

ENTANGLEMENT TENSOR FOR A GENERAL PURE MULTIPARTITE QUANTUM STATE

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We propose an entanglement tensor to quantitatively compute the entanglement of a general pure multipartite quantum state. We compare the ensuing tensor with the concurrence for bipartite state and apply the tensor measure to some interesting examples of entangled three-qubit and four-qubit states. It is shown that in defining the degree of entanglement of a multi-partite state, one needs to make assumptions about the willingness of the parties to cooperate. For such states our tensor becomes a measure of generalized entanglement of assistance. We also discuss the degree of entanglement and the concurrence of assistance of two generic multi-qubit states.

Keywords: quantum entanglement, multipartite state, concurrence

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1 Introduction

Quantum theory is a fundamental theory that can describe the subatomic world with a fascinating accuracy. Since 1935, quantum entanglement [1, 2] has been central for the understanding of the foundations of quantum theory. Besides, its fundamental interest, entanglement has become an essential resource for quantum communication applications created in recent years, which have potential applications such as quantum cryptography [3, 4] and quantum teleportation [5]. One widely used measure of entanglement for a pair of qubits is the concurrence, that gives an analytic formula for the entanglement of formation [6, 7, 8]. In recent years, there have been proposals to generalize this measure to general bipartite states, e.g., Uhlmann [9] has generalized the concept of concurrence by considering arbitrary conjugation, then Audenaert, Verstraete, and De Moor [10], in the spirit of Uhlmann's work, generalized the measure by defining a concurrence vector for pure states. Another generalization of concurrence has been done by Rungta *et al.* [11] based on the idea of a super operator called universal state inversion. Moreover, Gerjuoy [12], Albeverio and Fei [13], and Miyake and

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Wadati [14, 15], gave an explicit expression of generalized concurrence in terms of the coefficients of a general, pure, bipartite state. It is, therefore, interesting to be able to generalize this measure from bipartite to multipartite systems [16, 17]. Quantifying entanglement of multipartite states [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], is a complicated task. In [32, 33, 34] we proposed a measure of entanglement for a general pure multipartite state. In this paper we will take one step further in generalizing of concurrence to multipartite states by giving an explicit formula for this measure and verifying the well established result for bipartite states. Then, we give some results related to the simplest example of a multipartite state, namely a tripartite state. Moreover, we give some examples of four-partite states in such way as to show the idea behind our measure. Finally, we will discuss Dür *et al's* conjecture [28], and Koashi *et al's* subsequent proof [35], about the general multi-qubit $|W_M\rangle$ state.

The concurrence of a two qubit, bipartite state is defined as $\mathcal{C}(\Psi) = |\langle \Psi | \tilde{\Psi} \rangle|$, where the tilde represents the "spin-flip" operation $|\tilde{\Psi}\rangle = \sigma_y \otimes \sigma_y |\Psi^*\rangle$, $|\Psi^*\rangle = \sum_{l,k=1}^2 \alpha_{k,l}^* |k, l\rangle$ is the complex conjugate of $|\Psi\rangle = \sum_{l,k=1}^2 \alpha_{k,l} |k, l\rangle$, and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is a Pauli spin-flip operator. It was noted by Peres [36] that for a separable state, $|\tilde{\Psi}\rangle$ is orthogonal to $|\Psi\rangle$, whereas for any Bell-state, the states are parallel. The concurrence can also be written as follows

$$\mathcal{C}(\Psi) = |\langle \Psi | \tilde{\Psi} \rangle| = |\text{Tr}(\sigma_y \otimes \sigma_y |\Psi\rangle \langle \Psi^*|)|. \quad (1)$$

In the following section we will introduce an entanglement tensor, of a form similar to concurrence. For a bipartite qubit state, our measure coincides with the concurrence, or by the generalization of concurrence developed in [12, 13, 14]. In contrast, e.g., the bipartite entanglement of a tripartite state depends on what action the third party takes [37, 38, 39]. It is possible to find states where the third party can vary the remaining two parties' state continuously between a separable state and a maximally entangled state. Only the M -partite entanglement of an M -partite state can be defined unambiguously.

2 Entanglement tensor for general pure multipartite quantum state

In this section, we will give an expression for the entanglement of a general pure multipartite state. The derivation of the measure is tedious, and follows almost exactly that of our measure based on the density matrix of a pure state [34]. Therefore, it will not be repeated here. It suffices to point out that the mathematical derivation of the measure is based on the relative-phase correlations between a quantum system's various sub-systems.

