}_{max}\rangle$, is it possible to use further less amount of entanglement to achieve the target state, and if yes, then what is the minimum amount of entanglement required for such conversion. It is interesting to note here that in all the pairs of incomparable states $|\psi\rangle$ and $|\phi\rangle$, the first and last Schmidt coefficients of the states are intricately related with each other. We complete our analysis of minimum pure entanglement required for 3×3 system of states, as it is the minimum and possibly the hardest dimension to deal with. We also observe that the maximally entangled state of any finite Schmidt rank together with a suitable pure entangled state is able to produce the next higher rank maximally entangled state under deterministic LOCC. Surprisingly we found that to require a maximally entangled state of Schmidt rank d we need only $d-1$ number of suitable 2×2 pure entangled states.

In the next section, we shall show another interesting phenomenon that there may be two pairs of incomparable states such as $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ and $\mid \chi\rangle \not\leftrightarrow \mid \eta\rangle$ but $|\psi\rangle \otimes |\chi\rangle \longrightarrow |\phi\rangle \otimes |\eta\rangle$ is possible under LOCC with certainty. That is if we require $|\phi\rangle$, $|\eta\rangle$ but we have $|\psi\rangle$, $|\chi\rangle$ then this collective operation may be possible. We call this phenomenon as mutual co-operation. Obviously this is a general kind of mutual catalysis which is the most preferred one. Beside giving some numerical evidences, we provide analytically an auxiliary pair of incomparable states for every pair of incomparable states in 3×3 system such that the joint transformation under LOCC is always possible with certainty. The concluding part of this section reflects the feature of the 3×3 system that by collective local operation on two copy of a state $|\psi\rangle$, almost in all cases, we are able to get two different states $|\phi_1\rangle$ and $|\phi_2\rangle$ both are incomparable with $|\psi\rangle$. Now if we fix any one of $|\phi_1\rangle$ or $|\phi_2\rangle$ as our target state, then we provide a good range of the possible existence of the other state.

3 Assistance by Entanglement

Suppose we have a pair of incomparable states $\mid \psi\rangle, \mid \phi\rangle$ in $d \times d$ system, where the source state $\mid \psi\rangle = \sum_{i=1}^{d} \sqrt{a_i} \mid ii\rangle$, and the target state $\mid \phi\rangle = \sum_{i=1}^{d} \sqrt{b_i} \mid ii\rangle$, are taken in their most general form with $a_i \geq a_{i+1} \geq 0$ and $b_i \geq b_{i+1} \geq 0$, $\forall i = 1, 2, \cdots, (d-1)$ together with $\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i = 1$. In the whole processes we would not taken into account the amount of entanglement of the states. Now, consider the $(d-1) \times (d-1)$ maximally entangled state $\mid \Psi^{d-1}_{max}\rangle = \frac{1}{\sqrt{d-1}} \sum_{i=d+1}^{2d-1} \mid ii\rangle$, and the product state $\mid P\rangle = \mid 00\rangle$. We want to make possible the joint transformation $\mid \psi\rangle \otimes \mid \Psi^{d-1}_{max}\rangle \rightarrow \mid \phi\rangle \otimes \mid P\rangle$ under LOCC with certainty. For this we must have,

$$\frac{a_1 k}{d-1} \leq \sum_{i=1}^{k} b_i; \forall\ k = 1, 2, \ldots, d-1. \tag{1}$$

To prove this, we first state a theorem which shows an intricate relation holds between first and last Schmidt coefficients of any two incomparable states.

Theorem. For any pair of incomparable states $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ in $d \times d$ system, where $\mid \psi\rangle = \sum_{i=1}^{d} \sqrt{a_i} \mid ii\rangle$, and $\mid \phi\rangle = \sum_{i=1}^{d} \sqrt{b_i} \mid ii\rangle$, with $a_i \geq a_{i+1} \geq 0, \sum_{i=1}^{d} a_i = 1$ and

$b_i \geq b_{i+1} \geq 0, \sum_{i=1}^{d} b_i = 1$, the following always holds:

$$a_1 + b_d < 1,\ b_1 + a_d < 1. \tag{2}$$

Proof of this theorem follows from Nielsen's criteria if we analyze incomparability condition critically.

