Quantum Information and Computation, Vol. 3, No. 4 (2003) 359–375

UNIVERSAL COMPRESSION OF ERGODIC QUANTUM SOURCES [a]

ALXEI kALTCHENKO[b] and EN-HUI YANG[c]
E&CE Department, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada

Received March 17, 2003
Revised June 23, 2003

For a real number $r > 0$, let $F(r)$ be the family of all stationary ergodic quantum sources with von Neumann entropy rates less than r. We prove that, for any $r > 0$, there exists a blind, source-independent block compression scheme which compresses every source from $F(r)$ to rn qubits per input block length n with arbitrarily high fidelity for all large n.

As our second result, we show that the stationarity and the ergodicity of a quantum source $\{\rho_m\}_{m=1}^{\infty}$ are preserved by any trace-preserving completely positive linear map of the tensor product form $\mathcal{E}^{\otimes m}$, where a copy of $\mathcal{E}$ acts locally on each spin lattice site. We also establish ergodicity criteria for so called classically-correlated quantum sources.

Keywords: Universal quantum data compression, source coding, block codes, ergodicity
Communicated by: R Jozsa & R Laflamme

1 Introduction

The quantum ergodicity is as instrumental in studying quantum information systems as is the classical ergodicity in studying classical information systems. It is quite remarkable that there are the quantum analogs of Shannon's noiseless compression theorem[d] and Shannon-McMillan theorem for stationary ergodic quantum sources[1, 2, 3]. Thus, for any such source, one can always construct a source-dependent compression code (scheme) which compresses the source to its von Neumann entropy rate with arbitrarily high fidelity.

As in classical information systems[5], the next step would be to see if there exists a compression scheme which does just the same, but is source-independent, i.e. universal. A universal scheme for the family of all i.i.d. quantum sources with a known entropy upper bound was first introduced in [6] and then extended to the family of all i.i.d sources[7, 8]. The work[7] was also concerned with the scheme's optimality and performance evaluation for every finite block length and provided an explicit exponential bound on the encoding-decoding error probability.

---

[a]This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grants RGPIN203035-98 and RGPIN203035-02, by the Premier's Research Excellence Award, by the Canada Foundation for Innovation, by the Ontario Distinguished Research Award, and by the Canada Research Chairs Program.
[b]e-mail: akaltche@bbcr.uwaterloo.ca
[c]e-mail: ehyang@bbcr.uwaterloo.ca
[d]The quantum analog for i.i.d. (independently and identically distributed) sources was first formulated and proved in [4].

In this work we study stationary and ergodic properties of quantum sources and present a universal compression scheme for the family of all stationary ergodic sources with a known upper bound on their entropy rates.

Our paper is organized as follows. In Section 2 we review the mathematical formalism and notation for stationary ergodic quantum sources. In Section 3 we show that if a stationary ergodic (weakly mixing or strongly mixing) quantum source $\{\rho_m\}_{m=1}^{\infty}$ is subjected to a trace-preserving completely positive linear transformation (map) of the tensor product form $\mathcal{E}^{\otimes m}$, where a copy of $\mathcal{E}$ locally acts on each spin lattice site, then all the listed source properties are preserved. Such maps describe the effect of a transmission via a memoryless quantum channel as well as the effect of memoryless coding, both lossless and lossy ones. We also establish ergodicity criteria for so called *classically-correlated* quantum sources. In Section 4 we briefly review quantum block compression schemes and then introduce a so-called *universal projector sequence* $\{p^{(n)}\}_{n=1}^{\infty}$ *with asymptotical rate* $r > 0$, where $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{tr}(p^{(n)}) = r$. Loosely speaking, for every sufficiently large n, the range subspace of $p^{(n)}$ contains the typical subspace (or high probability subspace in another notation) for every stationary ergodic source with von Neumann entropy rate below r. This property implies the existence of a universal compression scheme for these sources. In Section 5 we prove in a constructive way that the universal sequence of projectors does exist for any given r. The basic idea of our universal sequence construction is as follows. We select a suitable classical subsystem[e] of our quantum system and restrict a given stationary ergodic quantum source to this classical subsystem, thus obtaining a classical source. This classical source is also stationary ergodic, and it is well-known in classical information theory that there exist universal compression codes for classical ergodic sources. So we select a suitable universal classical code and then use it to construct the universal projector sequence.

2 Quantum Sources: Mathematical Formalism and Notation

Before we define a general quantum source, we give an informal, intuitive definition of a so-called *classically correlated* quantum source as a triple[10] consisting of *quantum messages*, a *classical probability distribution* for the messages, and the *time shift*. Such a triple uniquely determines a state of a one-dimensional quantum lattice system. If quantum-mechanical correlation between the messages exists, one gets the notion of a general quantum source. While any given state corresponds to infinitely many different quantum sources, the quantum state formalism essentially captures all the information-theoretic properties of a corresponding quantum source. Thus, the notion of "quantum source" is usually identified with the notion of "state" of the corresponding lattice system and used interchangeably.

Let Q be an infinite quantum spin lattice system over lattice $\mathbb{Z}$ of integers. To describe Q, we use the standard mathematical formalism introduced in [11, sec. 2.6, defn. 2.6.3] [12, sec. 6.2.1] and [13, sec. 1.33 and sec. 7.1.3] and borrow notation from [1] and [3]. Let $\mathfrak{A}$ be a C^*-algebra[f] with identity of all bounded linear operators $\mathcal{B}(\mathcal{H})$ on a d-dimensional Hilbert space $\mathcal{H}$, $d < \infty$. To each $\mathbf{x} \in \mathbb{Z}$ there is associated an algebra $\mathfrak{A}_{\mathbf{x}}$ of observables for a spin

[e] In fact, we select an infinite family of classical subsystems. This approach was first used to prove the quantum analog of Shannon-McMillan theorem for completely ergodic quantum sources[3, 9] and later extended[1] to all ergodic sources.

[f] The algebra of all bounded linear operators may be simply thought of as the algebra of all square matrices with the standard matrix operations including conjugate-transpose.

located at site $\mathbf{x}$, where $\mathfrak{A}_{\mathbf{x}}$ is isomorphic to $\mathfrak{A}$ for every $\mathbf{x}$. The local observables in any finite subset $\Lambda \subset \mathbb{Z}$ are those of the finite quantum system

$$\mathfrak{A}_\Lambda := \bigotimes_{\mathbf{x}\in\Lambda} \mathfrak{A}_{\mathbf{x}}$$

The quasilocal algebra $\mathfrak{A}_\infty$ is the operator norm completion of the normed algebra $\bigcup_{\Lambda\subset\mathbb{Z}} \mathfrak{A}_\Lambda$, the union of all local algebras $\mathfrak{A}_\Lambda$ associated with finite $\Lambda \subset \mathbb{Z}$. A state of the infinite spin system is given by a normed positive functional

$$\varphi \ : \ \mathfrak{A}_\infty \to \mathbb{C}.$$

We define a family of states $\{\varphi^{(\Lambda)}\}_{\Lambda\subset\mathbb{Z}}$, where $\varphi^{(\Lambda)}$ denotes the restriction of the state φ to a finite-dimensional subalgebra $\mathfrak{A}_\Lambda$, and assume that $\{\varphi^{(\Lambda)}\}_{\Lambda\subset\mathbb{Z}}$ satisfies the so called *consistency* condition[1, 10], that is

$$\varphi^{(\Lambda)} = \varphi^{(\Lambda^{'})} \restriction \mathfrak{A}_\Lambda \tag{1}$$

for any $\Lambda \subset \Lambda^{'}$. The consistent family $\{\varphi^{(\Lambda)}\}_{\Lambda\subset\mathbb{Z}}$ can be thought of as a quantum-mechanical counterpart of a consistent family of cylinder measures. Since there is one-to-one correspondence between the state φ and the family $\{\varphi^{(\Lambda)}\}_{\Lambda\subset\mathbb{Z}}$, any physically realizable transformation of the infinite system Q, including coding and transmission of quantum messages, can be well formulated using the states $\varphi^{(\Lambda)}$ of finite subsystems. When the subset $\Lambda \in \mathbb{Z}$ needs to be explicitly specified, we will use the notation $\Lambda(n)$, defined as

$$\Lambda(n) := \big\{x \in \mathbb{Z} : x \in \{1, \dots, n\}\big\}$$

Let γ (or γ^{-1}, respectively) denote a transformation on $\mathfrak{A}_\infty$ which is induced by the right (or left, respectively) shift on the set $\mathbb{Z}$. Then, for any $l \in \mathbb{N}$, γ^l (or γ^{-l}, respectively) denotes a composition of l right (or left, respectively) shifts. Now we are equipped to define the notions of stationarity and ergodicity of a quantum source.