Let

$$|\Psi\rangle = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, k_2, \dots, k_m} |k_1, k_2, \dots, k_m\rangle, \quad (2)$$

be a general pure state defined on the Hilbert space $\mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$. To simplify subsequent notation we introduce projection probabilities by projecting the state $|\Psi\rangle$ onto the basis states in one or more of the subspaces $\mathcal{H}_{\mathcal{Q}_r}$ and computing the norm of the projection. E.g., reducing the j :th subspace, we get the probabilities

$$p_{k_j} = \langle \Psi | k_j \rangle \langle k_j | \Psi \rangle. \quad (3)$$

In the same vein, the projection probabilities if we project onto the j :th and r :th subspace are

$$p_{k_j, k_r} = \langle \Psi | k_j, k_r \rangle \langle k_j, k_r | \Psi \rangle. \quad (4)$$

We also need an index permutation operator P_j operating on the state coefficient product $\alpha_{k_1, k_2, \dots, k_j, \dots, k_m} \alpha_{l_1, l_2, \dots, l_j, \dots, l_m}$ as follows:

$$P_j(\alpha_{k_1, k_2, \dots, k_j, \dots, k_m} \alpha_{l_1, l_2, \dots, l_j, \dots, l_m}) = \alpha_{k_1, k_2, \dots, k_j, \dots, k_m} \alpha_{l_1, l_2, \dots, l_j, \dots, l_m} - \alpha_{k_1, k_2, \dots, l_j, \dots, k_m} \alpha_{l_1, l_2, \dots, k_j, \dots, l_m}. \quad (5)$$

In an M -partite state, there are many ways to share entanglement. There are e.g. $M(M-1)/2$ different kinds of bipartite entanglement, entanglement that can be shared between parties 1 and 2, 1 and 3, et.c. until parties $M-1$ and M . In general, there are

$$\binom{M}{D} = \frac{M!}{D!(M-D)!} \quad (6)$$

different kinds of D -partite entanglement in an M -partite state, where $M \geq D$. Each of these components have an associated entanglement tensor coefficient. Using our permutation operator above, we can define a D -partite tensor coefficient $c_{r, \dots, z}$, containing information about the entanglement between the D parties r, \dots, z , where parties r, \dots, z can be chosen any way among the M , as

$$c_{r, \dots, z} = \left(\mathcal{N}_D \sum_{k_1=1}^{N_1} \cdots \sum_{k_{r-1}=1}^{N_{r-1}} \sum_{k_{z+1}=1}^{N_{z+1}} \cdots \sum_{k_M=1}^{N_M} (p_{k_1, \dots, k_{r-1}, k_{z+1}, \dots, k_M})^{-1} \cdot \sum_{l_r > k_r}^{N_r-1} \sum_{k_r=0}^{N_r-1} \cdots \sum_{l_z > k_z}^{N_z-1} \sum_{k_z=0}^{N_z-1} |P_{r+1} P_{r+2} \cdots P_z(\alpha_{k_1, k_2, \dots, k_{M-1}, k_M} \alpha_{k_1, \dots, k_{r-1}, l_r, \dots, l_z, k_{z+1}, k_M})|^2 \cdots \right)^{1/2}. \quad (7)$$

Assume that we have a state where subsystem j is separable from all other subsystems. In such a case, it holds that $\alpha_{k_1, k_2, \dots, k_j, \dots, k_m} \alpha_{l_1, l_2, \dots, l_j, \dots, l_m} = \alpha_{k_1, k_2, \dots, l_j, \dots, k_m} \alpha_{l_1, l_2, \dots, k_j, \dots, l_m}$. That is, every entanglement tensor component involving the entanglement between subsystem j and any other subsystem(s) is identically zero. Hence, separability of any subsystem can be detected directly by looking at all entanglement tensor components associated with a certain subsystem. Note that one needs to look through all different kinds of entanglement (bipartite, tripartite, etc.) to ensure separability. We also see that the expression for $c_{r, \dots, z}$ is independent of local phase-transformations, e.g. transformations of the type

$$\sum_{k_j=1}^{N_j} e^{i\phi_{k_j}} |k_j\rangle \langle k_j|, \quad (8)$$

where ϕ_{k_j} are real numbers, because such a transformation will result in the same change of phase in the factors $\alpha_{k_1, k_2, \dots, k_j, \dots, k_m} \alpha_{l_1, l_2, \dots, l_j, \dots, l_m}$ and $\alpha_{k_1, k_2, \dots, l_j, \dots, k_m} \alpha_{l_1, l_2, \dots, k_j, \dots, l_m}$.