The theorem above readily implies that $a_1 < \sum_{i=1}^{d-1} b_i$, from which we have

$$\frac{ka_1}{(d-1)} < \frac{k}{d-1}\sum_{i=1}^{d-1} b_i < \sum_{i=1}^{k} b_i,\ \forall k = 1, 2, \cdots, d-1. \tag{3}$$

So the incomparability condition itself implies that the required joint transformation is possible under LOCC with certainty. i.e., $\mid \phi\rangle \otimes \mid \Psi_{max}^{d-1}\rangle \rightarrow \mid \psi\rangle \otimes \mid P\rangle$ is possible under LOCC with certainty. Therefore for any pair of incomparable states with a given Schmidt rank the maximally entangled state of the next lower rank is sufficient to assist the joint transformation under LOCC.

Next we show that instead of using lower rank maximally entangled state, the conversion may be possible under LOCC if we use lower rank non-maximally entangled states so that we need as much as minimum use of the resource. We found explicitly the minimum amount of entanglement that is required for the local transformation of any 3×3 incomparable pairs. Suppose $\mid \psi\rangle, \mid \phi\rangle$ be a pair of incomparable states such that $\mid \psi\rangle = \sum_{i=1}^{3} \sqrt{a_i} \mid ii\rangle$ and $\mid \phi\rangle = \sum_{i=1}^{3} \sqrt{b_i} \mid ii\rangle$, where $a_1 \geq a_2 \geq a_3 \geq 0, \sum_{i=1}^{3} a_i = 1$ and $b_1 \geq b_2 \geq b_3 \geq 0, \sum_{i=1}^{3} b_i = 1$; and consider a 2×2 pure entangled state $\mid \chi\rangle = \sqrt{c} \mid 44\rangle + \sqrt{1-c} \mid 55\rangle, 1 > c \geq \frac{1}{2}$, and a 2×2 product state $\mid \eta\rangle = \mid 44\rangle$. Then the collective operation under LOCC $\mid \psi\rangle \otimes \mid \chi\rangle \rightarrow \mid \phi\rangle \otimes \mid \eta\rangle$, occurs with certainty if $\mid \chi\rangle$ is specified according with the amount of entanglement used in the process; i.e., $E(\chi) = -c\log_2 c - (1-c)\log_2(1-c)$. Hence to minimize E, we have to find the largest possible value of c, obviously which is not 1. Now, we discuss separately two different classes of 3×3 incomparable pair of states.

Type-1: When $a_1 < b_1, a_1 + a_2 > b_1 + b_2$, then we must have $c \leq \frac{b_1+b_2}{a_1+a_2}$; i.e., the minimum amount of entanglement required in this process to achieve $\mid \phi\rangle$ from $\mid \psi\rangle$ is $E = E_0$ corresponding to the value $c = c_0 = \frac{b_1+b_1}{a_1+a_2}$.

Type-2: When $a_1 > b_1, a_1 + a_2 < b_1 + b_2$, then we must have $c \leq \frac{b_1}{a_1}$; i.e., the minimum amount of entanglement required in this process to achieve $\mid \phi\rangle$ from $\mid \psi\rangle$ is $E = E_0$ corresponding to the value $c = c_0 = \frac{b_1}{a_1}$.

We conclude this section with an interesting result that in any dimension from a non-maximally pure entangled state $\mid \psi^d\rangle$ of $d \times d$ system $(d \geq 3)$, we are able to reach the maximally entangled state $\mid \phi\rangle = \mid \Psi_{max}^{d}\rangle$ of the same dimension by the use of the next lower dimensional maximally entangled state $\mid \Psi_{max}^{d-1}\rangle$ through collective local operation with certainty.