Definition 2.1 *A state φ is called N-stationary for an integer N if $\varphi \circ \gamma^N = \varphi$. For $N = 1$, an N-stationary state is called stationary.*

Definition 2.2 *A N-stationary state is called N-ergodic if it is an extremal point in the set of all N-stationary states. For $N = 1$, N-ergodic state is called ergodic.*

The following lemma which provides a practical method of demonstrating the ergodicity of a state is due to [13, propos. 6.3.5, Lem. 6.5.1].

Lemma 2.1 *The following conditions are equivalent:*

(a) A stationary state φ on $\mathfrak{A}_\infty$ is ergodic.

(b) For all $a, b \in \mathfrak{A}_\infty$, it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(a \ \gamma^i(b)) = \varphi(a) \ \varphi(b). \tag{2}$$

(c) For every selfajoint $a \in \mathfrak{A}_\infty$, it holds

$$\lim_{n\to\infty} \varphi\left(\left(\frac{1}{n}\sum_{i=1}^{n}\gamma^i(a)\right)^2\right) = \varphi^2(a).$$

Now we state a series of definitions[11] which provide "stronger" notions of ergodicity:

Definition 2.3 *A state is called completely ergodic if it is N-ergodic for every integer N.*

Definition 2.4 *A stationary state φ on $\mathfrak{A}_\infty$ is called weakly mixing if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left|\varphi(a\ \gamma^i(b)) - \varphi(a)\ \varphi(b)\right| = 0, \quad \forall a,b \in \mathfrak{A}_\infty. \tag{3}$$

Definition 2.5 *A stationary state φ on $\mathfrak{A}_\infty$ is called strongly mixing if*

$$\lim_{i\to\infty} \varphi(a\ \gamma^i(b)) = \varphi(a)\ \varphi(b), \quad \forall a,b \in \mathfrak{A}_\infty. \tag{4}$$

It is straightforward to see that (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

Let $\mathrm{tr}_{\mathfrak{A}_\Lambda}(\cdot)$ denote the canonical trace on $\mathfrak{A}_\Lambda$ such that $\mathrm{tr}_{\mathfrak{A}_\Lambda}(e) = 1$ for all one-dimensional projections e in $\mathfrak{A}_\Lambda$. Where an algebra on which the trace is defined is clear from the context, we will omit the trace's subscript and simply write $\mathrm{tr}(\cdot)$. For each $\varphi^{(\Lambda)}$ there exists a unique density operator $\rho_\Lambda \in \mathfrak{A}_\Lambda$, such that $\varphi^{(\Lambda)}(a) = \mathrm{tr}(\rho_\Lambda a)$, $a \in \mathfrak{A}_\Lambda$. Thus, any stationary state φ is uniquely defined by the consistent family of density operators $\{\rho_{\Lambda(m)}\}_{m=1}^{\infty}$. Where no confusion arises, we will use the following abbreviated notation for the rest of the paper. For all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathfrak{A}^{(n)} &:= \mathfrak{A}_{\Lambda(n)}\\
\psi^{(n)} &:= \psi^{\Lambda(n)}\\
\rho_n &:= \rho_{\Lambda(n)}
\end{aligned}$$

It is well-known[14] in quantum mechanics that for every stationary state φ the limit

$$s(\varphi) := \lim_{n\to\infty} \frac{1}{n} S\big(\varphi^{(n)}\big) \tag{5}$$

exists, where $S\big(\varphi^{(n)}\big)$ is the von Neumann entropy of the state $\varphi^{(n)}$. In quantum statistical mechanics, the quantity $s(\varphi)$ is called the *mean* (von Neumann) entropy of φ, while in quantum information theory it is natural to call it the *entropy rate* of the stationary quantum source. It is not difficult to see that the existence of the limit (5) for any stationary state implies the existence of $\lim_{n\to\infty}\frac{1}{n}S\big(\psi^{(Nn)}\big)$ for any N-stationary state ψ and any fixed integer N. Thus, it makes possible to define a mean entropy with respect to N-shift as follows

$$s(\psi, N) := \lim_{n\to\infty} \frac{1}{n} S\big(\psi^{(Nn)}\big) \tag{6}$$

We note that if a state is stationary, then it is also N-stationary for any integer N. Therefore, the following equality holds for any stationary state φ:

$$s(\varphi, N) = N s(\varphi)$$

3 Invariance of Stationary and Ergodic Properties

In this section we present a sequence of lemmas and a theorem which help to establish the ergodicity of a state. But first we shall reformulate the stationary ergodic properties of an infinite spin lattice system in terms of its finite subsystems. By rewriting the consistency condition (1), Definition 2.1, and the equations (2–4) in terms of density operators, we obtain the following three elementary lemmas[g]

Lemma 3.1 *A family $\{\rho_m\}_{m=1}^{\infty}$ on $\mathfrak{A}_\infty$ is consistent if and only if, for all positive integers $m, i < \infty$ and every $a \in \mathfrak{A}^{(m)}$, the following holds:*

$$tr(\rho_m \ a) = tr\big(\rho_{m+i} \ (a \otimes I^{\otimes i})\big), \tag{7}$$

where $I^{\otimes i}$ stands for the i-fold tensor product of the identity operators acting on respective spins.

Lemma 3.2 *A quantum source $\{\rho_m\}_{m=1}^{\infty}$ on $\mathfrak{A}_\infty$ is stationary if and only if, for all positive integers $m, i < \infty$ and every $a \in \mathfrak{A}^{(m)}$, the following equality is satisfied:*

$$tr(\rho_m \ a) = tr\big(\rho_{m+i} \ (I^{\otimes i} \otimes a)\big), \tag{8}$$

Lemma 3.3 *A stationary quantum source $\{\rho_m\}_{m=1}^{\infty}$ on $\mathfrak{A}_\infty$ is ergodic (weakly mixing or strongly mixing, respectively) if and only if, for every positive integer $m < \infty$ and all $a, b \in \mathfrak{A}^{(m)}$, the equality (9) $\big($(10) or (11), respectively$\big)$ holds:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=m}^{n} tr\big(\rho_{m+i} \ (a \otimes I^{\otimes(i-m)} \otimes b)\big) = tr(\rho_m a) \ tr(\rho_m b), \tag{9}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=m}^{n} \Big| tr\big(\rho_{m+i} \ (a \otimes I^{\otimes(i-m)} \otimes b)\big) - tr(\rho_m a) \ tr(\rho_m b) \Big| = 0, \tag{10}$$

$$\lim_{i\to\infty} tr\big(\rho_{m+i} \ (a \otimes I^{\otimes(i-m)} \otimes b)\big) = tr(\rho_m a) \ tr(\rho_m b), \tag{11}$$

We now need to fix some additional notation. Let $\mathcal{E}$ be an arbitrary trace-preserving completely positive linear (TPCPL) map[15] which takes $\mathcal{B}(\mathcal{H})$ as its input. Without loss of generality we assume that the output space for $\mathcal{E}$ is also $\mathcal{B}(\mathcal{H})$. Next, we define a composite map

$$\mathcal{E}^{\otimes m} \ : \ \mathfrak{A}^{(m)} \to \mathfrak{A}^{(m)}, \qquad \forall m > 0.$$

We point out that such a tensor product map is the most general description of a *quantum memoryless channel*[16].

Theorem 3.1 *If $\{\rho_m\}_{m=1}^{\infty}$ is a stationary and ergodic (weakly mixing or strongly mixing, respectively) source, then so is the source $\big\{\mathcal{E}^{\otimes m}(\rho_m)\big\}_{m=1}^{\infty}$. The proof of this theorem is given in the appendix C.*

Remark 1 *This theorem can be viwed is the quantum generalization of a well-known classical information-theoretic result[17, chap. 7] for memoryless channels, and we strongly believe that the theorem can be extended to the case of quantum Markov channels[18].*

[g] In what follows we abusively use the same symbol to denote both an operator (or superoperator), confined to a lattice box $\Lambda(m)$, and its "shifted" copy, confined to a box $\{1+j, \dots , \ m+j\}$, where the value of integer j will be understood from the context.

Definition 3.1 *We define a classically correlated quantum source* $\{\rho_m^{cls}\}_{m=1}^{\infty}$ *by an equation*

$$\rho_m^{cls} := \sum_{x_1,x_2,\dots,x_m} p(x_1,x_2,\dots,x_m)|x_1\rangle\langle x_1| \otimes |x_2\rangle\langle x_2| \otimes \cdots \otimes |x_m\rangle\langle x_m|, \tag{12}$$

where $p(\cdot)$ *stands for a probability distribution, and for every* i, $|x_i\rangle$ *belongs to some fixed linearly-independent set* $S := \{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_d\rangle\}$ *of vectors in the Hilbert space* $\mathcal{H}$. *We recall that* $\mathcal{H}$ *is the support space for the operators in* $\mathfrak{A}$. *The set* S *is sometimes called a* quantum alphabet.