3 Concurrence for bipartite quantum states

As we have already mentioned, there has been considerable progress to generalize concurrence for bipartite states in arbitrary dimensions [12, 13, 14, 15, 16, 17, 34]. As our first example, we show that our entanglement tensor component (there is only one component for a bipartite state) coincide with the well established formula for the generalized concurrence of a bipartite state. Let $|\Psi\rangle = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \alpha_{k_1, k_2} |k_1, k_2\rangle$ be a general pure state defined on a bipartite Hilbert space $\mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}$. Then, the bipartite entanglement tensor component of the state is given by

$$c_{12} = \left(\mathcal{N}_2 \sum_{l_1 > k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2 > k_2}^{N_2} \sum_{k_2=1}^{N_2} |\alpha_{k_1, k_2} \alpha_{l_1, l_2} - \alpha_{k_1, l_2} \alpha_{l_1, k_2}|^2 \right)^{\frac{1}{2}}, \quad (9)$$

where, if we choose the normalization constant $\mathcal{N}_2 = 4$, that is, a normalization constant based on setting the entanglement of an EPR-pair to unity, we get identically the concurrence of the state [12, 13, 14, 16, 17]. In particular, for a pair of qubits [8], we have

$$c_{12} = \mathcal{N}_2 |\alpha_{1,1} \alpha_{2,2} - \alpha_{1,2} \alpha_{2,1}|. \quad (10)$$

The component is independent of any unitary operations, local to subsystems 1 and 2.

4 Entanglement of tripartite quantum states

The first step towards the more complex states goes through the tripartite state, which is the “simplest” state that can be called a multipartite state.

Let $|\Psi\rangle = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \sum_{k_3=0}^{N_3-1} \alpha_{k_1, k_2, k_3} |k_1, k_2, k_3\rangle$ be a general pure state. This state has three bipartite entanglement tensor components and one tripartite tensor component. They are:

$$c_{12} = \left(\mathcal{N}_2 \sum_{k_3=1}^{N_3} p_{k_3}^{-1} \sum_{l_1 > k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2 > k_2}^{N_2} \sum_{k_2=1}^{N_2} |\alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, k_3} - \alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, k_3}|^2 \right)^{1/2}, \quad (11)$$

$$c_{13} = \left(\mathcal{N}_2 \sum_{k_2=1}^{N_2} p_{k_2}^{-1} \sum_{l_1 > k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_3 > k_3}^{N_3} \sum_{k_3=1}^{N_3} |\alpha_{k_1, k_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, k_2, l_3} \alpha_{l_1, k_2, k_3}|^2 \right)^{1/2}, \quad (12)$$

$$c_{23} = \left(\mathcal{N}_2 \sum_{k_1=1}^{N_1} p_{k_1}^{-1} \sum_{l_2 > k_2}^{N_2} \sum_{k_2=1}^{N_2} \sum_{l_3 > k_3}^{N_3} \sum_{k_3=1}^{N_3} |\alpha_{k_1, k_2, k_3} \alpha_{k_1, l_2, l_3} - \alpha_{k_1, k_2, l_3} \alpha_{k_1, l_2, k_3}|^2 \right)^{1/2}, \quad (13)$$

and

$$c_{123} = \left(\mathcal{N}_3 \sum_{l_1 > k_1}^{N_1-1} \sum_{k_1=0}^{N_1-1} \sum_{l_2 > k_2}^{N_2-1} \sum_{k_2=0}^{N_2-1} \sum_{l_3 > k_3}^{N_3-1} \sum_{k_3=0}^{N_3-1} \{ |\alpha_{k_1, k_2, k_3} \alpha_{l_1, l_2, l_3} - \alpha_{k_1, k_2, l_3} \alpha_{l_1, l_2, k_3}|^2 \right. \\ \left. - |\alpha_{k_1, l_2, k_3} \alpha_{l_1, k_2, l_3} - \alpha_{k_1, l_2, l_3} \alpha_{l_1, k_2, k_3}|^2 \} \right)^{1/2}. \quad (14)$$

In this case, the bipartite tensor components are, in general, dependent of local unitary transformations (except for local phase shifts). The entanglement of multipartite states depends, in general, both on local operations and on whether or not the parties choose to cooperate.