Corollary-1. $\mid \psi^d\rangle \otimes \mid \Psi_{max}^{d-1}\rangle \rightarrow \mid \Psi_{max}^{d}\rangle \otimes \mid P\rangle$, where $\mid P\rangle$ is a product state, is possible under LOCC with certainty, if the largest Schmidt coefficient, a_1 of $\mid \psi^d\rangle$ satisfies the relation $a_1 \leq \frac{d-1}{d}$.

This result follows directly from the theorem above. In fact, instead of using the state $\mid \psi^d\rangle$, the above transformation is possible by a 2×2 state only.

Corollary-2. The transformation $\mid \psi\rangle \otimes \mid \Psi_{max}^{d-1}\rangle \rightarrow \mid \Psi_{max}^{d}\rangle \otimes \mid P\rangle$, is possible under LOCC with certainty, if we take $\mid \psi\rangle$, as a 2×2 state with Schmidt coefficients $(\frac{d-1}{d}, \frac{1}{d})$.

Corollary-2 immediately suggests that it is possible to achieve a maximally entangled state of any Schmidt rank d, $d \geq 3$ by using a finite number of 2×2 states only.

Corollary-3. The transformation,

$\mid \psi_1\rangle \otimes \mid \psi_2\rangle \otimes \cdots \mid \psi_{d-1}\rangle \rightarrow \mid \Psi^d_{max}\rangle \otimes \mid P\rangle$,

is possible under LOCC with certainty, where $\mid \psi_i\rangle$, $\forall i = 1, 2, \cdots, d-1$ are 2×2 states with Schmidt coefficients $(\frac{d-i}{d-i+1}, \frac{1}{d-i+1})$, respectively.

4 Mutual Co-operation

In this section our main goal is to provide an auxiliary incomparable pair so that the collective operation enables us to find the desired states; i.e., given a pair $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ we want to find an auxiliary pair $\mid \chi\rangle \not\leftrightarrow \mid \eta\rangle$ such that $\mid \psi\rangle \otimes \mid \chi\rangle \rightarrow \mid \phi\rangle \otimes \mid \eta\rangle$, is possible under LOCC deterministically. There are several ways to find nontrivial $(\mid \chi\rangle, \mid \eta\rangle)$. We first provide some examples that will show such features and then in two subsections we shall give analytical results for 3×3 system of incomparable states. We explicitly provide the form of the auxiliary pair for all possible incomparable pair $(\mid \psi\rangle, \mid \phi\rangle)$ in 3×3 system. One of the interesting feature of such incomparable pairs is that we are unable to say that which state has greater entanglement than the other. So in this way we may resolve the incomparability of $(\mid \psi\rangle, \mid \phi\rangle)$ with $E(\mid \psi\rangle) < E(\mid \phi\rangle)$ by mutual co-operation which obviously claims that $E(\mid \chi\rangle) > E(\mid \eta\rangle)$. Other interesting part we have studied analytically in 3×3 system is the following :

From two copy of a pure entangled state we are able to find two different pure entangled states, both of which are incomparable with the source state. Let us begin with an example of mutual co-operation.

Example 1.– Consider a pair of pure entangled states of the form

$$\mid \psi\rangle = \sqrt{0.4} \mid 00\rangle + \sqrt{0.4} \mid 11\rangle + \sqrt{0.2} \mid 22\rangle,$$

$$\mid \phi\rangle = \sqrt{0.48} \mid 00\rangle + \sqrt{0.26} \mid 11\rangle + \sqrt{0.26} \mid 22\rangle,$$

$$\mid \chi\rangle = \sqrt{0.49} \mid 33\rangle + \sqrt{0.255} \mid 44\rangle + \sqrt{0.255} \mid 55\rangle,$$

$$\mid \eta\rangle = \sqrt{0.41} \mid 33\rangle + \sqrt{0.41} \mid 44\rangle + \sqrt{0.18} \mid 55\rangle.$$

It is easy to check that $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ and $\mid \chi\rangle \not\leftrightarrow \mid \eta\rangle$; whereas, $E(\mid \psi\rangle) \approx 1.5219 > E(\mid \phi\rangle) \approx 1.5188, E(\mid \chi\rangle) \approx 1.5097 > E(\mid \eta\rangle) \approx 1.5001$; and if we allow collective operations locally on the joint systems, then the transformation $\mid \psi\rangle \otimes \mid \chi\rangle \rightarrow \mid \phi\rangle \otimes \mid \eta\rangle$ is possible with certainty, i.e., we see that the two pairs which are incomparable, will co-operate with each other and make the joint transformation possible.