Corollary 3.1 *If a classical probability distribution* $p(\cdot)$ *in Definition 3.1 is a stationary and ergodic (weakly mixing or strongly mixing, respectively), then so is the quantum source* $\{\rho_m^{cls}\}_{m=1}^{\infty}$. *The proof of this corollary is given in the appendix C.*

4 Universal Compression with Asymptotically Perfect Fidelity

We begin this section with the introduction of the so-called *quantum block compression scheme* which consists of a sequence $\{\mathcal{C}^{(n)}, \mathcal{D}^{(n)}\}_{n=1}^{\infty}$ of TPCPL compression and decompression maps

$$\begin{aligned} \mathcal{C}^{(n)} &: B(\mathcal{H}^{\otimes n}) \to B\left(\mathcal{H}_c^n\right) \\ \mathcal{D}^{(n)} &: B\left(\mathcal{H}_c^n\right) \to B(\mathcal{H}^{\otimes n}), \end{aligned}$$

where, for every integer $n > 0$, $\mathcal{H}_c^n$ is a subspace of a Hilbert space $\mathcal{H}^{\otimes n}$, and $B(\cdot)$ stands for the set of all linear operators on a respective Hilbert space. For every n, we also define the *compression rate* $r\left(\mathcal{C}^{(n)}\right)$ by the equation[h]

$$r\left(\mathcal{C}^{(n)}\right) := \frac{1}{n} \log \dim \mathcal{H}_c^n.$$

Although a broad class of classical sources can be compressed and decompressed without distortion i.e. with perfect fidelity, quantum sources, with some exceptions, are not compressible without errors[19]. That is, for a quantum source $\{\rho_m\}_{m=1}^{\infty}$, we have, in general,

$$\rho_m \neq \mathcal{D}^{(m)} \circ \mathcal{C}^{(m)}(\rho_m).$$

However, one may still be interested in compression schemes where the states ρ_m and $\mathcal{D}^{(m)} \circ \; C^{(m)}(\rho_m)$ are sufficiently close to each other. Such the "closeness" can be quantified by special measures. In this paper, we will use the so-called *entanglement fidelity*[20] measure which turns out to be the "strongest" of all fidelity notions that are applicable to encoding-decoding schemes. "Strongest" means that if the entanglement fidelity of a compression schemes converges to unity, then all the other applicable fidelities converge to unity, too[20]. Moreover, the higher the entanglement fidelity of a map $\mathcal{D}^{(m)} \circ \mathcal{C}^{(m)}$ is, the better is preserved[20] the entanglement of the source system with an external system. In order to define the entanglement fidelity, we first need to introduce the *fidelity of states.* The fidelity $F(\cdot,\cdot)$ of states with density matrices ϕ and σ is defined to be

$$F(\phi,\sigma) := \mathrm{tr}\sqrt{\phi^{\frac{1}{2}}\sigma\phi^{\frac{1}{2}}}$$

[h]All logarithms in this paper are to base 2.

Let ϕ be the density matrix of a state on a Hilbert space $\mathcal{H}$ which is subjected to a TPCPL map $\mathcal{E}$. Let $|\Theta\rangle\langle\Theta| \in \mathcal{H}\otimes\tilde{\mathcal{H}}$ be a purification of ϕ, where $\tilde{\mathcal{H}}$ is a Hilbert space for the reference system arising from purification procedure[21]. Then the entanglement fidelity $F_e(\cdot,\cdot)$ is defined by

$$F_e(\phi,\mathcal{E}) := F^2\big(|\Theta\rangle\langle\Theta|, (\tilde{\mathcal{I}}\otimes\mathcal{E})(|\Theta\rangle\langle\Theta|)\big),$$

where $\tilde{\mathcal{I}}$ is the identity map (superoperator) on the state space of the reference system. The fidelity of states and the entanglement fidelity have many interesting properties[16, 20], of which we just state the following

1. For all ϕ and $\mathcal{E}$, we have the relations

$$0 \leqslant F_e(\phi,\mathcal{E}) \leqslant F(\phi,\mathcal{E}(\phi)) \leqslant 1.$$

2. For all ϕ and σ, the equality $F(\phi,\sigma) = 1$ holds if and only if $\phi = \sigma$.

3. For all ϕ and $\mathcal{E}$, the equality $F_e(\phi,\mathcal{E}) = 1$ holds if and only if for all pure states $|\psi\rangle$ lying in the support of ϕ,

$$\mathcal{E}(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|.$$

Let $\{p^{(n)}\}_{n=1}^{\infty}$, where $p^{(n)} \in B(\mathcal{H}^{\otimes n})$, be a sequence of orthogonal projectors. Then we explicitly define two compression schemes[22, 23] $\{\mathcal{C}_1^{(n)},\mathcal{D}_1^{(n)}\}_{n=1}^{\infty}$ and $\{\mathcal{C}_2^{(n)},\mathcal{D}_2^{(n)}\}_{n=1}^{\infty}$ as follows

$$\mathcal{C}_1^{(n)}(\sigma) := p^{(n)}\sigma p^{(n)} + \sum_i A_i\sigma A_i^{\dagger},$$

$$\mathcal{C}_2^{(n)}(\sigma) := \frac{p^{(n)}\sigma p^{(n)}}{\mathrm{tr}\big(p^{(n)}\sigma p^{(n)}\big)},$$

where A_i is defined by $A_i := |0\rangle\langle i|$, and $\{|i\rangle\}$ is an orthonormal basis for the orthocomplement of the subspace $\mathcal{H}_{c_1}^n = \mathcal{H}_{c_2}^n := p^{(n)}\mathcal{H}^{\otimes n}$. The decompression maps $\mathcal{D}_1^{(n)}$ and $\mathcal{D}_2^{(n)}$ are just the identities on $B(\mathcal{H}_{c_1}^n)$ and $B(\mathcal{H}_{c_2}^n)$, respectively.

Definition 4.1 *We call* $\{p^{(n)}\}_{n=1}^{\infty}$ a universal projector sequence *with asymptotical rate* $r \in \mathbb{R}$, *if the following two conditions are satisfied:*

1. *there holds the limit*

$$\lim_{n\to\infty}\frac{1}{n}\log tr(p^{(n)}) = r;$$

2. *for* every *stationary ergodic source* $\{\rho_n\}_{n=1}^{\infty}$ *with von Neumann entropy rate below* r, *the following limit also holds*

$$\lim_{n\to\infty} tr(p^{(n)}\rho_n) = 1.$$

Theorem 4.1 *(i) A universal projector sequence exists for any asymptotical rate* $r \in \big(0,\ \log\dim(\mathcal{H})\big]$.

(ii) Let $\{p^{(n)}\}_{n=1}^{\infty}$ be a universal projector sequence with an asymptotical rate r, then for every *stationary ergodic source $\{\rho_n\}_{n=1}^{\infty}$ with von Neumann entropy rate below r, the following limits hold*

$$\lim_{n\to\infty} F_e(\rho_n, \mathcal{D}_1^{(n)} \circ \mathcal{C}_1^{(n)}) = 1 \qquad \lim_{n\to\infty} r(\mathcal{C}_1{}^{(n)}) = r$$
$$\lim_{n\to\infty} F_e(\rho_n, \mathcal{D}_2^{(n)} \circ \mathcal{C}_2^{(n)}) = 1 \qquad \lim_{n\to\infty} r(\mathcal{C}_2{}^{(n)}) = r,$$

where for every n, the maps $\mathcal{C}_1^{(n)}$ and $\mathcal{C}_1^{(n)}$ are constructed with $p^{(n)}$.

Proof: Part (i) is obtained in Lemma 5.2 via explicit construction of the universal projector sequence (27). Part (ii) follows immediately from the definition of a universal projector sequence and the so-called intrinsic expression[20] for $F_e(\cdot,\cdot)$ as shown in the proof of the theorem [24, chap. 7, theor. 21]. □

Remark 2 *Theorem 4.1 can be viewed as a quantum extension of universal noiseless fixed-rate coding[25, 26] of classical ergodic sources.*

5 Universal Projector Sequence Construction

We begin this section with stating the important results from [1]. Since work[1] deals with more general, multi-dimensional quantum lattice systems, we reformulate the results for one-dimensional lattice, while staying as close as possible to the original notation.