That is, the local operations one party chooses to perform on his subsystem, and the extent to which he chooses to communicate his result, determines the entanglement of the remaining state. This has been called hidden entanglement by Cohen [37] and entanglement of assistance by DiVincenzo *et al.* [38]. Laustsen *et al.* separate between two kinds of entanglement of assistance, denoted “sharp” and “flat” [39], where the former kind includes joint measurements on subsystems, whereas the latter only includes local measurements on each subsystem. Our entanglement tensor essentially represents flat entanglement of assistance, but, as we shall see below, in a certain respect it goes a little bit further than the work in [39].

A necessary requirement for an entanglement measure is its monotonicity under local operations and classical communication. The measure should not increase under such transformations. As discussed in, e.g., [37, 38, 39], if one makes a local measurement on a multipartite state, both the amount and the form of the entanglement may be changed. In the preceding paragraph we observed that our entanglement tensor component as given by (8), is not monotonic under local transformations. Hence, the entanglement tensor must be defined as the supremum of (8) under all unitary transformations. If this is not done, then our bipartite entanglement tensor components will show the structure of the state’s associated entanglement graph [40, 41]. That is, for a given pure state it is possible to immediately see what subsystems have bipartite entanglement. Entanglement cannot be shared freely between any of M parties [42]. It is therefore interesting to find states that share entanglement in ways represented by the different topological M -partite entanglement graphs. Our tensor can assist in the analysis of pure systems, but is probably less suited for the synthesis of states associated with a certain graph.

To complicate the issue of multipartite entanglement, there is an intrinsic problem with an optimization over local operations. It is well known that, e.g., tripartite entanglement may be transformed into bipartite entanglement and vice versa. Neither transformation is reversible. One can get a maximum of one EPR-state per initial GHZ state. At the same time, in the limit of many EPR-states, we can only obtain 2 GHZ-states from 3 EPR-states [25]. The optimal conversion rates between most tripartite and higher-partite states are still unknown. Before such conversion rates are known, (and a classification of the irreversible sets of states is done [28, 43]) it is not possible to give appropriate weights to the tripartite, fourpartite, etc. tensor components. This implies that until then, it is only possible to find the supremum of our entanglement measure for each kind of entanglement separately [34]. This precludes a unique entanglement quantification of, e.g. the state

$$(|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle + |1, 0, 0\rangle)/2, \quad (15)$$

a state that contains both bipartite and tripartite entanglement and that cannot be converted by invertible local operators neither to a W-state nor to a GHZ-state [43]. In this respect, however, our tensor goes further than the entanglement of assistance, that is defined as the maximum remaining bipartite entanglement in a state after the other parties have cooperated to “concentrate” as much entanglement as possible to the remaining two subsystems [37, 38, 39]. Our measure also quantifies the “entanglement of assistance” for three, four, etc., remaining subsystems. The problem, as just discussed, is that the different kinds of remaining entanglement cannot be directly compared for the lack of an operational measure of multipartite entanglement.

Let us first study the W-state $|W_3\rangle$ that is given by

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle). \quad (16)$$

The tripartite entanglement tensor component c_{123} of this state is zero, and it can be shown that it remains zero under all local transformations. Each of the state's three bipartite tensor components' supremal values are equal to

$$c_{12} = c_{13} = c_{23} = \sqrt{\frac{\mathcal{N}_2}{6}}. \quad (17)$$

The state is known for its robustness under loss of one qubit. If any of the three qubits is traced out, the ensuing mixed two-qubit state has the same average entanglement as the original pure three-qubit state.

Next, consider the GHZ-state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 1\rangle). \quad (18)$$

The bipartite tensor components of the state in this basis are all zero, whereas the tripartite tensor component attains its maximal value $c_{123} = \sqrt{\mathcal{N}_3}/2$. Now assume that a Hadamard transformation is made on the first qubit. The ensuing state becomes

$$\frac{1}{2}(|0, 0, 0\rangle + |1, 0, 0\rangle + |0, 1, 1\rangle - |1, 1, 1\rangle). \quad (19)$$

The state in this basis has $c_{123} = c_{12} = c_{13} = 0$ and $c_{23} = \sqrt{\mathcal{N}_2}/2$. The component c_{23} reaches its supremum in this basis. This result can easily be interpreted. If the leftmost qubit is measured in the computational basis, the results zero and unity will occur with equal probability, $1/2$. If one obtains the result zero, the remaining state will be in the EPR-state $(|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$. If one obtains the result unity, then the remaining state will be in the EPR-state $(|0, 0\rangle - |1, 1\rangle)/\sqrt{2}$, orthogonal to the one above. However, if the measurement result is communicated to the parties holding the remaining two qubits, either party can convert one of the EPR-states to the other using local operations (a local phase shift). Therefore, irrespective of the measurement result, the remaining state can be made to be a deterministic EPR-state, and this is what our result predicts. If, on the other hand, the measurement result is not communicated, then the ensuing bipartite, qubit mixed state is separable.