If we looked upon the whole thing in a little more physically then something more comes out. We see here that the comparability of the joint operation actually comes through the co-operation with the comparable class, i.e., this four states are related in such a way that $\mid \psi\rangle \rightarrow \mid \eta\rangle$ and $\mid \phi\rangle \rightarrow \mid \chi\rangle$. So here we reduce the incomparability of two states by choosing some class of states comparable with them. It is obvious that such a pair of states always exist for any incomparable pair, i.e., incomparable pairs can always be made to compare. Without going into details of the proof, we state that this approach resolves the incomparability of the

3×3 states. At this moment someone may think that this result imply that only with the help of comparable classes we destroy the incomparability. Obviously, the answer is in the negative. The next example is given in support of this.

Example 2.– Consider two pairs of pure entangled states $(\mid \psi\rangle, \mid \phi\rangle)$ and $(\mid \chi\rangle, \mid \eta\rangle)$ of the form

$$\mid \psi\rangle = \sqrt{0.41} \mid 00\rangle + \sqrt{0.38} \mid 11\rangle + \sqrt{0.21} \mid 22\rangle,$$

$$\mid \phi\rangle = \sqrt{0.4} \mid 00\rangle + \sqrt{0.4} \mid 11\rangle + \sqrt{0.2} \mid 22\rangle,$$

$$\mid \chi\rangle = \sqrt{0.45} \mid 33\rangle + \sqrt{0.34} \mid 44\rangle + \sqrt{0.21} \mid 55\rangle,$$

$$\mid \eta\rangle = \sqrt{0.48} \mid 33\rangle + \sqrt{0.309} \mid 44\rangle + \sqrt{0.211} \mid 55\rangle.$$

It is quite surprising to see that not only $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ and $\mid \chi\rangle \not\leftrightarrow \mid \eta\rangle$ but also $\mid \psi\rangle \not\leftrightarrow \mid \eta\rangle$, $\mid \chi\rangle \not\leftrightarrow \mid \phi\rangle$. Beside this we also get the extra facility to prepare $|\chi\rangle$ from $|\psi\rangle$ as $\mid \psi\rangle \rightarrow \mid \chi\rangle$. From the informative point of view the picture is although, $E(\mid \psi\rangle) \approx 1.5307 > E(\mid \phi\rangle) \approx 1.5219$, and $E(\mid \chi\rangle) \approx 1.5204 > E(\mid \eta\rangle) \approx 1.50544$, but still independently we can not convert $\mid \psi\rangle$ to either one of $\mid \phi\rangle$ or $\mid \eta\rangle$ and also $\mid \chi\rangle$ to either one of $\mid \phi\rangle$ or $\mid \eta\rangle$ with certainty under LOCC. Therefore although the resource states have greater information content, the individual pairs aren't convertible, but treating them together we break their incomparability. Here we didn't fix our eyes only on the transformation of the first pair and recover as much as possible amount of entanglement from second pair rather we have tried to reduce the incomparability of both the two pairs of states together.

To give rise the fact that mutual co-operation also exists in other dimensions, we are providing other two sets of incomparable pairs in 4×4 system which are strongly incomparable so that deterministic local conversions are not possible by assisting also catalytic states and 2×2 mutual catalytic states but co-operate each to make the joint transformation possible.