Theorem 5.1 ([1, theor. 3.1]) *Let ψ be a stationary ergodic state. Then, for every integer $l > 1$, there exists a $k(l) \in \Lambda(l)$ and a unique convex decomposition of ψ into l-ergodic states $\psi_{\mathbf{x},l}$:*

$$\psi = \frac{1}{k(l)} \sum_{\mathbf{x}=0}^{k(l)-1} \psi_{\mathbf{x},l}, \tag{13}$$

where $\psi_{\mathbf{x},l}$ has the following properties:

1. $\psi_{\mathbf{x},l} = \psi_{0,l} \circ \gamma^{\mathbf{x}}$
2. $s(\psi_{\mathbf{x},l}, l) = s(\psi, l)$

Lemma 5.1 ([1, lem 3.1]) *Let ψ be an ergodic state. For a real $\eta > 0$, we define a set of integers $A_{l,\eta}$:*

$$A_{l,\eta} := \left\{\mathbf{x} \in \mathbb{Z} \ : \ 0 \leqslant \mathbf{x} < k(l) \ \& \ \frac{1}{l} S(\psi_{\mathbf{x},l}^{(\Lambda(l))}) \geqslant s(\psi) + \eta\right\}. \tag{14}$$

Then, the limit

$$\lim_{l\to\infty} \frac{|A_{l,\eta}|}{k(l)} = 0$$

holds for every $\eta > 0$, where $|A_{l,\eta}|$ denotes the cardinality of the set $A_{l,\eta}$.

Let ψ be a stationary ergodic state on $\mathfrak{A}^{\infty}$, and let $\{\psi_{\mathbf{x},l}\}_{\mathbf{x}=0}^{k(l)-1}$ be the l-ergodic decomposition of ψ. In what follows, unless otherwise specified, we assume that $\mathbf{x}$ and $\mathbf{y}$ are integers from the set $\{0, 1, \ldots, k(l) - 1\}$, where l is also an integer as defined in Theorem 5.1. For every $\mathbf{x}$, let $\{\rho_{\mathbf{x},l,n}\}_{n=1}^{\infty}$ be the family of density operators for $\psi_{\mathbf{x},l}$, and let $\mathfrak{C}_{\mathbf{x},l}$ be the maximal abelian C^*-subalgebra of $\mathfrak{A}_{\Lambda(l)}$ which is generated by the spectral eigenprojections of $\rho_{\mathbf{x},l}$. Then, the following well-known[27] relation holds:

$$S(\psi_{\mathbf{x},l}^{\Lambda(l)} \restriction \mathfrak{C}_{\mathbf{x},l}) = S(\psi_{\mathbf{x},l}^{\Lambda(l)}) \tag{15}$$

For a fixed l and every $\mathbf{x}$, we define the abelian quasi-local C^*-algebra $\mathfrak{C}_{\mathbf{x},l}^{\infty}$, which is constructed with the copies of $\mathfrak{C}_{\mathbf{x},l}$ over the sub-lattice $l \cdot \mathbb{Z}$ (in the same way as $\mathfrak{A}^{\infty}$ was constructed with $\mathfrak{A}$ over lattice $\mathbb{Z}$) and viewed as C^*-subalgebra of $\mathfrak{A}^{\infty}$, and set

$$m_{\mathbf{x},l} := \psi_{\mathbf{x},l} \restriction \mathfrak{C}_{\mathbf{x},l}^{\infty} \tag{16}$$

$$m_{\mathbf{x},l}^{(n)} := \psi_{\mathbf{x},l} \restriction \mathfrak{C}_{\mathbf{x},l}^{(n)}. \tag{17}$$

To avoid possible confusion, we emphasize that the states $m_{\mathbf{x},l}^{(n)}$ are confined to the box $\Lambda(n)$ in the sub-lattice $l \cdot \mathbb{Z}$ which corresponds to the box $\Lambda(nl)$ in the lattice $\mathbb{Z}$. By Lemma B.1 in the appendix, the states $m_{\mathbf{x},l}$ are stationary ergodic with respect to the sub-lattice $l \cdot \mathbb{Z}$ and therefore are l-stationary l-ergodic with respect to the lattice $\mathbb{Z}$. By Gelfand isomorphism and Riesz representation, for every $\mathbf{x}$, quasilocal algebra $\mathfrak{C}_{\mathbf{x},l}^{\infty}$ is identified with a measurable space which we denote by $\big(\mathcal{Z}_l^{\infty}, \mathcal{P}^{\infty}(\mathcal{Z}_l)\big)$ with the following properties:

(a) the sample space $\mathcal{Z}_l^{\infty}$ is the direct product of replicas of an abstract set $\mathcal{Z}_l$ with the cardinality d^l over the the sub-lattice $l \cdot \mathbb{Z}$, and $\mathcal{P}^{\infty}(\mathcal{Z})$ is the corresponding direct product σ-field;

(b) there is a bijective map $f_{\mathbf{x},l} : \Pi(\mathfrak{C}_{\mathbf{x},l}^{\infty}) \to \mathcal{P}^{\infty}(\mathcal{Z}_l)$, where $\Pi(\mathfrak{C}_{\mathbf{x},l}^{\infty})$ denotes the set of all projections in $\mathfrak{C}_{\mathbf{x},l}^{\infty}$;

(c) every state $m_{\mathbf{x},l}$ on $\mathfrak{C}_{\mathbf{x},l}^{\infty}$ corresponds to a positive measure on $\big(\mathcal{Z}_l^{\infty}, \mathcal{P}^{\infty}(\mathcal{Z}_l)\big)$ which we denote by $\mu_{\mathbf{x},l}$ such that $m_{\mathbf{x},l}(p) = \mu_{\mathbf{x},l}\big(f_{\mathbf{x},l}(p)\big)$ for every $p \in \Pi(\mathfrak{C}_{\mathbf{x},l}^{\infty})$.

In fact, the tuple $(\mathfrak{C}_{\mathbf{x},l}^{\infty}, m_{\mathbf{x},l})$ and the triple $\big(\mathcal{Z}_l^{\infty}, \mathcal{P}^{\infty}(\mathcal{Z}_l), \mu_{\mathbf{x},l}\big)$ are just two equivalent descriptions[13] of a given classical stochastic process (see the appendix A for more details). Unsurprisingly, the measure $\mu_{\mathbf{x},l}$ is stationary ergodic with respect to the sub-lattice $l \cdot \mathbb{Z}$, and the following relation holds by Proposition A.1:

$$S(m_{\mathbf{x},l}^{(n)}) = H(\mu_{\mathbf{x},l}^{n}), \tag{18}$$

where $H(\mu_{\mathbf{x},l}^{n})$ denotes the Shannon entropy of the probability distribution on $\mathcal{Z}_{\mathbf{x},l}^{n}$ defined by measure $\mu_{\mathbf{x},l}$. It is well-known(cf. [28]) in (classical) information theory that the limit $\lim_{n\to\infty} \frac{1}{n} H(\mu_{\mathbf{x},l}^{n})$ exists due to stationarity of $\mu_{\mathbf{x},l}$ and is called the *Shannon entropy rate* of measure $\mu_{\mathbf{x},l}$ and denoted by $h(\mu_{\mathbf{x},l})$. Shannon entropy rate possess[28] the following important property

$$h(\mu_{\mathbf{x},l}) = \lim_{n\to\infty} \frac{1}{n} H(\mu_{\mathbf{x},l}^{n}) = \inf_{n} \frac{1}{n} H(\mu_{\mathbf{x},l}^{n}). \tag{19}$$

For all integers $L, n > 0$, let $\mathcal{U}_L$ and $\mathcal{U}_L^n$ stand for a set of L symbols (alphabet) and a direct product of n replicas of this set, respectively. Then we define a mapping[26]

$$C_{L,R}^{n} \,:\, \mathcal{U}_L^{n} \to \mathcal{U}_2^{nR},$$

where nR is an integer, and $R > 0$. If $R < \log_2 L$, then some of the elements of $\mathcal{U}_L^n$ are mapped to the same elements of $\mathcal{U}_2^{nR}$. Let G_L^n be a subset of $\mathcal{U}_L^n$ for which the mapping $C_{L,R}^n$ is bijective. Clearly, the cardinality $|G_L^n|$ cannot exceed 2^{nR}, and we only consider mappings $C_{L,R}^n$ for which $|G_L^n|$ is maximized, that is

$$|G_L^n| = 2^{nR}.$$

We will denote each such set by $G_{L,R}^n$ and call it a *block code*[i] of *rate R*. Now we are ready to define a so-called *universal sequence of block codes* on a measurable space $\big(\mathcal{U}_L^\infty, \mathcal{F}^\infty(\mathcal{U}_L)\big)$, where $\mathcal{F}^\infty(\mathcal{U}_L)$ is the usual product σ-field of subsets of $\mathcal{U}_L^\infty$.