In order to use the state's symmetry to the fullest, now suppose that all three qubits of the GHZ-state are Hadamard-transformed. The ensuing state is

$$|\bar{W}\rangle = \frac{1}{2}(|0, 0, 0\rangle + |0, 1, 1\rangle + |1, 0, 1\rangle + |1, 1, 0\rangle). \quad (20)$$

This state has $c_{123} = 0$ and $c_{12} = c_{13} = c_{23} = \sqrt{\mathcal{N}_2}/2$, and this is the basis in which all three components c_{12} , c_{13} , and c_{23} simultaneously attain their suprema. In this case, measurement of the value of any of the three qubits and subsequent communication of the result will enable the parties holding the remaining two qubits to transform their state into a deterministic

EPR-state. We see that the entanglement tensor components give the entanglement of the corresponding state, provided that the parties assist each other. In this case, the entanglement of each bipartite subsystem is equal to that of a EPR-state. Hence, the average entanglement of the $|\bar{W}\rangle$ state is higher than that of the W-state, a state that is sometimes referred to as the most bipartite entangled tripartite state. The latter statement is true if one assumes that one of the qubits is simply discarded, corresponding to a trace-operation.

In earlier papers [32, 33, 34], we have defined the entanglement in a way that can be interpreted as a tensor norm. Such a crude measure has some merit. However, as only one number is obtained, a large norm does not signal whether or not the state is highly entangled (a GHZ-state being a simple example), or if the state is not highly entangled, but has entanglement “all over the place” (such as a W-state). In other words, the entanglement tensor norm gives no hint about the state’s entanglement graph. Giving all the entanglement tensor components rather than the norm of the tensor of course gives more information about the particular type of entanglement of a state, such as its graph.

5 Entanglement of four-partite quantum states

As a first example of four qubit state, let us consider the state

$$|\Psi\rangle = \frac{1}{2}(|0, 1, 1, 0\rangle + |1, 0, 0, 1\rangle + |0, 1, 1, 1\rangle + |1, 0, 0, 0\rangle). \quad (21)$$

The state has no four-partite entanglement and, in the given computational basis, it has no bipartite entanglement. The tripartite entanglement tensor components c_{124}, c_{134} , and c_{234} are all zero, while, inserting the state’s expansion coefficients in (8) we have

$$\begin{aligned} c_{123} &= (2\mathcal{N}_3[|\alpha_{1,2,2,1}\alpha_{2,1,1,1}|^2 + |\alpha_{1,2,2,2}\alpha_{2,1,1,2}|^2])^{\frac{1}{2}} \\ &= (2\mathcal{N}_3[\frac{1}{16} + \frac{1}{16}])^{\frac{1}{2}} = \sqrt{\frac{\mathcal{N}_3}{4}}. \end{aligned} \quad (22)$$

It is quite clear that this is the supremal value of this tripartite tensor component. The result can most easily be checked by writing the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0, 1, 1\rangle + |1, 0, 0\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

In this case, any local action on the rightmost qubit will not change the state’s its entanglement. However, as shown in the previous section, local actions on the remaining three qubits may transform the tripartite entanglement to various degrees of bipartite entanglement.

The state

$$|\psi\rangle = \frac{1}{2\sqrt{2}} (|[0, 0\rangle + |1, 1\rangle] \otimes |[0, 1\rangle + |1, 0\rangle] + |[0, 1\rangle + |1, 0\rangle] \otimes |[0, 0\rangle + |1, 1\rangle]), \quad (23)$$

is an example of a state that has nested entanglement. That is, the state is a (bipartite) entangled state of (bipartite) entangled states. Computing the entanglement for this state in the given basis, we find that the state has no four-partite entanglement, no tripartite entanglement, whereas all six bipartite entanglement tensor components are equal to $\sqrt{\mathcal{N}_2}/2$, indicating EPR-type entanglement. Again, assistance between the parties is needed to exploit

this entanglement. However, this state has the feature that we also can see it as a bipartite $\mathcal{H}_4 \otimes \mathcal{H}_4$ state, if each of two parties have access to two of the qubits. The bipartite $\mathcal{H}_4 \otimes \mathcal{H}_4$ entanglement of the state can also be obtained by the expression (8). In this particular case we get the supremal value $\sqrt{\mathcal{N}_2}/2$.