Example 3.– Consider two pairs of pure entangled states $(\mid \psi\rangle, \mid \phi\rangle)$ and $(\mid \chi\rangle, \mid \eta\rangle)$ of the form

$$\mid \psi\rangle = \sqrt{0.4} \mid 00\rangle + \sqrt{0.3} \mid 11\rangle + \sqrt{0.2} \mid 22\rangle + \sqrt{0.1} \mid 33\rangle,$$

$$\mid \phi\rangle = \sqrt{0.45} \mid 00\rangle + \sqrt{0.29} \mid 11\rangle + \sqrt{0.14} \mid 22\rangle + \sqrt{0.12} \mid 33\rangle,$$

$$\mid \chi\rangle = \sqrt{0.5} \mid 44\rangle + \sqrt{0.25} \mid 55\rangle + \sqrt{0.2} \mid 66\rangle + \sqrt{0.05} \mid 77\rangle,$$

$$\mid \eta\rangle = \sqrt{0.48} \mid 44\rangle + \sqrt{0.36} \mid 55\rangle + \sqrt{0.12} \mid 66\rangle + \sqrt{0.04} \mid 77\rangle.$$

It is easy to check that $\mid \psi\rangle \not\leftrightarrow \mid \phi\rangle$ and $\mid \chi\rangle \not\leftrightarrow \mid \eta\rangle$, and $E(\mid \psi\rangle) \approx 1.846 > E(\mid \phi\rangle) \approx 1.800$, $E(\mid \chi\rangle) \approx 1.680 > E(\mid \eta\rangle) \approx 1.592$. However, one may check $\mid \psi\rangle\otimes \mid \chi\rangle \longrightarrow \mid \phi\rangle\otimes \mid \eta\rangle$, is possible under LOCC.

Example 4.– Consider two pairs of pure entangled states $(\mid \psi\rangle, \mid \phi\rangle)$ and $(\mid \chi\rangle, \mid \eta\rangle)$ of the form

$$\mid \psi\rangle = \sqrt{0.4} \mid 00\rangle + \sqrt{0.3} \mid 11\rangle + \sqrt{0.2} \mid 22\rangle + \sqrt{0.1} \mid 33\rangle,$$

$$| \phi\rangle = \sqrt{0.45} \mid 00\rangle + \sqrt{0.29} \mid 11\rangle + \sqrt{0.14} \mid 22\rangle + \sqrt{0.12} \mid 33\rangle,$$

$$| \chi\rangle = \sqrt{0.5} \mid 44\rangle + \sqrt{0.23} \mid 55\rangle + \sqrt{0.22} \mid 66\rangle + \sqrt{0.05} \mid 77\rangle,$$

$$| \eta\rangle = \sqrt{0.48} \mid 44\rangle + \sqrt{0.36} \mid 55\rangle + \sqrt{0.12} \mid 66\rangle + \sqrt{0.04} \mid 77\rangle.$$

Here also it is easy to verify that $| \psi\rangle \not\rightarrow | \phi\rangle$, $| \psi\rangle \not\rightarrow | \eta\rangle$, $| \chi\rangle \not\rightarrow | \phi\rangle$ and $| \chi\rangle \not\rightarrow | \eta\rangle$. But surprisingly $| \psi\rangle \rightarrow | \chi\rangle$. Now it is very interesting that we can prepare the state of co-operation from the state in our hand. The relations between the entanglement of those states are, $E(| \psi\rangle) \approx 1.846 > E(| \phi\rangle) \approx 1.800$, and $E(| \chi\rangle) \approx 1.684 > E(| \eta\rangle) \approx 1.592$, and $| \psi\rangle \otimes | \chi\rangle \longrightarrow | \phi\rangle \otimes | \eta\rangle$, is possible under LOCC with certainty. All the examples we are providing are non-trivial one. Next we show some analytical results for 3×3 system of incomparable states.

4.1 Local conversion of 3×3 *incomparable pairs by auxiliary* 3×3 *incomparable pairs*

Now we concentrate to the case of incomparable pairs in 3×3 system of states. We shall show for every pair of incomparable pure entangled states $(| \psi_1\rangle, | \phi_1\rangle)$ there is always a pair of incomparable pure entangled states $(| \psi_2\rangle, | \phi_2\rangle)$ such that $| \psi_1\rangle \otimes | \psi_2\rangle \longrightarrow | \phi_1\rangle \otimes | \phi_2\rangle$, is possible under LOCC with certainty. The main idea of this portion is, assuming $| \psi_1\rangle$ as the source state and $| \phi_1\rangle$ as the target state, we choose the nontrivial auxiliary incomparable pair $(| \psi_2\rangle, | \phi_2\rangle)$ such that by collective LOCC the joint transformation of both pairs is possible with certainty.