Definition 5.1 *A sequence of block codes* $\{G_{L,R}^n\}_{n=1}^\infty$ *on* $\big(\mathcal{U}_L^\infty, \mathcal{F}^\infty(\mathcal{U}_L)\big)$ *is called* universal *if for every stationary ergodic measure* μ *on* $\big(\mathcal{U}_L^\infty, \mathcal{F}^\infty(\mathcal{U}_L)\big)$ *with* $h(\mu) < R$ *there holds the limit*

$$\lim_{n\to\infty} \mu\big(G_{L,R}^n\big) = 1. \tag{20}$$

The existence of a universal sequence for any real $R > 0$ was shown[25, 26] in the framework of the universal (classical) compression of stationary ergodic sources. It is not difficult to see that, for any $R > 0$, a universal sequence $\{G_{L,R}^n\}_{n=1}^\infty$ can be constructed[29] in such a way that for any integer $i > 0$, the subsequence $\{G_{L,R}^{i\cdot j}\}_{j=1}^\infty$ gives rise to another universal sequence $\{G_{L^i,iR}^j\}_{j=1}^\infty$. More specifically, if we partition every sequence in $G_{L,R}^{i\cdot j}$ into non-overlapping blocks of length i, and view it as the sequence of the supersymbols, then we get exactly the set $G_{L^i,iR}^j$. From now on, we will be only considering universal sequences with this property.

Now, for all l and $R > 0$, let $\left\{\Omega_{l,R}^{(n)}\right\}_{n=1}^\infty$ be a universal sequence on $\big(\mathcal{Z}_l^\infty, \mathcal{P}^\infty(\mathcal{Z}_l)\big)$, and, for every $\mathbf{x}$, let $p_{\mathbf{x},l,R}^{(n)}$ be a projector in $\mathfrak{C}_{\mathbf{x},l}^{(n)}$ that corresponds to the set $\Omega_{l,R}^{(n)}$, that is

$$p_{\mathbf{x},l,R}^{(n)} := f_{\mathbf{x},l}^{-1}\left(\Omega_{l,R}^{(n)}\right) \tag{21}$$

Let $\tilde{\psi}$ be an arbitrary stationary ergodic state on $\mathfrak{A}^\infty$, and we convert all the notation we introduce in connection with the sate ψ to the notation for $\tilde{\psi}$ by adding the symbol $\tilde{}$. We know that given any two (faithful) states on $\mathfrak{A}^{(l)}$, for any l, the eigenbasis of the density operator of one state can be obtained from the other state's eigenbasis by applying some unitary operator in $\mathfrak{A}^{(l)}$. Therefore, for every pair $\mathbf{x}$ and $\mathbf{y}$, there exists a unitary operator $U_l \in \mathfrak{A}^{(l)}$ which satisfies the equality

$$\tilde{p}_{\mathbf{y},l,R}^{(n)} = U_l^{\otimes n} p_{\mathbf{x},l,R}^{(n)}\, U_l^{\dagger\otimes n}, \tag{22}$$

We define an auxiliary projector $w_{l,R}^{(ln)} \in \mathfrak{A}^{(ln)}$

$$w_{l,R}^{(ln)} := \bigvee_{U_l\in\mathfrak{A}^{(l)}} U_l^{\otimes n} p_{0,l,R}^{(n)}\, U_l^{\dagger\otimes n}, \tag{23}$$

---

[i] In some literature(cf. [26]) the term "block code" is reserved for the mapping $C_{L,R}^n$ rather than for the set $G_{L,R}^n$ since specifying $C_{L,R}^n$ is equivalent to specifying $G_{L,R}^n$ up to a permutation of the alphabet.

where $p_{0,l,R}^{(n)} := p_{\mathbf{x},l,R}^{(n)}\Big|_{\mathbf{x}=0}$. Then, for every $\mathbf{x}$ and all real $R > 0$, we have

$$\tilde{p}_{\mathbf{x},R}^{(n)} \leqslant w_{l,R}^{(ln)}. \tag{24}$$

Moreover, for all integer $i, j > 0$ and real $R > 0$, we have the inequality

$$w_{l,R}^{(l\cdot ij)} \leqslant w_{il,iR}^{(il\cdot j)} \tag{25}$$

due to the special relationship between the universal codes $G_{L,R}^{i\cdot j}$ and $G_{L^i,iR}^{j}$, which we discussed earlier. Finally, we construct, for any real $r > 0$, a projector sequence $\{q_r^{(m)}\}_{m=1}^{\infty}$ as follows. For every integer $m > 0$, let i_m be the integer-valued function of m which is defined by the inequality

$$2^{i_m} d^{3\cdot 2^{i_m}} \leqslant m < 2^{i_m+1} d^{3\cdot 2^{i_m+1}}, \tag{26}$$

and we also define integer-valued functions l_m, n_m and real-valued function R_m via equalities

$$\begin{aligned} l_m &:= 2^{i_m}, \\ n_m &:= \left\lfloor \frac{m}{l_m} \right\rfloor, \\ R_m &:= l_m \cdot r. \end{aligned}$$

Then $q_r^{(m)}$ is given by the expression

$$q_r^{(m)} := \begin{cases} w_{l_m,R_m}^{(l_m n_m)} & \text{if } m = 2^{i_m} d^{3\cdot 2^{i_m}}, \\ w_{l_m,R_m}^{(l_m n_m)} \otimes I^{\otimes (m - l_m n_m)} & \text{otherwise.} \end{cases} \tag{27}$$

Thus, projectors $q_r^{(m)}$ do not depend on either ψ or $\tilde{\psi}$ or any other state(s).

Lemma 5.2 *For any real $0 < r \leqslant \log d$ and stationary ergodic source ψ with $s(\psi) < r$, the following two limits hold:*

(i) $\lim_{m\to\infty} \psi^{(m)}(q_r^{(m)}) = 1$;

(ii) $\lim_{m\to\infty} \frac{1}{m} \log tr(q_r^{(m)}) = r$.

Proof: For all integer $m \geqslant \tilde{m} > 0$, we have the following sequence of relations

$$\begin{aligned} \psi^{(m)}\left(q_r^{(m)}\right) &\overset{1)}{=} \psi^{(l_m n_m)}\left(w_{l_m,R_m}^{(l_m n_m)}\right) \overset{2)}{\geqslant} \psi^{(l_m n_m)}\left(w_{l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m)}\right) \overset{3)}{=} \frac{1}{k(l_{\tilde{m}})} \sum_{\mathbf{x}=0}^{k(l_{\tilde{m}})-1} \psi_{\mathbf{x},l_{\tilde{m}}}^{(l_m n_m)}\left(w_{l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m)}\right) \\ &\geqslant \frac{1}{k(l_{\tilde{m}})} \sum_{\mathbf{x}\in A_{l_{\tilde{m}},\eta}^c} \psi_{\mathbf{x},l_{\tilde{m}}}^{(l_m n_m)}\left(w_{l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m)}\right) \geqslant \frac{|A_{l_{\tilde{m}},\eta}^c|}{k(l_{\tilde{m}})} \min_{\mathbf{x}\in A_{l_{\tilde{m}},\eta}^c} \psi_{\mathbf{x},l_{\tilde{m}}}^{(l_m n_m)}\left(w_{l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m)}\right) \\ &\overset{4)}{\geqslant} \frac{|A_{l_{\tilde{m}},\eta}^c|}{k(l_{\tilde{m}})} \min_{\mathbf{x}\in A_{l_{\tilde{m}},\eta}^c} \psi_{\mathbf{x},l_{\tilde{m}}}^{(l_m n_m)}\left(p_{\mathbf{x},l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m/l_{\tilde{m}})}\right) = \frac{|A_{l_{\tilde{m}},\eta}^c|}{k(l_{\tilde{m}})} \min_{\mathbf{x}\in A_{l_{\tilde{m}},\eta}^c} \mu_{\mathbf{x},l_{\tilde{m}}}\left(\Omega_{l_{\tilde{m}},R_{\tilde{m}}}^{(l_m n_m/l_{\tilde{m}})}\right) \end{aligned} \tag{28}$$

where 1) is due to (27), 2) is due to (25), 3) is due to Theorem 5.1, 4) is due to (24), and $\Omega^{(j)}_{l_{\tilde{m}},R_{\tilde{m}}} \equiv f_{\mathbf{x},l_{\tilde{m}}}\left(p^{(j)}_{\mathbf{x},l_{\tilde{m}},R_{\tilde{m}}}\right)$ for any integer $j > 0$ by definition (21).
Now we want to show that for every $\mathbf{x} \in A^c_{l_{\tilde{m}},\eta}$, where $\eta := r - s(\psi)$, the inequality

$$h(\mu_{\mathbf{x},l_{\tilde{m}}}) < R_{\tilde{m}} \tag{29}$$

holds. First, we upper-bound $h(\mu_{\mathbf{x},l})$, for all integer l and $\mathbf{x}$, as follows

$$h(\mu_{\mathbf{x},l}) \overset{a)}{\leqslant} H(\mu^{(1)}_{\mathbf{x},l}) \overset{b)}{=} S(\psi^{(l)}_{\mathbf{x},l} \upharpoonright \mathfrak{C}_{\mathbf{x},l}) \overset{c)}{=} S(\psi^{(l)}_{\mathbf{x},l}), \tag{30}$$

where $a)$ is due to (19), $b)$ is due to (17), (18), and $c)$ is due to (15). Then (30) and (14) imply (29).