It is obvious that, in general, a state's entanglement depend on the chosen Hilbert space factorization of the state. Operationally, this can be stated that the entanglement of the state depends on how the state's subsystems are shared among the parties because this division defines what operations are considered to be local. This is why, in this paper, we have made a distinction between subsystems and parties.

As a last example of a four-qubit state, consider the four qubit W-state

$$|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle). \quad (24)$$

Quite expectedly, the state has no four-partite, nor any tripartite entanglement. The supremal values of the six bipartite entanglement tensor components are all equal to $\sqrt{\mathcal{N}_2}/8$. The state is robust to the loss of any two qubits, and a rather obvious analysis show that the parties need not cooperate to get this result. Note, that the state $|\psi\rangle$ in Eq. (23), above, give a substantially higher average bipartite entanglement, but only if the parties cooperate.

6 Entanglement of multi-qubit W-states

As a very last example, we would like to show how to apply the formula (8) on generic classes of multipartite states. A simple case of a multipartite state is the generalization of $|W_3\rangle$ and $|W_4\rangle$ to $|W_M\rangle$, where M signifies M qubits. This state can symbolically be written as

$$|W_M\rangle = \frac{1}{\sqrt{M}}|M-1, 1\rangle, \quad (25)$$

where $|M-1, 1\rangle$ denotes the totally symmetric superposition state including $M-1$ zeros and 1 one. The entanglement of this state is, again, very robust against particle losses, i.e., the state $|W_M\rangle$ remains entangled even if any $M-2$ parties discards, or loses, the information about their particle.

In a paper by Dür, Vidal, and Cirac, [28], it was conjectured that the average value of the square of the concurrence for $|W_M\rangle$ is given by

$$\frac{2}{M(M-1)} \sum_k \sum_{k \neq l} C_{k,l}^2(W_M) = \frac{4}{M^2}. \quad (26)$$

It was also conjectured by Dür *et al.* that the $|W_M\rangle$ -state is the state with the highest average bipartite entanglement (if all other subsystems are traced out). This conjecture was proven by Koashi *et al.* [35]. The expression for the entanglement tensor, Eq. (8), confirms this result, as all tensor components are equal to zero, except for the bipartite components that all simultaneously can have the supremal values $\sqrt{\mathcal{N}_2}/2M$. As discussed in Sec. 3, we should set $\mathcal{N}_2 = 4$ to make our tensor components equal to unity for an EPR-state. Doing so, we obtain the value $2/M$ for all of the tensor components squared. That is, the average of the components squared is also $2/M$. The interpretation of this result is rather simple. If all but two qubits of the state are lost, we stand a $2/M$ chance of having an EPR-pair and a

$(M - 2)/M$ chance of having the state $|0, 0\rangle$. From a large number N of $|W_M\rangle$ -states, we can hence statistically obtain $2N/M$ pure EPR states. Assistance among the parties holding the “traced out” $M - 2$ qubits will not help. However, as demonstrated for tri- and four-partite systems, this is not the highest achievable average concurrence of assistance for a M -partite state. This (the concurrence squared) is instead $\mathcal{N}_2/4$, or unity if \mathcal{N}_2 is set to four. (Moreover, in this case, the sharp and the flat concurrence of assistance are equal).

7 Discussion and conclusion

In conclusion we have proposed an explicit formula for an entanglement tensor of a general, pure, multipartite quantum state. To demonstrate the nature of the measure, and some of the aspects involved in entanglement classification such as generalized entanglement of assistance, we have given some example for bipartite, tripartite, four-partite, and M -partite states. In Sec. 6 we reconfirm the conjecture by Dür *et al.* about the concurrence of multi-qubit $|W_M\rangle$ -states. However, we note that a higher value of the average concurrence of a state is possible, provided that the parties assist each other. That is, the entanglement of assistance is higher than the entanglement computed from the state’s entanglement graph.

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