Consider, $| \psi_1\rangle \equiv (a_1, a_2, a_3), | \phi_1\rangle \equiv (b_1, b_2, b_3)$ where $a_1 \geq a_2 \geq a_3 \geq 0, a_1 + a_2 + a_3 = 1, b_1 \geq b_2 \geq b_3 \geq 0, b_1 + b_2 + b_3 = 1$. There are two possible cases of incomparability that exist in this dimension, which are discussed and treated differently below.

Case-1: $a_1 > b_1, a_1 + a_2 < b_1 + b_2$. We choose $| \psi_2\rangle \equiv (\beta_1, \beta_1, \beta_2), | \phi_2\rangle \equiv (\alpha_1, \alpha_2, \alpha_2)$ where $\beta_1 > \beta_2 > 0, 2\beta_1 + \beta_2 = 1, \alpha_1 > \alpha_2 > 0, \alpha_1 + 2\alpha_2 = 1, \beta_1 < \alpha_1, 2\beta_1 > \alpha_1 + \alpha_2$, such that

$$\max\{\frac{a_1}{a_2}, \frac{b_1}{b_3}\} < \frac{\alpha_1}{\alpha_2} \tag{4}$$

and

$$\frac{a_3}{(2a_1 + a_3)} > \beta_2 > \max\{\frac{\alpha_2 b_3}{a_3}, \alpha_2(b_2 + 2b_3), \frac{\alpha_2(2 - b_1) - a_3}{(1 - a_3)}, 1 - \frac{\alpha_1(b_1 + b_2)}{a_1}\} \tag{5}$$

Under such a choice the required joint transformation is always possible.

In the above process there may arise a similar condition like our first example. For this type of choice we have always $| \psi_1\rangle \rightarrow | \phi_2\rangle$. Except this choice we further require that those cross pairs $(| \psi_1\rangle, | \phi_2\rangle)$ or $(| \psi_2\rangle, | \phi_1\rangle)$, remain incomparable too. To fulfill this requirement the state $| \psi_2\rangle$ is chosen slight differently, as $| \psi_2\rangle \equiv (\beta_1, \beta_2, \beta_3)$, where $\beta_1 > \beta_2 > \beta_3 > 0, \beta_1 + \beta_2 + \beta_3 = 1$, such that $a_1\beta_3 > \beta_1 a_3 > b_1\alpha_2$, $a_1\beta_1 < b_1\alpha_1, a_3\beta_3 > b_3\alpha_2$ and $\{(\beta_1 a_3 - a_2\beta_3) - (a_3 - b_3)\} < \min\{0, (\alpha_1 b_3 - \alpha_2 b_2), (a_2\beta_2 - \alpha_2 b_2)\}$. After such a choice the pair $(| \psi_2\rangle, | \phi_1\rangle)$ became incomparable except when $a_2 = a_3$. But whenever we face the case $b_1 = b_2$ and $a_2 = a_3$ then correspondingly we see that $| \psi_1\rangle \rightarrow | \phi_2\rangle$ and $| \psi_2\rangle \rightarrow | \phi_1\rangle$.

Case-2: $a_1 < b_1, a_1 + a_2 > b_1 + b_2$. In this case we choose $|\psi_2\rangle \equiv (\beta_1, \beta_2, \beta_3), |\phi_2\rangle \equiv (\alpha_1, \alpha_1, \alpha_2)$ where $\beta_1 > \beta_2 > \beta_3 > 0, \beta_1 + \beta_2 + \beta_3 = 1, \alpha_1 > \alpha_2 > 0, 2\alpha_1 + \alpha_2 = 1, \beta_1 > \alpha_1, \beta_1 + \beta_2 > 2\alpha_1$. Now consider two subcases.