Since l_m is a non-decreasing function of m, and there holds the limit

$$\lim_{m\to\infty} l_m = \infty, \tag{31}$$

Lemma 5.1 implies the existence of the limit

$$\lim_{m\to\infty} \frac{|A^c_{l_m,\eta}|}{k(l_m)} = 1.$$

That is, for any $\epsilon > 0$, there exists an integer $\tilde{m}_{\epsilon,\eta} > 0$ which satisfies the inequality

$$\frac{|A^c_{l_{\tilde{m}_{\epsilon,\eta}},\eta}|}{k(l_{\tilde{m}_{\epsilon,\eta}})} > 1 - \epsilon. \tag{32}$$

On the other hand, for every integer $m > \tilde{m}_{\epsilon,\eta}$, the expression $l_m n_m / l_{\tilde{m}}$ is a non-decreasing integer-valued function of m, and there holds the limit

$$\lim_{m\to\infty} l_m n_m = \infty.$$

Then by (20) there exists an integer $M_{\epsilon,\eta} > \tilde{m}_{\epsilon,\eta}$ such that for every integer $m > M_{\epsilon,\eta}$ and every $\mathbf{x} \in A^c_{l_{\tilde{m}_{\epsilon,\eta}},\eta}$, there holds the inequality

$$\mu_{\mathbf{x},l_{\tilde{m}_{\epsilon,\eta}}}\left(\Omega^{(l_m n_m / l_{\tilde{m}_{\epsilon,\eta}})}_{l_{\tilde{m}_{\epsilon,\eta}},R_{\tilde{m}_{\epsilon,\eta}}}\right) > 1 - \epsilon. \tag{33}$$

Thus, combining (28), (32), and (33), we obtain the first part of the lemma.

To prove the second part of the lemma, we will make use of the simple upper bound[6] on the dimensionality of a so-called *symmetrical subspace* of a linear space. We define a space

$$SYM\big(\mathfrak{A}^{(ln)}\big) := span\big\{A^{\otimes n} : A \in \mathfrak{A}^{(l)}\big\},$$

which is the symmetrical subspace of $\mathfrak{A}^{(ln)}$ over sub-lattice box $l \cdot \Lambda(n)$. Then the dimensionality of $SYM\big(\mathfrak{A}^{(ln)}\big)$ is upper-bounded[6] by $(n+1)^{d^{2l}}$. Thus, for all integer $m > 0$ and real $r > 0$, we have

$$\begin{aligned}\operatorname{tr}\left(q^{(m)}_r\right) &= \operatorname{tr}\left(w^{(l_m n_m)}_{l_m,R_m}\right) \cdot \operatorname{tr}\left(I^{\otimes(m - l_m n_m)}\right) \leqslant SYM\big(\mathfrak{A}^{(l_m n_m)}\big) \cdot \operatorname{tr}\left(p^{(n_m)}_{0,l_m,R_m}\right) \cdot d^{l_m} \\ &\leqslant (n_m+1)^{d^{2l_m}} \cdot \left|\Omega^{(n_m)}_{0,l_m,R_m}\right| \cdot d^{l_m} = (n_m+1)^{d^{2l_m}} \cdot 2^{n_m R_m} \cdot d^{l_m},\end{aligned}$$

and

$$\frac{1}{m}\log\operatorname{tr}\left(q_r^{(m)}\right) \leqslant \frac{1}{l_m n_m}\log\operatorname{tr}\left(q_r^{(m)}\right) \overset{1)}{\leqslant} \frac{d^{2l_m}\log(d^{3l_m}+1)}{l_m d^{3l_m}} + r + \frac{\log d}{d^{3l_m}}, \tag{34}$$

where 1) is due to the fact that inequality $n_m \geqslant d^{3l_m}$ holds for all integers $m > 0$. On the other hand,

$$\frac{1}{m}\log\operatorname{tr}\left(q_r^{(m)}\right) \overset{1)}{\geqslant} \frac{n_m R_m + (m - l_m n_m)\log d}{m} \overset{2)}{\geqslant} r, \tag{35}$$

where 1) is due (24) and 2) is due to the relation $rl_m \equiv R_m \leqslant l_m \log d$ which holds for all m. Combining (31), (34), and (35), we obtain the second part of the lemma. □

6 Conclusion

We prove that, for any real number $r > 0$, there exists a sequence $\{p^{(n)}\}_{n=1}^{\infty}$ of orthogonal projectors such that for any stationary ergodic source with von Neumann entropy rate below r and all sufficiently large n, the range subspace of $p^{(n)}$ approximately contains the source's typical subspace. Thus, we can compress the source by projecting it into the range subspace. Since $\{p^{(n)}\}_{n=1}^{\infty}$ does not depend on the source, we obtain a universal compression scheme for the family of all stationary ergodic sources with the entropy rates less than r. This extends the result[6] obtained by Jozsa et al. for independently and identically distributed quantum sources.

We also show invariance of stationary and ergodic properties under completely positive linear transformations that describe the effect of a transmission via a quantum memoryless channel. As the corrolarly of our invariance result, we establish ergodicity criteria for classically-correlated quantum sources. This can be viewd as a step towards the studies on how the properties of a quantum source are changed after transmission through a quantum channel, and which subclasses of stationary ergodic quantum sources are invariant under certain transformations.

References

1. I. Bjelaković, T. Krüger, R. Siegmund-Schultze, and A. Szkoła, "The Shannon-McMillan Theorem for Ergodic Quantum Lattice Systems", *LANL e-print* `http://lanl.arxiv.org/math.DS/0207121`
2. I. Bjelaković and A. Szkoła, "The Data Compression Theorem for Ergodic Quantum Information Sources," *LANL e-print* `http://lanl.arxiv.org/quant-ph/0301043`
3. D. Petz and M. Mosonyi, "Stationary Quantum Source Coding", *J. Math. Phys*, Vol. 42, pp. 4857–4864, 2001, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9912103`
4. B. Schumacher, "Quantum Coding," *Phys. Rev. A*, Vol. 51, No. 4, pp. 2738–2747, Apr. 1995.
5. L. Davisson, "Universal Noiseless Coding," *IEEE Trans. Inform. Theory*, Vol. 19, No. 6, pp. 783–795, Nov. 1973.
6. R. Jozsa, M. Horodecki, P. Horodecki, and R. Horodecki, "Universal Quantum Information Compression", *Phys. Rev. Lett*, Vol. 81, pp. 1714–1717, 1998, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9805017`
7. M. Hayashi and K. Matsumoto, "Simple construction of quantum universal variable-length source coding," *LANL e-print* `http://lanl.arxiv.org/quant-ph/0209124`
8. R. Jozsa and S. Presnell, "Universal quantum information compression and degrees of prior knowledge," *LANL e-print* `http://lanl.arxiv.org/quant-ph/0210196`

9. F. Hiai and D. Petz, "The Proper Formula for Relative Entropy and its Asymptotics in Quantum Probability," *Commun. Math. Phys*, Vol. 143, pp. 99–114, 1991.
10. C. King and A. Leśniewski, "Quantum Sources and a Quantum Coding Theorem", *J. Math. Phys*, Vol. 39 (1), pp. 88–101, 1998, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9511019`
11. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, New York, 1979.
12. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, New York, 1981.
13. D. Ruelle, *Statistical Mechanics*, W.A. Benjamin, New York, 1969.
14. O. Lanford and D. Robinson, "Mean Entropy of States in Quantum Statistical Mechanics", *J. Math. Phys*, Vol. 9, pp. 1120–1125, 1968.
15. K. Kraus, *States, Effects, and Operations*, Berlin, Germany: Springer-Verlag, 1983.
16. H. Barnum, E. Knill, and M. Nielsen, "On quantum fidelities and channel capacities", *IEEE Trans. Inform. Theory*, Vol. 46, No. 4, pp. 1317–1329, July, 2000.
17. T. Berger, *Rate distortion theory; a mathematical basis for data compression*, Englewood Cliffs, N.J., Prentice-Hall, 1971.
18. M. Hamada, "A Lower Bound on the Quantum Capacity of Channels with Correlated Errors," *J. Math. Phys*, Vol. 43, No. 9, pp. 4382–4390, Sept. 2002, `http://lanl.arxiv.org/quant-ph/0201056`
19. M. Koashi and N. Imoto, "Quantum Information is Incompressible Without Errors," *Phys. Rev. Lett*, Vol. 89, No. 9, 097904, Aug. 2002, *LANL e-print* `http://lanl.arxiv.org/quant-ph/0203045`
20. B. Schumacher, "Sending quantum entanglement through noisy channels", *Phys. Rev. A*, Vol. 54, No. 7, pp. 2614–2628, Oct. 1996, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9604023`
21. H. Barnum, M. Nielsen, and B. Schumacher, "Information transmission through a noisy quantum channel", *Phys. Rev. A*, Vol. 57, No. 7, pp. 4153–4175, June 1998, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9702049`
22. H. Barnum, C. Fuchs, R. Jozsa, and B. Schumacher, "General fidelity limit for quantum channels," *Phys. Rev. A*, Vol. 54, No. 6, pp. 4707–4711, Dec. 1996, *LANL e-print* `http://lanl.arxiv.org/quant-ph/9603014`
23. R. Jozsa and B. Schumacher, "A new proof of the quantum noiseless coding theorem," *J. Mod. Optics*, Vol. 41, pp. 2343–2349, 1994.
24. M. Nielsen, "Quantum information theory," Ph.D. thesis, Univ. of New Mexico, Albuquerque, 1998, *LANL e-print* `http://lanl.arxiv.org/quant-ph/0011036`
25. J. Kieffer, "A unified approach to weak universal source coding". *IEEE Trans. Inform. Theory*, Vol. 24, pp. 674–682, Nov. 1978.
26. J. Ziv, "Coding of sources with unknown statistics–I: Probability of encoding error," *IEEE Trans. Inform. Theory*, Vol. 18, No. 3, pp. 384–389, May 1972.
27. M. Ohya and D. Petz, *Quantum Entropy and its Use*, Springer, Berlin, 1993.
28. R. Gallager, *Information Theory and Reliable Communication*, Wiley & Sons, New York, 1968.
29. E.-H. Yang, A. Kaltchenko, and J. Kieffer, "Universal lossless data compression with side information by using a conditional MPM grammar transform," *IEEE Trans. Inform. Theory*, Vol. 47, No. 6, pp. 2130–2150, Sep. 2001.
30. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973
31. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987
32. A. Kolmogorov, *Foundations of the Theory of Probability*, Chelsea, New York, 1950.

Appendix A States on Quasilocal Commutative C^*-algebras

Let $\mathfrak{B}$ be an arbitrary commutative k-dimensional C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $\mathfrak{B}_\infty$ be a quasilocal algebra $\mathfrak{B}_\infty$ over lattice $\mathbb{Z}$ with local algebras $\mathfrak{B}_{\mathbf{x}}$ isomorphic to $\mathfrak{B}$ for every $\mathbf{x} \in \mathbb{Z}$, i.e., $\mathfrak{B}_\infty$ is constructed in the same way as is $\mathfrak{A}_\infty$ in Section 2. Then, for any $\Lambda \subset \mathbb{Z}$, every minimal projector in $\mathfrak{B}_\Lambda$ is necessarily one-dimensional, and the density operator for every pure state $\varphi^{(\Lambda)}$ on $\mathfrak{B}_\Lambda$ is exactly a one-dimensional projector. Let $\{|z_i\rangle\langle z_i|\}_{i=1}^{k}$ be a collection of the density operators for all the distinct pure states on $\mathfrak{B}$. We then define an abstract set $\mathcal{Z} := \{z_i\}_{i=1}^{k}$, where every element z_i symbolically corresponds to the operator $|z_i\rangle\langle z_i|$, and $z_i \neq z_j$ for all $i \neq j$. For every finite lattice subset $\Lambda \in \mathbb{Z}$, we define the Cartesian product

$$\mathcal{Z}^\Lambda := \mathop{\times}_{\mathbf{x}\in\Lambda} \mathcal{Z}_\mathbf{x},$$

i.e., the elements ω of $\mathcal{Z}^{\Lambda(n)}$ have the form $\omega = \omega_1 \dots \omega_n$, $\omega_i \in \mathcal{Z}$. It is easy to see that, for every $\Lambda \in \mathbb{Z}$, the set $\mathcal{Z}^\Lambda$ and the set of all one-dimensional projectors in $\mathfrak{B}_\Lambda$ are in one-to-one correspondence: $\omega \longleftrightarrow |\omega\rangle\langle\omega|$. Consequently, there is one-to-one correspondence between the set of all projectors in $\mathfrak{B}_\Lambda$ and $\mathcal{P}^\Lambda(\mathcal{Z})$, the Cartesian product of the power sets of $\mathcal{Z}$. In particular, every projector $p \in \mathfrak{B}_\Lambda$ corresponds to a set $\{\omega : \omega \in \mathcal{Z}^\Lambda, |\omega\rangle\langle\omega| \leqslant p\}$. We note that, equipped with the product of the discrete topologies of the sets $\mathcal{Z}_\mathbf{x}$, $\mathcal{Z}^\Lambda$ is a compact space, and the pair $(\mathcal{Z}^\Lambda, \mathcal{P}^\Lambda(\mathcal{Z}))$ defines a measurable space. Thus, by Gelfand-Naimark theorem[30, chap. 11] and Riesz representation theorem[31, sec. 2.14], for any pure or mixed state $\varphi^{(\Lambda)}$ on $\mathfrak{B}_\Lambda$, there exists a unique positive measure on $(\mathcal{Z}^\Lambda, \mathcal{P}^\Lambda(\mathcal{Z}))$, denoted by μ_Λ, such that the following equality holds for any projector $p \in \mathfrak{B}_\Lambda$:

$$\varphi^{(\Lambda)}(p) = \sum_{|\omega\rangle\langle\omega| \leqslant p} \mu_\Lambda(\omega) \tag{A.1}$$

Combining (A.1) and (7) and setting $a := |\omega_1 \dots \omega_m\rangle\langle\omega_m \dots \omega_1|$ in the latter, we obtain, for any $m, i \in \mathbb{N}$ and any $\omega_1 \dots \omega_m \in \mathcal{Z}^{\Lambda(m)}$,

$$\mu_{\Lambda(m)}(\omega_1 \dots \omega_m) = \sum_{\omega_{m+1} \dots \omega_{m+i}} \mu_{\Lambda(m+i)}(\omega_1 \dots \omega_m \omega_{m+1} \dots \omega_{m+i}) \tag{A.2}$$

The equality (A.2) is called the (classical) *consistency* condition. Thus, $\{\mu_\Lambda\}_{\Lambda\subset\mathbb{Z}}$ is a consistent family of probability measures, and μ_Λ extends to a probability measure on $(\mathcal{Z}^\infty, \mathcal{P}^\infty(\mathcal{Z}))$ by the Kolmogorov extension theorem[32]. The extended measure is denoted by μ.

Proposition A.1 *If a state φ on $\mathfrak{B}_\infty$ is stationary and ergodic (weakly mixing or strongly mixing, respectively), then so is the corresponding measure μ on $(\mathcal{Z}^\infty, \mathcal{P}^\infty(\mathcal{Z}))$, and the following entropy relations hold:*

$$S(\varphi^{(n)}) = H(\mu^n) \tag{A.3}$$

$$s(\varphi) = h(\mu) \tag{A.4}$$

where $H(\mu^n)$ and $h(\mu)$ denote the Shannon entropy of the probability distribution on $\mathcal{Z}^n$ defined by measure μ and the Shannon entropy rate of μ, respectively. The converse is also true.

Proof: The result follows immediately from Lemma 3.2, Lemma 3.3, and the equality (A.1). □

Appendix B Conditional expectation

Let $\tilde{\mathfrak{A}}$ be a C^*-subalgebra of $\mathfrak{A}$, and let $E : \mathfrak{A} \to \tilde{\mathfrak{A}}$ be a linear mapping which sends the density of every state φ on $\mathfrak{A}$ to the density of the state $\varphi \upharpoonright \tilde{\mathfrak{A}}$. Such a mapping is usually called a *conditional expectation* and has the following properties[27, propos. 1.12]:

(a) if $a \in \mathfrak{A}$ is positive operator, then so is $E(a) \in \tilde{\mathfrak{A}}$;

(b) $E(b) = b$ for every $b \in \tilde{\mathfrak{A}}$;

(c) $E(ab) = E(a)b$ for every $a \in \mathfrak{A}$ and $b \in \tilde{\mathfrak{A}}$;

(d) for every $a \in \mathfrak{A}$, it holds

$$\mathrm{tr}_{\mathfrak{A}}(a) = \frac{\mathrm{tr}_{\mathfrak{A}}(I)}{\mathrm{tr}_{\tilde{\mathfrak{A}}}(I)} \mathrm{tr}_{\tilde{\mathfrak{A}}}\big(E(a)\big),$$

where I stands for identity operator.