Firstly, when $a_1 < \frac{1}{2}$, we choose the state $(|\psi_2\rangle, |\phi_2\rangle)$ in such a way that $\alpha_1 b_3 > a_1 \beta_3 > \beta_1 a_3$ and

$$\alpha_1 > \max\{\frac{\beta_1 a_1}{b_1}, \frac{\beta_1(a_1 + a_2) + a_1\beta_2}{2b_1 + b_2}, \frac{(1-\beta_3)(1-a_3)}{2(1-b_3)}\} \tag{6}$$

Secondly, when $a_1 \geq \frac{1}{2}$, we choose the state $(|\psi_2\rangle, |\phi_2\rangle)$ in such a way that $\beta_1 = \frac{1}{2}$ and $\alpha_1 b_3 > a_1\beta_3 > \beta_1 a_3$,

$$\alpha_1 > \max\{\frac{a_1}{2b_1}, \frac{a_1 + a_2 + 2a_1\beta_2}{2(2b_1 + b_2)}, \frac{(0.5 + \beta_2)(1 - a_3)}{2(1 - b_3)}, \frac{2a_1 + a_2}{4(b_1 + b_2)}, \frac{a_1 + a_2 - a_2\beta_3}{2 - b_3}\} \tag{7}$$

It is interesting to note that in the first subcase when $a_1 = a_2$ then $|\psi_1\rangle \rightarrow |\phi_2\rangle$. Except this, our choice maintains $|\psi_i\rangle \not\leftrightarrow |\phi_j\rangle, \ \forall i, j = 1, 2$.

4.2 Two incomparability with the same initial state may be broken jointly

At the beginning of this section we want to present the special result for 3×3 system which is as follows :

For any source state $|\psi\rangle$ in 3×3 system, with distinct Schmidt coefficients there always exist two states $(|\chi\rangle, |\eta\rangle)$ such that both of them are incomparable with $|\psi\rangle$ but from two copy of $|\psi\rangle$ we are able to get them by collective LOCC with certainty.

Suppose the source state is $|\psi\rangle \equiv (a_1, a_2, a_3)$ with $a_1 > a_2 > a_3 > 0, \ a_1 + a_2 + a_3 = 1$ and the other states are $|\chi\rangle \equiv (b_1, b_2, b_3)$ and $|\eta\rangle \equiv (c_1, c_2, c_3)$ with $b_1 > b_2 > b_3 > 0$, $b_1 + b_2 + b_3 = 1$ and $c_1 > c_2 > c_3 > 0$, $c_1 + c_2 + c_3 = 1$. We first impose the incomparability conditions as $a_1 > b_1, a_1 + a_2 < b_1 + b_2$ and $a_1 < c_1, a_1 + a_2 > c_1 + c_2$. Then it follows from Nielsen's condition that there is always a possible range of $(|\chi\rangle, |\eta\rangle)$ such that $|\psi\rangle^{\otimes 2} \longrightarrow |\chi\rangle \otimes |\eta\rangle$, under LOCC with certainty. It should be noted that the cases of failure of this general result is only the small number of cases where irrespective of the incomparability condition, the Schmidt coefficients of the source state are not all distinct, i.e., either $a_1 = a_2$ or $a_2 = a_3$.

This result is very important because we must keep in our mind the fact, that multiple copy transformation is not possible for states in 3×3 system. Now with this result in our hand, let us try to fix $|\chi\rangle$ as our target state and find the possible range (if exists at all) of $|\eta\rangle$; i.e., in a quite general sense we assume that there is two copy of the source state $|\psi\rangle$ in our hand, where $|\psi\rangle \not\leftrightarrow |\chi\rangle$. Our aim is to find a $|\eta\rangle$; such that $|\psi\rangle \not\leftrightarrow |\eta\rangle$, and to make possible the joint transformation $|\psi\rangle^{\otimes 2} \longrightarrow |\chi\rangle \otimes |\eta\rangle$, under LOCC with certainty.

Like the previous section here also we have two cases of incomparability.

Case-1: When $a_1 > b_1, a_1 + a_2 < b_1 + b_2$, we take $|\eta\rangle \equiv (\alpha_1, \alpha_2, \alpha_2)$ where $\alpha_1 > \alpha_2 > 0, \alpha_1 + 2\alpha_2 = 1, a_1 < \alpha_1, a_1 + a_2 > \alpha_1 + \alpha_2$. Then the condition for such transformation is, $a_3 < \frac{1}{2}(1 - \frac{{a_1}^2}{b_1})$. Under this condition we have not only one $|\eta\rangle$, but a range of it specified either by the relation

$$\alpha_2 < \min\{\frac{a_1 a_3}{b_1}, \frac{a_3^2}{b_3}, \frac{a_3(2a_2 + a_3)}{(b_2 + 2b_3)}\}, \text{ for } {a_2}^2 > a_1 a_3, \tag{8}$$

or by the relation

$$\alpha_2 < \min\{a_3 + \frac{{a_2}^2 - {a_3}^2}{2}, \frac{a_3^2}{b_3}, \frac{a_3(2a_2 + a_3)}{(b_2 + 2b_3)}\}, \text{ for } {a_2}^2 < a_1 a_3 \tag{9}$$

Case-2: $a_1 < b_1, a_1 + a_2 > b_1 + b_2$. Here we take $|\,\eta\rangle \equiv (\alpha_1, \alpha_1, \alpha_2)$ where $\alpha_1 > \alpha_2 > 0, 2\alpha_1 + \alpha_2 = 1, a_1 > \alpha_1, a_1 + a_2 < 2\alpha_1$. Consider two subcases separately.

Firstly, when ${a_2}^2 > a_1 a_3$ then such a joint transformation occurs if; $\frac{(a_1+a_2)^2}{2(b_1+b_2)} < a_1$ and the range of $|\,\eta\rangle$ is specified by the relation

$$\alpha_1 > \max\{a_1 - \frac{{a_1}^2 - {a_2}^2}{2}, \frac{(a_1 + a_2)^2}{2(b_1 + b_2)}, \frac{(a_1)^2}{b_1}, \frac{a_1(a_1 + 2a_2)}{(2b_1 + b_2)}\} \tag{10}$$

Next, when ${a_2}^2 < a_1 a_3$ then the condition for such transformation is; $a_1 + 2a_2 < 2b_1 + b_2$. Under this condition range of $|\,\eta\rangle$ is specified by the relation

$$\alpha_1 > \max\{a_1 - \frac{{a_1}^2 - {a_2}^2}{2}, \frac{a_1(2 - a_1)}{(2 - b_3)}, \frac{(a_1)^2}{b_1}, \frac{a_1(a_1 + 2a_2)}{(2b_1 + b_2)}\} \tag{11}$$

Finally we must mention that this process works for most of the cases of incomparability. But, it is not always successful; i.e., choosing any arbitrary incomparable pair we might not be able to reach the target state by this method. This small range of failure of the process is possibly due to the fact that we didn't ever bother about the amount of entanglement contained into the states. It is possible that $E(|\,\psi\rangle) \ll E(|\,\chi\rangle)$; for which there doesn't exists such a state $|\,\eta\rangle$, incomparable with $|\,\psi\rangle$ and $E(|\,\psi\rangle^{\otimes 2}) > E(|\,\chi\rangle \otimes |\,\eta\rangle)$.

In conclusion we have succeeded in providing a method by which any incomparable pair of pure bipartite entangled states in any finite dimension, can be maid to compare(i.e., transform one to another), under LOCC with certainty, by providing some pure entanglement. We observed that mutual co-operation is an useful process to break the incomparability of two pairs under LOCC. This is not only discussed as an abstract or rather complicated theory, but we provide the algorithmic structure by which this goal can be really achieved. This work supports the possibility of reaching any pure bipartite entangled states by deterministic LOCC.

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