Lemma B.1 *Let $\mathfrak{A}^{\infty}$ be the quasi-local C^*-algebra, which is constructed with the copies of the finite-dimensional C^*-algebra $\mathfrak{A}$ over the lattice $\mathbb{Z}$ as described in Section 2. Let $\mathfrak{C}$ be a maximal abelian C^*-subalgebra of $\mathfrak{A}$, and let $\mathfrak{C}^{\infty} \subset \mathfrak{A}^{\infty}$ be the abelian quasi-local C^*-algebra which is constructed with the copies of $\mathfrak{C}$ over the lattice $\mathbb{Z}$. Then, for every stationary ergodic state φ on $\mathfrak{A}^{\infty}$, the state $\varphi \upharpoonright \mathfrak{C}^{\infty}$ is also stationary ergodic.*

Proof: For any integer $m > 1$, let $E_m : \mathfrak{A}^{(m)} \to \mathfrak{C}^{(m)}$ be the conditional expectation mapping which sends the density of $\varphi^{(m)}$ to the density of $\varphi^{(m)} \upharpoonright \mathfrak{C}^{(m)}$, and let $\{\rho_m\}_{m=1}^{\infty}$ be the family of density operators for φ. Since $\mathfrak{C}^{(m)}$ is a maximal abelian subalgebra of $\mathfrak{A}^{(m)}$, we have $\mathrm{tr}_{\mathfrak{A}^{(m)}}(I) = \mathrm{tr}_{\mathfrak{C}^{(m)}}(I)$. Then, the following equalities hold by the properties of conditional expectation for all positive integers $m < i < \infty$ and all $a, b \in \mathfrak{C}^{(m)}$:

$$\mathrm{tr}_{\mathfrak{A}^{(m+i)}}\Big(\rho_{m+i}\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big) = \mathrm{tr}_{\mathfrak{C}^{(m+i)}}\bigg(E_{m+i}\Big(\rho_{m+i}\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big)\bigg)$$

$$= \mathrm{tr}_{\mathfrak{C}^{(m+i)}}\Big(E_{m+i}(\rho_{m+i})\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big),$$

$$\mathrm{tr}_{\mathfrak{A}^{(m)}}(\rho_m a) = \mathrm{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\rho_m a)\big) = \mathrm{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\rho_m)a\big),$$

$$\mathrm{tr}_{\mathfrak{A}^{(m)}}(\rho_m b) = \mathrm{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\rho_m b)\big) = \mathrm{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\rho_m)b\big).$$

Thus, the family $\{E_m(\rho_m)\}_{m=1}^{\infty}$ is consistent, stationary, and ergodic by the lemmas 3.1, 3.2, and 3.3. □

Appendix C Proofs

Proof of Theorem 3.1:

For any TPCPL map there exists a so-called "operator-sum representation"[16],[15]. Then, an m-fold tensor product map $\mathcal{E}^{\otimes m}$ has the following representation:

$$\mathcal{E}^{\otimes m}(\rho_m) = \sum_{j_1, j_2, \cdots, j_m} \big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big)\rho_{[1,m]}\big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big)^{\dagger} \tag{C.1}$$

with

$$\sum_i A_i^{\dagger} A_i = I, \quad A_i, I \in \mathfrak{A}, \tag{C.2}$$

where I stands for identity operator.

Due to (C.1) and (C.2), the following three equalities hold for all positive integers $m < i < \infty$

and all $a, b \in \mathfrak{A}^{(m)}$

$$\begin{aligned}
&\operatorname{tr}\big(\mathcal{E}^{\otimes(m+i)}(\rho_{m+i})\ (a \otimes I^{\otimes(i-m)} \otimes b)\big) = \operatorname{tr}\big(\rho_{m+i}\ (\tilde{a} \otimes I^{\otimes(i-m)} \otimes \tilde{b})\big),\\
&\operatorname{tr}(\mathcal{E}^{\otimes m}(\rho_m)a) = \operatorname{tr}(\rho_m \tilde{a}),\\
&\operatorname{tr}(\mathcal{E}^{\otimes m}(\rho_{\Lambda(m)})b) = \operatorname{tr}(\rho_m \tilde{b}),
\end{aligned} \tag{C.3}$$

where $a, b \in \mathfrak{A}^{(m)}$ and $\tilde{a}$ and $\tilde{b}$ are defined as follows:

$$\tilde{a} := \sum_{j_1, j_2, \cdots, j_m} \big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big)^{\dagger} a \big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big),$$

$$\tilde{b} := \sum_{j_1, j_2, \cdots, j_m} \big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big)^{\dagger} b \big(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}\big).$$

Combining (C.3) with Lemma 3.3, we obtain the ergodicity (weakly mixing or strongly mixing, respectively) of $\{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty}$. In a similar manner, the application of Lemma 3.1 establishes the consistency of $\{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty}$, and the application of Lemma 3.2 establishes the stationarity of $\{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty}$. □

Proof of Corollary 3.1:

Let $S_{\perp} := \{|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle\}$ be any orthonormal basis in $\mathcal{H}$, and let $\{\tilde{\rho}_m^{cls}\}_{m=1}^{\infty}$ be the source with alphabet $S_{\perp}$ and distribution $p(\cdot)$. For $i = 1, \ldots, d$, we define a set $\{A_i\}$ of linear operators as follows

$$A_i := |\psi_i\rangle\langle e_i|. \tag{C.4}$$

Then, set $\{A_i\}$ satisfies (C.2), and we define a TPCPL map $\mathcal{E}^{\otimes m}$ as in (C.1). Consequently, we have $(\rho_m^{cls}) = \mathcal{E}^{\otimes m}(\tilde{\rho}_m^{cls})$. Thus, to complete the proof, we need to show that $\{\tilde{\rho}_m^{cls}\}_{m=1}^{\infty}$ on $\mathfrak{A}_{\infty}$ is ergodic (weakly mixing or strongly mixing, respectively). Let $\mathfrak{C}$ be a subalgebra of $\mathfrak{A}$ spanned by the set $\{|e_i\rangle\langle e_i| \ :\ |e_i\rangle \in S_{\perp}\}$. We extend $\mathfrak{C}$ to a quasilocal algebra $\mathfrak{C}_{\infty} \subset \mathfrak{A}_{\infty}$ over lattice $\mathbb{Z}$ in the same way we did for $\mathfrak{A}_{\infty}$. The algebra $\mathfrak{C}_{\infty}$ is abelian due to the orthogonality of the set $S_{\perp}$. For any integer $m > 1$, let $E_m : \mathfrak{A}^{(m)} \to \mathfrak{C}^{(m)}$ denote the conditional expectation. Since $\mathfrak{C}^{(m)}$ is a maximal abelian subalgebra of $\mathfrak{A}^{(m)}$, we have $\operatorname{tr}_{\mathfrak{A}^{(m)}}(I) = \operatorname{tr}_{\mathfrak{C}^{(m}}(I)$. Moreover, by our construction, $\tilde{\rho}_m^{cls}$ is an element of algebra $\mathfrak{C}^{(m)} \subset \mathfrak{A}^{(m)}$ for every m. Then, the following equalities hold by the properties of conditional expectation for all positive integers $m < i < \infty$ and all $a, b \in \mathfrak{A}^{(m)}$:

$$\begin{aligned}
&\operatorname{tr}_{\mathfrak{A}^{(m+i)}}\Big(\tilde{\rho}_{m+i}^{cls}\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big) = \operatorname{tr}_{\mathfrak{C}^{(m+i)}}\Big(E_{m+i}\Big(\tilde{\rho}_{m+i}^{cls}\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big)\Big)\\
&= \operatorname{tr}_{\mathfrak{C}^{(m+i)}}\Big(\tilde{\rho}_{m+i}^{cls} E_{m+i}\big(a \otimes I^{\otimes(i-m)} \otimes b\big)\Big),\\
&\operatorname{tr}_{\mathfrak{A}^{(m)}}(\tilde{\rho}_m^{cls} a) = \operatorname{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\tilde{\rho}_m^{cls} a)\big) = \operatorname{tr}_{\mathfrak{C}^{(m)}}\big(\tilde{\rho}_m^{cls} E_m(a)\big),\\
&\operatorname{tr}_{\mathfrak{A}^{(m)}}(\tilde{\rho}_m^{cls} b) = \operatorname{tr}_{\mathfrak{C}^{(m)}}\big(E_m(\tilde{\rho}_m^{cls} b)\big) = \operatorname{tr}_{\mathfrak{C}^{(m)}}\big(\tilde{\rho}_m^{cls} E_m(b)\big).
\end{aligned}$$

Thus, if $\{\tilde{\rho}_m^{cls}\}_{m=1}^{\infty}$ is consistent, stationary, and ergodic (weakly mixing or strongly mixing, respectively) on $\mathfrak{C}_{\infty}$, then it also holds on $\mathfrak{A}_{\infty}$ by the lemmas 3.1, 3.2, and 3.3. Finally, we note that since $\mathfrak{C}_{\infty}$ is abelian, $\{\tilde{\rho}_m^{cls}\}_{m=1}^{\infty}$ on $\mathfrak{C}_{\infty}$ is ergodic (weakly mixing or strongly mixing, respectively) if and only if so is $p(\cdot)$ by Proposition A.1 from the appendix A. □